Initial data for two Kerr-like black holes

Sergio Dain
Albert-Einstein-Institut, Max-Planck-Institut für Gravitationsphysik, Am Mühlenberg 1, D-14476 Golm, Germany
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We prove the existence of a family of initial data for the Einstein vacuum equation which can be interpreted as the data for two Kerr-like black holes in arbitrary location and with spin in arbitrary direction. When the mass parameter of one of them is zero, this family reduces exactly to the Kerr initial data. The existence proof is based on a general property of the Kerr metric which can be used in other constructions as well. Further generalizations are also discussed.

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Introduction. —Black-hole collisions are considered as one of the most important sources of gravitational radiation that may be observable with the gravitational wave detectors currently under construction. The first step in the study of a black-hole collisions is to provide proper initial data for the Einstein vacuum equation. Initial data for two black holes were first constructed by Misner [20], shortly after Brill and Lindquist [8] studied a similar data but with a different topology which considerable simplifies the construction. Bowen-York [5] included linear and angular momentum. Generalizations of these data were studied in [4] [3] [2] [7] (see also the review [11] and the references therein).

These families of initial data depend on the mass, the momentum, the spin, and the location of each black hole. When the mass parameter of one of the holes is zero, one obtains initial data for only one black hole. It is physically reasonable to require that these black hole data be stationary, i.e.; a slice of Schwarzschild or Kerr space time. If this is not the case, it means that spurious gravitational radiation is present in the initial data. In the case of [20] and [8], this gives the Schwarzschild initial data. But for the other cases, where angular momentum is included, one does not obtain the initial data of the Kerr metric.

The Kerr initial data are not included in the families considered above because the restrictions imposed on the conformal 3-metric are too strong. In most cases it is assumed that the conformal metric is flat. However, it appears that the Kerr metric admits no conformally flat slices (in fact, in [15] it has been shown that there does not exist axisymmetric, conformally flat foliations of the Kerr spacetime that smoothly reduce, in the Schwarzschild limit, to slices of constant Schwarzschild time). Much weaker conditions have been imposed on the 3-metric which nevertheless exclude Kerr data; for example in [4] it is required that the conformal metric admit a smooth compactification. We will see that, at least for the Boyer-Lindquist slices, this condition is also strong enough to exclude Kerr data.

The purpose of this article is to generalize these constructions above in order to include the Kerr initial data as a particular case in which the mass parameter of one black hole is zero. This family of initial data is the natural generalization of the one found in [8]. For the axisymmetric case, similar data have been calculated numerically [18]. Recently, a different type of initial data has been also calculated numerically [19]. The existence proof is based on general property of the Kerr initial data, which may also be useful in other constructions. The plan of this article is as follows: first we prove this key property of the Kerr metric; second we give a remarkably simple application of it in the construction of data that can be interpreted as data for a Schwarzschild and a Kerr black hole; then we construct more complicate initial data, which can be interpreted as the data for two Kerr black holes. Finally, certain generalizations are discussed.

The Kerr initial data. —Consider the Kerr metric in the Boyer-Lindquist coordinates $(t, \tilde{r}, \vartheta, \phi)$ [17] [6], with mass m and angular momentum a such that $m > a$. Take any slice $t = const$. Denote by $\tilde{h}^k_{ab}$ the intrinsic three metric of the slice and by $\tilde{\Psi}^{ab}_k$ its extrinsic curvature. These slices are maximal, i. e. $\tilde{h}^k_{ab}\,\tilde{\Psi}^{ab}_k = 0$. The metric $\tilde{h}^k_{ab}$ is given in the coordinates $(\tilde{r}, \vartheta, \phi)$ by

$$\tilde{h}^k \equiv \frac{\Sigma}{\Delta} d\tilde{r}^2 + \Sigma d\vartheta^2 + \eta d\phi^2, \tag{1}$$

where

$$\Sigma = \tilde{r}^2 + a^2 \cos^2\vartheta, \quad \Delta = \tilde{r}^2 + a^2 - 2m\tilde{r}, \tag{2}$$

and

$$\eta = \sin^2\theta(\Sigma + a^2 \sin^2\theta(1 + \hat{\sigma})), \quad \hat{\sigma} = \frac{2m\tilde{r}}{\Sigma}. \tag{3}$$

The metric is singular where Δ or Σ vanishes. The zeros of the function Δ are given by

$$\tilde{r}_+ = m + \delta, \quad \tilde{r}_- = m - \delta, \tag{4}$$

with $\delta = \sqrt{m^2 - a^2}$.

Consider the coordinate transformation

$$\tilde{r} = \frac{\alpha^2 \cos^2(\psi/2) + \delta^2 \sin^2(\psi/2)}{\alpha \sin\psi} + m, \quad 0 \le \psi \le \pi, \tag{5}$$

where α is a positive constant. This transformation is the composition of the transformation to the quasi isotropical radius $\bar{r}$ and a stereographic projection, i.e.

$$\tilde{r} = \bar{r} + m + \frac{\delta^2}{4\bar{r}}, \quad \bar{r} = \frac{\alpha\cos(\psi/2)}{2\sin(\psi/2)}. \tag{6}$$

It is defined for for $\tilde{r} > \tilde{r}_+$, and becomes singular at $\tilde{r}_+$. We consider (ψ, ϑ, ϕ) as standard coordinates on S^3. The south pole is given by $\psi = 0$ and the north pole by $\psi = \pi$, we will denote them by $\{0\}$ and $\{\pi\}$ respectively. Due to the isometry $\bar{r} \to \delta^2/(4\bar{r})$, the transformation (5) maps one copy of the region $\tilde{r} > \tilde{r}_+$ into the region $\psi > \psi_+$ of S^3, where $\psi_+ = 2\arctan(\alpha/\delta)$, and another copy into $\psi < \psi_+$. In the new coordinates the metric (1) extend to a smooth metric in $S^3 - \{0\} - \{\pi\}$. This manifold defines a space like hypersurface in the Kerr space time which, in figure 28 of [16], can be indicated by horizontal straight line going from one apex of a region I to the opposite apex of the adjacent region I. The poles $\{0\}$ and $\{\pi\}$ are precisely these apexes, they represent the space like infinities of the initial data. This hypersurface is a Cauchy surface for an asymptotically flat region of the Kerr space time (comprising two regions I and II respectively).

Using the conformal factor

$$\theta_k = \frac{\Sigma^{1/4}}{\sqrt{\sin\psi}}, \tag{7}$$

define the conformal metric h^k_{ab} by

$$h^k_{ab} = \theta_k^{-4}\tilde{h}^k_{ab}. \tag{8}$$

The conformal factor θ_k is singular at $\{0\}$ and $\{\pi\}$,

$$\lim_{\psi\to\pi}(\psi - \pi)\theta_k = \sqrt{\delta}, \quad \lim_{\psi\to 0}\psi\theta_k = \sqrt{\alpha}. \tag{9}$$

The metric h^k_{ab} has the form

$$h^k_{ab} = h^0_{ab} + a^2 f v_a v_b, \tag{10}$$

where h^0_{ab} is the standard metric of S^3, the smooth vector field v_a is given by $v_a \equiv \sin^2\psi\sin^2\vartheta(d\phi)_a$, and the function f, which contains the non-trivial part of the metric, is given by

$$f = \frac{(1+\hat{\sigma})}{\Sigma\,\sin^2\psi}. \tag{11}$$

The function f depends on $a, m, \sin\psi, \cos\vartheta$. It is smooth in $S^3 - \{0\} - \{\pi\}$. In order to analyze the differentiability of f at the poles, take a normal coordinate system x^i with respect to the metric h^k_{ab}, centered at one of the poles, define the radius $|x| = (\sum_{i=1}^3 (x^j)^2)^{1/2}$. In terms of these coordinates the function ψ, given by $\psi = |x|$, is seen to be a C^α function of x^i. From the expression (11) one can prove that the function f has the form

$$f = f_1 + f_2\sin^3\psi, \tag{12}$$

where f_1 and f_2 are smooth functions in the neighborhood of the poles, with respect to the coordinates x^i. Since $\sin^3\psi \in W^{4,p}$, $p < 3$, (see e.g. [1] for the definitions of the Sobolev and Hölder spaces $W^{s,p}$ and $C^{m,\alpha}$) from expression (12) we see that

$$h^k_{ab} \in W^{4,p}(S^3), \quad p < 3. \tag{13}$$

This is the crucial property of the metric that will be used in the existence proof. In fact, it is the only property of the Kerr metric that we will need. It implies, in particular, that the metric is in $C^{2,\alpha}(S^3)$. Since the poles $\{0\}$ and $\{\pi\}$ are the infinities of the data, the expression (12) characterizes the fall-off behavior of the Kerr initial data near space like infinity. The Ricci scalar R of the metric h^k_{ab} is a continuous function of the parameter a, and for $a = 0$ we have that $R = 6$, the scalar curvature of h^0_{ab}. Thus, if a is sufficiently small, R will be a positive function on S^3. In the following we will assume the latter condition to be satisfied.

It remains to analyze the extrinsic curvature of the Kerr initial data. Define Ψ^{ab}_k by

$$\Psi^{ab}_k = \theta^{10}_k\tilde{\Psi}^{ab}_k. \tag{14}$$

The tensor Ψ^{ab}_k is smooth in $S^3 - \{0\} - \{\pi\}$ and at the poles it has the form

$$\Psi^{ab}_k = \Psi^{ab}_J + Q^{ab} \tag{15}$$

where $\Psi^{ab}_J = O(|x|^{-3})$ and it is trace-free and divergence free with respect to the flat metric (it contains the angular momentum of the data and the explicit form of this tensor is given in [5]). The tensor Q^{ab} is $O(|x|^{-1})$. If $a = 0$ then $\Psi^{ab}_k = 0$.

The coordinate transformation (5) simplifies considerably if we choose $\alpha = \delta$. This choice makes the metric (10) symmetric with respect to $\psi = \pi/2$, it is useful in explicit calculations. Nevertheless, this choice is inconvenient for our present purpose, since it is singular when $\delta = 0$ and we want to have the flat initial data in this limit. In the following we will assume $\alpha = 1$.

Initial data with Schwarzschild-like and Kerr-like asymptotic ends —The conformal approach to find solutions of the constraint equations with many asymptotically flat end points i_n is the following (cf. [9], [10] and the reference given there. The setting outlined here, where we have to solve (17), (19) on the compact manifold has been studied in [4], [13], [14]). Let S be a compact manifold (in our case it will be S^3), denote by i_n a finite number of points in S, and define the manifold $\tilde{S}$ by $\tilde{S} = S\backslash\bigcup i_n$. We assume that h_{ab} is a positive definite metric on S, with covariant derivative D_a, and Ψ^{ab} is a trace-free symmetric tensor, which satisfies

$$D_a\Psi^{ab} = 0 \quad \text{on} \quad \tilde{S}. \tag{16}$$

Let θ a solution of

$$L_h\theta = -\frac{1}{8}\Psi_{ab}\Psi^{ab}\theta^{-7} \quad \text{on} \quad \tilde{S}, \tag{17}$$

where $L_h = D^aD_a - R/8$. Then the physical fields $(\tilde{h}, \tilde{\Psi})$ defined by $\tilde{h}_{ab} = \theta^4 h_{ab}$ and $\tilde{\Psi}^{ab} = \theta^{-10}\Psi^{ab}$ will satisfy the vacuum constraint equations on $\tilde{S}$. To ensure asymptotic flatness of the data at the points i_n we require at each point i_n

$$\Psi^{ab} = O(|x|^{-4}) \quad \text{as} \quad x \to 0, \tag{18}$$

$$\lim_{|x|\to 0} |x|\theta = c_n, \tag{19}$$

where the c_n are positive constants, and x^i are normal coordinates centered at i_n.

Since (h^k, Ψ_k) are obtained from the Kerr solution, they satisfy equations (16) and (17); and also the boundary conditions (18) and (19) at each of the poles, since they satisfy equations (9) and (15). The Kerr metric h^k satisfies (13), then the coefficients of the elliptic operator L_{h^k} satisfy the hypothesis of the existence theorems proved in [12], in particular they are in $C^\alpha(S^3)$. From these theorems, it follows that, for arbitrarily chosen point $i \in S^3$, there exists a unique, positive function θ_i, which satisfies

$$L_{h^k}\theta_i = 0, \text{ in } S^3 - \{i\}, \tag{20}$$

and at i

$$\lim_{x\to 0} |x|\theta_i = 1, \tag{21}$$

$|x|$ denoting the distance from i.

We denote by θ_0, θ_π the solutions so obtained by choosing the point i to be $\{0\}$ and $\{\pi\}$ respectively and write the Kerr conformal factor θ_k in the form

$$\theta_k = \theta_0 + \sqrt{\delta}\theta_\pi + u_k.$$

The function u_k is then in $C^\alpha(S^3)$.

In order to produce another asymptotic end in the initial data above, take an arbitrary point $i_1 \in S^3$, different from $\{0\}$ and $\{\pi\}$, with coordinates $(\psi_1, \vartheta_1, \phi_1)$, and consider the corresponding function θ_1 which satisfies (20) and (21) in i_1. Define the function θ_{sk} by

$$\theta_{sk} = \theta_0 + \sqrt{\delta}\theta_\pi + \sqrt{m_1}\theta_1 \sin(\psi_1/2) + u_{sk}, \tag{22}$$

where m_1 is an arbitrary, positive, constant. Insert this in equation (17), where we use Ψ^{ab}_k in place of Ψ^{ab} and h^k_{ab} in place of h_{ab}. Observing that we used (20), we obtain an equation for u_{sk} on S^3. The right hand side of this equation is in $L^2(S^3)$, since the singular behavior of Ψ^{ab} at the poles is canceled by the negative power of θ. In [12] it has been proven that this equation has a unique, positive, solution, under the conditions stated above. In this particular case, since there is no linear momentum in any of the asymptotic ends, the extrinsic curvature Ψ^{ab}_k is $O(|x|^{-3})$ at the poles, and then the right hand side of (17) is in $C^\alpha(S^3)$. But it is important to recall that the existence theorem also applies when a term with linear momentum is included at the ends. We will come back to this point later on. It follows that, for arbitrarily chosen point i_1, the tensors

$$\tilde{h}^{sk}_{ab} = \theta^{-4}_{sk} h^k_{ab}, \quad \tilde{\Psi}^{ab}_{sk} = \theta^{10}_{sk}\Psi^{ab}_k, \tag{23}$$

define a solution of the vacuum constraint equations.

These initial data have three asymptotic ends $\{0\}$, $\{\pi\}$ and $\{i_1\}$, i.e.; they have the same topology as the data obtained in [8]. Moreover, when $a = 0$ we obtain exactly the same solution obtained in [8] with mass m and m_1 (this is the reason for the factor $\sin(\psi_1/2)$ in (22)). Then, at least for small a, we expect the same behavior of the apparent horizons as the one discussed there. That is, when the mass parameter m and m_1 are small with respect of the separations between the ends, only two apparent horizons will appear, surrounding $\{\pi\}$ and $\{i_1\}$. This makes a geometric distinction between the ends $\{\pi\}$ and $\{i_1\}$, which have an apparent horizon around them, and $\{0\}$, which has not. When the separation of the ends is comparable with the masses, we expect that another apparent horizon appears around $\{0\}$. The evolution of these data will presumably contain an event horizon, the final picture of the whole space time will be similar to the one shown in figure 60 of [16], which represents a collision and merging of two black holes. The asymptotic end $\{i_1\}$ can be interpreted as the 'Schwarzschild' end, since when $m = a = 0$ (i.e.; when the end $\{\pi\}$ is not present) we obtain exactly the Schwarzschild initial data with mass m_1. One can expect that the geometry near $\{i_1\}$ approximates, in some sense, the Schwarzschild geometry. In an analogous way, when $m_1 = 0$, we obtain the Kerr initial data with mass m and angular momentum a. We then say that $\{\pi\}$ is the 'Kerr' asymptotic end. If we chose $\vartheta_1 \neq 0, \pi$ the data will be non-axially symmetric. It is remarkable that, in order to construct these data, the only new function that one has to compute is the conformal factor θ_{sk}, the conformal metric (8) and the conformal extrinsic curvature (14) being given explicity by the Kerr geometry.

Initial data with two Kerr-like asymptotic ends. — Take the Kerr initial data in coordinates $(\bar{r}, \vartheta, \phi)$. Make a rigid rotation such that the spin point in the direction of an arbitrary unit vector S^a_1, and make a shift of the origin $\bar{r} = 0$ to the coordinate position of an arbitrary point i_1. Let the mass and the modulus of angular momentum of this data be m_1 and a_1. We apply the stereographic projection (6) and the conformal rescaling (7). Then, we obtain a rescaled metric $h^{k_1}_{ab} = h^0_{ab} + a_1^2 f_1 v^1_a v^1_b$, where f_1 and v^1_a are obtained from f and v_a by the rotation and

the shift of the origin, they depend on the coordinates of the point i_1 and the vector S_1^a. In S^3, this coordinate transformation is a smooth conformal mapping with a fixed point at $\{0\}$. In an analogous way we define the corresponding rescaled extrinsic curvature $\Psi_{k_1}^{ab}$. Take another vector S_2^a and another point i_2 and make the same construction. We define the following metric

$$h_{ab}^{kk} = h_{ab}^0 + a_1^2 f_1 v_a^1 v_b^1 + a_2^2 f_2 v_a^2 v_b^2. \tag{24}$$

By (13) we have that this metric is in $W^{4,p}(S^3)$. Is also clear that for small a_1 and a_2 the scalar curvature is positive.

Set $\bar{\Psi}_{k_1}^{ab}$ to be the trace-free part of $\Psi_{k_1}^{ab}$ with respect to the metric h_{ab}^{kk}. Define the tensor Ψ_{kk}^{ab} by

$$\Psi_{kk}^{ab} = \bar{\Psi}_{k_1}^{ab} + \bar{\Psi}_{k_2}^{ab} + (lw)^{ab}, \tag{25}$$

where $(lw)^{ab}$ is the conformal Killing operator l, with respect to the metric h_{ab}^{kk}, acting on a vector w^a. In [12] we have proved that there exist a unique $w^a \in W^{2,p}(S^3)$ such Ψ_{ab}^{kk} satisfies (16). When a_1 or a_2 is equal to zero, then Ψ_{kk}^{ab} is equal to $\Psi_{k_1}^{ab}$ or $\Psi_{k_2}^{ab}$ respectively, since the solution is unique.

Define the conformal factor θ_{kk} by

$$\theta_{kk} = \theta_0 + \sqrt{\delta_1}\theta_1 \sin(\psi_1/2) + \sqrt{\delta_2}\theta_2 \sin(\psi_2/2) + u_{kk}, \tag{26}$$

where θ_1 and θ_2 satisfy (20) and (21) for i_1 and i_2 respectively, with respect to the metric (24). Using again the existence theorem proved in [12], we have that there exist a unique, positive, solution u_{kk} of equation (17), where we have replaced h_{ab} by h_{ab}^{kk} and Ψ^{ab} by Ψ_{kk}^{ab}. Then, we have constructed a solution

$$\tilde{h}_{ab}^{kk} = \theta_{kk}^{-4} h_{ab}^{kk}, \quad \tilde{\Psi}_{kk}^{ab} = \theta_{kk}^{10} \Psi_{kk}^{ab}. \tag{27}$$

of the constraint equation. This solution has also three asymptotic ends $\{0\}$, $\{i_1\}$ and $\{i_2\}$, and when $a_1 = a_2 = 0$ we obtain the solution [8] with mass m_1 and m_2. Then, at least for small a_1 and a_2, we expect that the same behavior of the apparent horizons as the one discussed before. The main difference is that now both ends i_1 and i_2 are 'Kerr' ends, since when $m_1 = a_1 = 0$ we obtain the Kerr initial data, and the same is true for $m_2 = a_2 = 0$. We can expect that the geometry near each of this ends is similar, in some sense, to the geometry of the Kerr initial data, when the mass are small with respect to the separation. Numerical comparison for the conformal factor, which exhibits this behavior, has been made in [18] for the axisymmetric case.

Conclusion. We have constructed a family of initial data that can be interpreted as representing two Kerr black holes. It reduces exactly to the Kerr initial data when the mass of one of them is zero. This is the first rigorous proof of the existence of such a class of initial data. We have chosen the ansatz (24), which is perhaps the simplest one, but other choices are possible too. The only requirement we must impose on the conformal metric (24) is that it reduces to the conformal Kerr metric when a_1 or a_2 is equal to zero and satisfies (13), this is a very mild condition. It is also possible to add an extra term in the extrinsic curvature (25) which contains the linear momentum of each black hole. The existence proof is exactly the same (see [12]). However, we will not have either Kerr or Schwarzschild when only one black hole is present, since the Boyer-Lindquist slices are not boosted, this is exactly the same situation as for the boosted data given in [5]. The conformal Kerr metric has the special form (12), this can be used to prove additional regularity properties of the initial data, this will be done in future work. In order to see whether the gravitational waves emitted in the case of our data differ in a significant way from the waves observed for Bowen-York data, it would be interesting to compare the numerical evolution of the corresponding space-times.

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[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] R. Beig. Generalized Bowen-York initial data. In S. Cotsakis and G. Gibbons, editors, *Mathematical and Quantum Aspects of Relativity and Cosmology*, volume 537, pages 55–69. Springer Lecture Note in Physics, Berlin, 2000. also in gr-qc/0005043.

[3] R. Beig and S. Husa. Initial data for general relativity with toroidal conformal symmetry. *Phys. Rev. D*, 50:7116–7118, 1994.

[4] R. Beig and N. O'Murchadha. Trapped surface in vacuum spacetimes. *Class. Quantum Grav.*, 11(2):419–430, 1994.

[5] J. M. Bowen and J. W. York, Jr. Time-asymmetric initial data for black holes and black-hole collisions. *Phys. Rev. D*, 21(8):2047–2055, 1980.

[6] R. H. Boyer and R. W. Lindquist. Maximal analytic extension of the Kerr metric. *J. Math. Phys.*, 8(2):265–281, 1967.

[7] S. Brandt and B. Brügmann. A simple construction of initial data for multiple black holes. *Phys. Rev. Lett.*, 78(19):3606–3609, 1997.

[8] D. Brill and R. W. Lindquist. Interaction energy in geometrostatics. *Phys.Rev.*, 131:471–476, 1963.

[9] Y. Choquet-Bruhat, J. Isenberg, and J. W. York, Jr. Einstein constraint on asymptotically euclidean manifolds. *gr-qc/9906095*, 1999.

[10] Y. Choquet-Bruhat and J. W. York, Jr. The Cauchy problem. In A. Held, editor, *General Relativity and Grav-*

itation, volume 1, pages 99–172. Plenum, New York, 1980.

[11] G. B. Cook. Initial data for numerical Relativity. *gr-qc*, page 0007085, 2000.

[12] S. Dain and H. Friedrich. Asymptotically flat initial data with prescribed regularity. 2000. in preparation.

[13] H. Friedrich. On static and radiative space-time. *Commun. Math. Phys.*, 119:51–73, 1988.

[14] H. Friedrich. Gravitational fields near space-like and null infinity. *J. Geom. Phys.*, 24:83–163, 1998.

[15] A. Garat and R. Price. Nonexistence of conformally flat slices of the kerr spacetime. *Phys. Rev. D*, 61:124011, 2000.

[16] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge University Press, Cambridge, 1973.

[17] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11:237–238, 1963.

[18] W. Krivan and R. Price. Initial data for superposed rotating black holes. *Phys. Rev. D*, 58:104003, 1998.

[19] P. Marronetti and R. A. Matzner. Soving the initial value problem of two black holes. *gr-qc/0009044 (to appear in Phys. Rev. Lett.)*, 2000.

[20] C. Misner. Wormhole initial conditions. *Phys.Rev.*, 118:1110, 1960.