Cubic couplings in $D=6$, $\mathcal{N}=4b$ supergravity on $\mathrm{AdS}_3\times\mathrm{S}^3$

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We determine the AdS exchange diagrams needed for the computation of 4-point functions of chiral primary operators in the SCFT_2 dual to the $D=6$, $\mathcal{N}=4b$ supergravity on the $\mathrm{AdS}_3\times\mathrm{S}^3$ background and compute the corresponding cubic couplings. We also address the issue of consistent truncation.

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I. INTRODUCTION AND SUMMARY

The AdS–conformal field theory correspondence provides information about the strong coupling behavior of some conformal field theories by studying their supergravity (string) duals [1–4]. In particular, the AdS-CFT duality relates type IIB string theory on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{M}^4$, where M^4 is either $K3$ or T^4, to a certain $\mathcal{N}=(4,4)$ supersymmetric two-dimensional conformal field theory (CFT) living on the boundary of AdS_3. A two-dimensional sigma model with the target space being a deformation of the orbifold symmetric product $S^N(M^4)=(M^4)^N/S_N$, $N\to\infty$, is believed to provide an effective description of this CFT [5].

An important class of operators in the supersymmetric CFT are the chiral primary operators (CPOs) since they are annihilated by 1/2 of the supercharges and in two dimensions their highest weight components of the R-symmetry group form a ring. On the gravity side the CPOs correspond to Kaluza-Klein (KK) modes of the type IIB supergravity compactification. Recently, using the orbifold technique developed in [6] the three-point functions of scalar CPOs were computed [7] in the CFT on the symmetric product $S^N(M^4)$ and, on the other hand, in $\mathrm{AdS}_3\times\mathrm{S}^3$ supergravity [8], and were found to disagree.[1] On the other hand, computations of quantities that are stable under deformations of the orbifold CFT, like the spectrum of the CPOs and the elliptic genus, were found to be in complete agreement [10]. Obviously this supports the expectation that the $\mathrm{AdS}_3\times\mathrm{S}^3$ background may correspond to some deformation of the target space $S^N(M^4)$ of the boundary CFT. However, even though one presently does not have an explicit sigma model formulation of the boundary CFT (see also [11] for recent developments), one may proceed to study the CFT by using directly the gravity dual description and the AdS-CFT correspondence [12].

In this paper we study the $\mathrm{AdS}_3\times\mathrm{S}^3$ compactification of the $D=6$, $\mathcal{N}=4b$ supergravity coupled to n tensor multiplets. In particular, the case $n=21$ corresponds to the theory obtained by dimensional reduction of type IIB supergravity on $K3$. Our final aim will be to find the 4-point correlation functions of the scalar CPOs in the supergravity approximation. This program was successfully performed for the $\mathcal{N}=4$ four-dimensional super Yang-Mills theory which is related to the $\mathrm{AdS}_5\times\mathrm{S}^5$ compactification of type IIB supergravity and led to an understanding of the structure of the operator product expansion in the field theory at strong coupling [9,13–16]. As a first necessary step in this direction we derive the effective gravity action on AdS_3 that contains all cubic couplings involving at least two gravity fields corresponding to CPOs in the boundary CFT.

Since the supergravity we consider is a chiral theory it suffers from the absence of a simple Lagrangian formulation. In principle, one may approach the problem of computing correlation functions by using the Pasti-Sorokin-Tonin formulation of the six-dimensional supergravity action, where the manifest Lorentz covariance is achieved by introducing an auxiliary scalar field a [17]. However, to obtain the action for physical fields one needs to fix the gauge symmetries, in particular the additional symmetry associated with the field a. This breaks the manifest Lorentz covariance and makes the problem of solving the noncovariant constraints imposed by gauge fixing unfeasible. Thus, we prefer to start with the covariant equations of motion of chiral six-dimensional supergravity [18] and obtain the quadratic, cubic and so on corrections to the equations of motion for physical fields by decomposing the original equations near the $\mathrm{AdS}_3\times\mathrm{S}^3$ background and partially fixing the gauge (diffeomorphism) symmetries. The equations obtained in this way are in general non-Lagrangian with higher derivative terms and we perform the nonlinear field redefinitions to remove higher derivative terms [13] and bring the equations to the Lagrangian form.

The spectrum of the $\mathrm{AdS}_3\times\mathrm{S}^3$ compactification of the $D=6$, $\mathcal{N}=4b$ supergravity coupled to n tensor multiplets was found in [19] and it is governed by the supergroup $SU(1,1|2)_L\times SU(1,1|2)_R$. Since we are interested in the quadratic and ultimately in cubic corrections to the equations of motion for the gravity fields we reconsider the derivation of the linearized equations of motion and recover the spectrum of [19]. According to [19] the scalar CPOs are divided into two classes. The first class contains CPOs σ that are

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[1]Computing the 2- and 3-point functions of CPOs in the supergravity approximation by using the prescription [3], which is known to be compatible with the Ward identities [9], one finds a result different from [8]. This however does not remove the disagreement between CFT and gravity calculations.

singlets with respect to the internal symmetry group $SO(n)$.[2] The corresponding gravity fields are mixtures of the trace of the graviton on S^3 and the sphere components of the self-dual form. The second class comprise the CPOs s^r transforming in the fundamental representation of $SO(n)$ and the corresponding gravity fields are mixtures of n from $5n$ scalars describing the coset space $SO(5,n)/SO(5)\times SO(n)$ and the sphere components of the n antiself-dual forms.

We find that the fields appearing in the exchange diagrams involving at least two CPOs include in addition to the CPOs themselves, also other scalars or vectors, either in the singlet or in the vector representation of $SO(n)$, and symmetric 2nd rank (massive) tensor fields. We determine the corresponding cubic couplings. By using the factorization property of the Maxwell operator in odd dimensions we diagonalize the equations for the vector fields which originate from components of the second order Einstein equation and the first order self-duality equation. This diagonalization is helpful to identify the vector fields propagating in the AdS exchange diagrams.

The cubic couplings exhibit the same vanishing property in the extremal case (e.g., for three scalar fields σ_k, where k denotes a Kaluza-Klein mode, the extremality condition is $k_1+k_2=k_3$ and permutations thereof) as the cubic couplings found in the compactification of type IIB supergravity on $AdS_5\times S^5$ [13].

In addition to the cubic couplings involving CPOs we also compute certain cubic couplings of vector fields, which allows us to check the consistency of the KK truncation of the three-dimensional action to the massless graviton multiplet. Recall that the bosonic part of this multiplet contains the graviton and the $SU(2)_L\times SU(2)_R$ gauge fields, all of them carrying nonpropagating (topological) degrees of freedom. Since the other multiplets contain the propagating modes, the graviton multiplet should admit a consistent truncation and we show that this is indeed the case. The truncated action coincides with the topological Chern-Simons action constructed in [20]. We also consider the problem of the KK truncation to the sum of two multiplets, one of them the massless graviton multiplet, whereas the second involves the fields corresponding to the lowest weight CPOs. Surprisingly, we have found indications that the sum of the massless graviton multiplet and the special spin-1/2 multiplet containing the lowest mode scalar CPOs may admit a consistent truncation.[3] This situation is reminiscent of, but is different from the $AdS_5\times S^5$ compactification, where the lowest weight CPOs occur in the stress tensor multiplet, which on the gravity side corresponds to the massless graviton multiplet, allowing a consistent truncation [15]. In the AdS_3 case the gauge degrees of freedom encoded in the graviton multiplet give rise to the $\mathcal{N}=(4,4)$ superconformal algebra of the boundary CFT [22].

[2] To ensure the wider applicability of our results we keep n unspecified. Except $n=21$ another case of interest is $n=5$. Dimensional reduction of type IIB supergravity on T^4 produces the nonchiral $D=6$, $\mathcal{N}=8$ theory, for which the equations of motion for the metric, the scalar fields and the two-forms are the same as for $D=6$, $\mathcal{N}=4b$ with $n=5$.

[3] Certainly the situation where the consistent KK truncation to the sum of the graviton and a certain lowest multiplet may exist is not limited to the example we consider. Recently strong indications were found [21] that $\mathcal{N}=3$ M-theory compactifications on $AdS_4\times X^7$ can be consistently truncated to the sum of the graviton and a long gravitino multiplet (a shadow of the graviton multiplet).

II. THE CUBIC EFFECTIVE ACTION IN AdS_3

Cubic couplings of chiral primaries may be derived from the quadratic corrections to the covariant equations of motion for $D=6$, $\mathcal{N}=4b$ supergravity coupled to n tensor multiplets [18]. All the bosonic fields—the graviton, the two-form potentials B^I_{MN}, $I=1,\ldots,5+n$ and the scalar sector—provide relevant contributions to the quadratic corrections. The scalar sector constitutes a sigma model over the coset space $SO(5,n)/SO(5)\times SO(n)$ with vielbein (V^i_I,V^r_I), $i=1,\ldots,5$, $r=1,\ldots,n$ which is parametrized by $5n$ scalar fields. The index I transforms under global $SO(5,n)$ transformations and is raised and lowered with the $SO(5,n)$ invariant metric $\eta=\mathrm{diag}(1_{5\times5},-1_{n\times n})$, whereas the indices (i,r) transform under local composite $SO(5)\times SO(n)$ transformations. We use the following indices: M, N for $D=6$, μ, ν for AdS_3 and a,b for S^3 coordinates.

Defining

$$dVV^{-1}=\begin{pmatrix} Q^{ij} & \sqrt{2}P^{is} \\ \sqrt{2}P^{rj} & Q^{rs}\end{pmatrix}, \tag{2.1}$$

the covariant derivative in the scalar sector is found by the Cartan-Maurer equation to be

$$D_M P^{ir}_N=\nabla_M P^{ir}_N-Q^{ij}_M P^{jr}_N-P^{is}_M Q^{sr}_N \tag{2.2}$$

and the equations of motion for the bosonic sector of $D=6$, $\mathcal{N}=4b$ supergravity are

$$R_{MN}=H^i_{MPQ}H^{iPQ}_N+H^r_{MPQ}H^{r\,PQ}_N+2P^{ir}_M P^{ir}_N, \tag{2.3}$$

$$D^M P^{ir}_M=\frac{\sqrt{2}}{3}H^i_{MPQ}H^{rMP}, \tag{2.4}$$

$$*H^i=H^i, \quad *H^r=-H^r, \tag{2.5}$$

where

$$H^i=G^I V^i_I, \quad H^r=G^I V^r_I \text{ and } G^I=dB^I. \tag{2.6}$$

In units where the radius of S^3 is set to unity, the $AdS_3\times S^3$ background solution is

$$ds^2=\frac{1}{x_0^2}(dx_0^2+\eta_{ij}dx^i dx^j)+d\Omega_3^2, \tag{2.7}$$

where η_{ij} is the 2-dimensional Minkowski metric. One of the self-dual field strengths is singled out and set equal to the Levi-Cevita tensor, while all others vanish:

$$H^i_{\mu\nu\rho}=\delta^{i5}\varepsilon_{\mu\nu\rho}, \quad H^i_{abc}=\delta^{i5}\varepsilon_{abc}, \quad H^r_{MNP}=0. \tag{2.8}$$

Here $\varepsilon_{\mu\nu\rho}$ and ε_{abc} are the volume forms on AdS_3 and S^3, respectively, so that $\varepsilon_{\mu\nu\rho}\varepsilon_{abc}$ is the volume form in six dimensions. The $SO(5,n)$ background vielbein is taken to be constant and by a global $SO(5,n)$ rotation it can be set to unity.

To construct the Lagrangian equations of motion we represent the fields as

$$g_{MN}=\overline{g}_{MN}+h_{MN}, \tag{2.9}$$

$$G^I=\overline{G^I}+g^I, \quad g^I=db^I, \tag{2.10}$$

and

$$V_I^i=\delta_I^i+\phi^{ir}\delta_I^r+\frac{1}{2}\phi^{ir}\phi^{jr}\delta_I^j, \tag{2.11}$$

$$V_I^r=\delta_I^r+\phi^{ir}\delta_I^i+\frac{1}{2}\phi^{ir}\phi^{is}\delta_I^s. \tag{2.12}$$

The gauge symmetry of the equations of motion allows one to impose the de Donder–Lorentz gauge:[4]

$$\nabla^a h_{\mu a}=\nabla^a h_{(ab)}=\nabla^a b^I_{Ma}=0. \tag{2.13}$$

Here and below the subscript (ab) denotes symmetrization of indices a and b with the trace removed.

This gauge choice does not fix all the gauge symmetry of the theory, for a detailed discussion of the residual symmetry, cf. [19]. The gauge condition (2.13) implies that the physical fluctuations are decomposed in spherical harmonics on S^3 as[5]

$$h_{\mu\nu}(x,y)=\sum h^{I_1}_{\mu\nu}(x)Y^{I_1}(y),$$

$$h_{\mu a}(x,y)=\sum h^{I_3\pm}_{\mu}(x)Y^{I_3\pm}_a(y),$$

$$h^a_a(x,y)=\sum \pi^{I_1}(x)Y^{I_1}(y),$$

$$h_{(ab)}(x,y)=\sum \varrho^{I_5\pm}(x)Y^{I_5\pm}_{(ab)}(y),$$

$$b^I_{\mu\nu}(x,y)=\sum \varepsilon_{\mu\nu}{}^{\rho}X^{II_1}_{\rho}(x)Y^{I_1}(y),$$

$$b^I_{ab}(x,y)=\sum \varepsilon_{abc}U^{II_1}(x)\nabla^c Y^{I_1}(y),$$

$$b^I_{\mu a}(x,y)=\sum Z^{II_3\pm}_{\mu}(x)Y^{I_3\pm}_a(y),$$

$$\phi^{ir}(x,y)=\sum \phi^{irI_1}(x)Y^{I_1}(y),$$

where we have represented

$$h_{ab}=h_{(ab)}+\frac{1}{3}\overline{g}_{ab}h^c_c, \quad \overline{g}^{ab}h_{(ab)}=0. \tag{2.14}$$

The various spherical harmonics transform in the following irreducible representations of $SO(4)\simeq SU(2)_L\times SU(2)_R$:

Scalar spherical harmonics Y^I: $(k/2,k/2)$, $k\geqslant 0$,

Vector spherical harmonics $Y^I_a=Y^{I+}_a+Y^{I-}_a$:

$$(\tfrac{1}{2}(k+1),\tfrac{1}{2}(k-1))\oplus(\tfrac{1}{2}(k-1),\tfrac{1}{2}(k+1)), \quad k\geqslant 1,$$

Tensor spherical harmonics $Y^I_{(ab)}=Y^{I+}_{(ab)}+Y^{I-}_{(ab)}$:

$$(\tfrac{1}{2}(k+2),\tfrac{1}{2}(k-2))\oplus(\tfrac{1}{2}(k-2),1/2(k+2)), \quad k\geqslant 2.$$

The upper index enumerates a basis in a given irreducible representation of $SO(4)$: $I_1=1,\dots,(k+1)^2$, $k\geqslant 0$; $I_3=1,\dots,(k+1)^2-1$, $k\geqslant 1$; $I_5=1,\dots,(k+1)^2-4$, $k\geqslant 2$. The action of the Laplacian is [23]

$$\nabla^2 Y^{I_1}=-\Delta Y^{I_1},$$

$$\nabla^2 Y^{I_3\pm}_a=(1-\Delta)Y^{I_3\pm}_a, \quad \nabla^a Y^{I_3\pm}_a=0,$$

$$\nabla^2 Y^{I_5\pm}_{(ab)}=(2-\Delta)Y^{I_5\pm}_{(ab)}, \quad \nabla^a Y^{I_5\pm}_{(ab)}=0, \quad \overline{g}^{ab}Y^{I_5\pm}_{(ab)}=0, \tag{2.15}$$

where $\Delta\equiv k(k+2)$. The vector spherical harmonics $Y^{I_3\pm}_a$ are also eigenfunctions of the operator $(*\nabla)^c_a\equiv\varepsilon_a{}^{bc}\nabla_b$:

$$(*\nabla)^c_a Y^{I_3\pm}_c=\pm(k+1)Y^{I_3\pm}_a. \tag{2.16}$$

We also need to make a number of field redefinitions, the simplest ones, required to diagonalize the linearized equations of motion, are

$$\phi^{5r}_I=2ks^r_I+2(k+2)t^r_I, \quad U^r_I=s^r_I-t^r_I, \tag{2.17}$$

$$\pi_I=-6k\sigma_I+6(k+2)\tau_I, \quad U^5_I=\sigma_I+\tau_I, \tag{2.18}$$

$$h_{\mu\nu I}=\varphi_{\mu\nu I}+\nabla_\mu\nabla_\nu\zeta_I+g_{\mu\nu}\eta_I, \tag{2.19}$$

$$\zeta_I=\frac{4}{k+1}(\tau_I-\sigma_I), \tag{2.20}$$

$$\eta_I=\frac{2}{k+1}(k(k-1)\sigma_I-(k+2)(k+3)\tau_I), \tag{2.21}$$

$$h^{\pm}_{\mu I}=\frac{1}{2}(C^{\pm}_{\mu I}-A^{\pm}_{\mu I}), \quad Z^{5\pm}_{\mu I}=\pm\frac{1}{4}(C^{\pm}_{\mu I}+A^{\pm}_{\mu I}). \tag{2.22}$$

[4] From now on all the covariant derivatives are understood to be in the background geometry.

[5] Here and in what follows we use normalized spherical harmonics, i.e., $\int Y^{I_1}Y^{J_1}=\delta^{I_1J_1}$, $\int\overline{g}^{ab}Y^{I_3\pm}_aY^{J_3\pm}_b=\delta^{I_3J_3}$, $\int\overline{g}^{ac}\overline{g}^{bd}Y^{I_5\pm}_{(ab)}Y^{J_5\pm}_{(cd)}=\delta^{I_5J_5}$.

Here s_I^r and σ_I are scalar chiral primaries [19]. Note also that we use an off-shell shift for $h_{\mu\nu}$ that first appeared in [24]. It differs from the on-shell shift used in [19] by higher order terms.

A. Cubic couplings of chiral primaries

To compute four-point functions involving only chiral primary operators in the boundary conformal field theory one needs the quartic couplings giving rise to contact diagrams and cubic couplings involving at least two chiral primaries, which contribute to the AdS exchange diagrams. Here we confine ourselves to the problem of determining the corresponding cubic couplings.

Obviously, fields like ϕ^{ir}, $i=1,\ldots,4$ that transform as vectors under the $SO(4)$ R-symmetry cannot contribute to these couplings. Therefore, we can set all these fields to zero and, to simplify the notation, we denote, e.g., ϕ^{5r} as ϕ^r, etc.

Then the action for the chiral primaries s^r and σ may be written in the form

$$
\begin{aligned}
S(s^r,\sigma)=\frac{N}{(2\pi)^3}\int d^3x\sqrt{-g_{AdS_3}}(&\mathcal{L}_2(s^r)+\mathcal{L}_2(t^r)+\mathcal{L}_2(\sigma)+\mathcal{L}_2(\tau)+\mathcal{L}_2(\varrho^{\pm})+\mathcal{L}_2(Z_\mu^{r\pm})+\mathcal{L}_2(A_\mu^{\pm},C_\mu^{\pm})\\
&+\mathcal{L}_2(\varphi_{\mu\nu})+\mathcal{L}_3^s(\sigma)+\mathcal{L}_3^s(\tau)+\mathcal{L}_3^s(\varrho^{\pm})+\mathcal{L}_3^\sigma(\sigma)+\mathcal{L}_3^\sigma(\tau)+\mathcal{L}_3^\sigma(\varrho^{\pm})+\mathcal{L}_3^{s\sigma}(t^r)+\mathcal{L}_3^s(A_\mu^{\pm},C_\mu^{\pm})\\
&+\mathcal{L}_3^\sigma(A_\mu^{\pm},C_\mu^{\pm})+\mathcal{L}_3^{s\sigma}(Z_\mu^{r\pm})+\mathcal{L}_3^s(\varphi_{\mu\nu})+\mathcal{L}_3^\sigma(\varphi_{\mu\nu})).
\end{aligned}\tag{2.23}
$$

The quadratic terms for the various scalar fields are

$$\mathcal{L}_2(s^r)=\sum 16k(k+1)\left(-\frac{1}{2}\nabla_\mu s_I^r\nabla^\mu s_I^r-\frac{1}{2}m_s^2(s_I^r)^2\right),\tag{2.24}$$

$$\mathcal{L}_2(\sigma)=\sum 16k(k-1)\left(-\frac{1}{2}\nabla_\mu\sigma_I\nabla^\mu\sigma_I-\frac{1}{2}m_\sigma^2(\sigma_I)^2\right),\tag{2.25}$$

$$\mathcal{L}_2(t^r)=\sum 16(k+1)(k+2)\left(-\frac{1}{2}\nabla_\mu t_I^r\nabla^\mu t_I^r-\frac{1}{2}m_t^2(t_I^r)^2\right),\tag{2.26}$$

$$\mathcal{L}_2(\tau)=\sum 16(k+2)(k+3)\left(-\frac{1}{2}\nabla_\mu\tau_I\nabla^\mu\tau_I-\frac{1}{2}m_\tau^2(\tau_I)^2\right),\tag{2.27}$$

$$\mathcal{L}_2(\varrho^{\pm})=\sum\left(-\frac{1}{4}\nabla_\mu\varrho_I^{\pm}\nabla^\mu\varrho_I^{\pm}-\frac{1}{4}m_\varrho^2(\varrho_I^{\pm})^2\right)\tag{2.28}$$

with masses

$$m_s^2=m_\sigma^2=k(k-2),\quad m_t^2=m_\tau^2=(k+2)(k+4),\quad m_\varrho^2=\Delta.\tag{2.29}$$

The quadratic Lagrangians for the vector fields can be written as

$$\mathcal{L}_2(Z_\mu^{r\pm})=\sum 16(k+1)\times\left(\mp\frac{1}{4}\varepsilon^{\mu\nu\rho}Z_{\mu I}^{r\pm}\partial_\nu Z_{\rho I}^{r\pm}+\frac{1}{4}m_Z Z_{\mu I}^{r\pm}Z_I^{r\pm\mu}\right)\tag{2.30}$$

for the fields $Z_\mu^{r\pm}$ with mass $m_Z=k+1$ and

$$\mathcal{L}_2(A_\mu^{\pm},C_\mu^{\pm})=\mathcal{L}_2(A_\mu^{\pm})+\mathcal{L}_2(C_\mu^{\pm})+\mathcal{L}_2^{\text{cross}}(A_\mu^{\pm},C_\mu^{\pm})\tag{2.31}$$

for the fields $A_\mu^{\pm}$ and $C_\mu^{\pm}$, where

$$
\begin{aligned}
\mathcal{L}_2(A_\mu^{\pm})=\sum\Big(&-\frac{1}{8}F_{\mu\nu I}(A^{\pm})F_I^{\mu\nu}(A^{\pm})\\
&-\frac{1}{4}(k+1)(k-1)A_{\mu I}^{\pm}A_I^{\pm\mu}\mp\frac{1}{2}\varepsilon^{\mu\nu\rho}A_{\mu I}^{\pm}\partial_\nu A_{\rho I}^{\pm}\\
&\mp\frac{k}{2}P_{k-1}^{\pm}(A^{\pm})_I^\mu A_{\mu I}^{\pm}\Big),
\end{aligned}\tag{2.32}
$$

$$
\begin{aligned}
\mathcal{L}_2(C_\mu^{\pm})=\sum\Big(&-\frac{1}{8}F_{\mu\nu I}(C^{\pm})F_I^{\mu\nu}(C^{\pm})\\
&-\frac{1}{4}(k+1)(k+3)C_{\mu I}^{\pm}C_I^{\pm\mu}\pm\frac{1}{2}\varepsilon^{\mu\nu\rho}C_{\mu I}^{\pm}\partial_\nu C_{\rho I}^{\pm}\\
&\mp\frac{1}{2}(k+2)P_{k+3}^{\pm}(C^{\pm})_I^\mu C_{\mu I}^{\pm}\Big),
\end{aligned}\tag{2.33}
$$

$$
\begin{aligned}
\mathcal{L}_2^{\text{cross}}(A_\mu^{\pm},C_\mu^{\pm})=\sum\Big(&\frac{1}{4}F_{\mu\nu I}(A^{\pm})F_I^{\mu\nu}(C^{\pm})\\
&-\frac{1}{2}(k-1)(k+3)A_{\mu I}^{\pm}C_I^{\pm\mu}\mp\frac{1}{2}\\
&\times(k+1)\varepsilon^{\mu\nu\rho}(A_{\mu I}^{\pm}\partial_\nu C_{\rho I}^{\pm}+C_{\mu I}^{\pm}\partial_\nu A_{\rho I}^{\pm})\Big).
\end{aligned}\tag{2.34}
$$

Here $F_{\mu\nu I}(V)=\partial_\mu V_{\nu I}-\partial_\nu V_{\mu I}$ and we have introduced the first order operators

$$(P^{\pm}_{m})^{\lambda}_{\mu} \equiv \varepsilon_{\mu}{}^{\nu\lambda}\nabla_{\nu} \pm m\,\delta^{\lambda}_{\mu} = (*\nabla)^{\lambda}_{\mu} \pm m\,\delta^{\lambda}_{\mu}. \tag{2.35}$$

Some comments are in order. Since the quadratic action for the vector fields $Z^{r\pm}_{\mu}$ is of the Chern-Simons form it vanishes on-shell, but however, in the boundary CFT we have to add certain boundary terms which give rise to nonvanishing two-point functions [25]. It is also worthwhile to note that the equations of motion for the vector fields $A^{\pm}_{\mu}$, $C^{\pm}_{\mu}$ are not the Proca equations, rather they are Proca-Chern-Simons equations containing both the usual and the topological mass terms. Indeed, the equations for $A^{\pm}_{\mu}$ and $C^{\pm}_{\mu}$ following directly from Eq. (2.31) are nondiagonal and both are of the second order. Adding them produces an equation of the first order (a constraint) that relates the fields $A^{\pm}_{\mu}$ and $C^{\pm}_{\mu}$:

$$P^{\pm}_{k-1}(A^{\pm})_{\mu I} + P^{\pm}_{k+3}(C^{\pm})_{\mu I} = 0. \tag{2.36}$$

This constraint is then used to obtain the closed Proca-Chern-Simons equations for the vector fields, e.g.,

$$\begin{aligned}\nabla^{\nu}F_{\nu\mu I}(A^{\pm}) - (k-1)(k+1)A^{\pm}_{\mu I} \mp 2\varepsilon_{\mu}{}^{\nu\rho}\partial_{\nu}A^{\pm}_{\rho I} \\ = P^{\mp}_{k+1}P^{\pm}_{k-1}(A^{\pm})_{\mu I} = 0.\end{aligned} \tag{2.37}$$

Thus, intrinsically one has a second order equation for one of the vector fields and a constraint on the second one. The number of physical degrees of freedom described by a massive pair $A_{\mu I}$, $C_{\mu I}$ is then three and this is in agreement with the discussion in [19]. The original equations being components of the second order Einstein equation and the first order self-duality equation are related to Eqs. (2.36) and (2.37) by simple linear transformations of the fields $A^{\pm}_{\mu I}$ and $C^{\pm}_{\mu I}$ [cf. Eq. (2.22)]. Note that the conformal dimensions of the operators in the boundary CFT dual to $A^{\pm}_{\mu I}$ and $C^{\pm}_{\mu I}$ are k, $k+2$ and $k+4$.

Finally, the quadratic Lagrangian for the symmetric second rank tensor field ($\varphi \equiv \varphi^{\mu}_{\mu}$) is

$$\begin{aligned}\mathcal{L}_2(\varphi_{\mu\nu}) = \sum \Big(& -\frac{1}{4}\nabla_{\mu}\varphi_{\nu\rho I}\nabla^{\mu}\varphi^{\nu\rho}_{I} + \frac{1}{2}\nabla_{\mu}\varphi^{\mu\nu}_{I}\nabla^{\rho}\varphi_{\rho\nu I} \\ & -\frac{1}{2}\nabla_{\mu}\varphi_{I}\nabla_{\nu}\varphi^{\nu\mu}_{I} + \frac{1}{4}\nabla_{\mu}\varphi_{I}\nabla^{\mu}\varphi_{I} + \frac{1}{4}(2-\Delta) \\ & \times(\varphi_{\mu\nu I})^2 + \frac{1}{4}\Delta(\varphi_I)^2\Big).\end{aligned} \tag{2.38}$$

The cubic couplings of scalar fields are

$$\mathcal{L}^{s}_{3}(\psi) = V^{ss\psi}_{I_1I_2I_3}s^{r}_{I_1}s^{r}_{I_2}\psi_{I_3}, \quad \mathcal{L}^{\sigma}_{3}(\psi) = V^{\sigma\sigma\psi}_{I_1I_2I_3}\sigma_{I_1}\sigma_{I_2}\psi_{I_3},$$
$$\mathcal{L}^{s\sigma}_{3}(t^{r}) = V^{st\sigma}_{I_1I_2I_3}s^{r}_{I_1}t^{r}_{I_2}\sigma_{I_3}, \tag{2.39}$$

with $\psi \in \{\sigma,\tau,\varrho^{\pm}\}$ and the vertices [the notation is explained in Eqs. (2.61) and (2.62)]

$$V^{ss\sigma}_{I_1I_2I_3} = -\frac{2^4(\Sigma-2)\Sigma(\Sigma+2)\alpha_1\alpha_2\alpha_3}{k_3+1}a_{I_1I_2I_3}, \tag{2.40}$$

$$V^{ss\tau}_{I_1I_2I_3} = \frac{2^6(\Sigma+2)(\alpha_1+1)(\alpha_2+1)\alpha_3(\alpha_3-1)(\alpha_3-2)}{k_3+1}a_{I_1I_2I_3}, \tag{2.41}$$

$$V^{\sigma\sigma\sigma}_{I_1I_2I_3} = -\frac{2^3(\Sigma-2)\Sigma(\Sigma+2)\alpha_1\alpha_2\alpha_3}{3(k_1+1)(k_2+1)(k_3+1)}(k_I^2+k_2^2+k_3^2-2)a_{I_1I_2I_3}, \tag{2.42}$$

$$V^{\sigma\sigma\tau}_{I_1I_2I_3} = \frac{2^5(\Sigma+2)(\alpha_1+1)(\alpha_2+1)\alpha_3(\alpha_3-1)(\alpha_3-2)}{(k_1+1)(k_2+1)(k_3+1)}[k_1^2+k_2^2+(k_3+2)^2-2]a_{I_1I_2I_3}, \tag{2.43}$$

$$V^{st\sigma}_{I_1I_2I_3} = \frac{2^7(\Sigma+2)(\alpha_1+1)\alpha_2(\alpha_2-1)(\alpha_2-2)(\alpha_3+1)}{k_3+1}a_{I_1I_2I_3}, \tag{2.44}$$

$$V^{ss\varrho^{\pm}}_{I_1I_2I_3} = 2^2\Sigma(\alpha_3-1)p^{\pm}_{I_1I_2I_3}, \tag{2.45}$$

$$V^{\sigma\sigma\varrho^{\pm}}_{I_1I_2I_3} = \frac{2\Sigma(\alpha_3-1)}{(k_1+1)(k_2+1)}[k_1^2+k_2^2-(k_3+1)^2-1]p^{\pm}_{I_1I_2I_3}. \tag{2.46}$$

In our notation, the vertices (2.40) and (2.42) are precisely the ones found in [8].

Cubic terms involving two chiral primaries and the vector fields $A^{\pm}_{\mu}$, $C^{\pm}_{\mu}$ can be represented as

$$\mathcal{L}^{s}_{3}(A^{\pm}_{\mu},C^{\pm}_{\mu}) = V^{ssA^{\pm}}_{I_1I_2I_3}s^{r}_{I_1}\nabla^{\mu}s^{r}_{I_2}A^{\pm}_{\mu I_3} + V^{ssC^{\pm}}_{I_1I_2I_3}s^{r}_{I_1}\nabla^{\mu}s^{r}_{I_2}C^{\pm}_{\mu I_3} \pm W^{s\pm}_{I_1I_2I_3}(P^{\pm}_{k_3-1}(s^{r}_{I_1}\nabla s^{r}_{I_2})^{\mu}A^{\pm}_{\mu I_3} - P^{\pm}_{k_3+3}(s^{r}_{I_1}\nabla s^{r}_{I_2})^{\mu}C^{\pm}_{\mu I_3}), \tag{2.47}$$

$$\mathcal{L}_3^{\sigma}(A_\mu^{\pm},C_\mu^{\pm}) = V_{I_1I_2I_3}^{\sigma\sigma A^{\pm}}\sigma_{I_1}\nabla^{\mu}\sigma_{I_2}A_{\mu I_3}^{\pm} + V_{I_1I_2I_3}^{\sigma\sigma C^{\pm}}\sigma_{I_1}\nabla^{\mu}\sigma_{I_2}C_{\mu I_3}^{\pm} \pm W_{I_1I_2I_3}^{\sigma\pm}(P_{k_3-1}^{\pm}(\sigma_{I_1}\nabla\sigma_{I_2})^{\mu}A_{\mu I_3}^{\pm} - P_{k_3+3}^{\pm}(\sigma_{I_1}\nabla\sigma_{I_2})^{\mu}C_{\mu I_3}^{\pm}), \tag{2.48}$$

whereas the interaction of s^r and σ with the fields $Z_\mu^{r\pm}$ is found to be

$$\mathcal{L}_3^{\sigma s}(Z_\mu^{r\pm}) = \pm V_{I_1I_2I_3}^{\sigma s Z^{\pm}}\sigma_{I_1}\nabla^{\mu}s_{I_2}^{r}Z_{\mu I_3}^{r\pm}. \tag{2.49}$$

These expressions describe the minimal interactions of vector fields with two scalars in three dimensions. Here the couplings are

$$V_{I_1I_2I_3}^{ssA^{\pm}} = -2(\Sigma+1)(\Sigma-1)t_{I_1I_2I_3}^{\pm}, \tag{2.50}$$

$$V_{I_1I_2I_3}^{ssC^{\pm}} = 2(2\alpha_3-1)(2\alpha_3-3)t_{I_1I_2I_3}^{\pm}, \tag{2.51}$$

$$W_{I_1I_2I_3}^{s\pm} = 2(k_3+1)t_{I_1I_2I_3}^{\pm}, \tag{2.52}$$

$$V_{I_1I_2I_3}^{\sigma\sigma A^{\pm}} = -\frac{(\Sigma+1)(\Sigma-1)}{(k_1+1)(k_2+1)}(k_1^2-k_2^2-k_3^2-1)t_{I_1I_2I_3}^{\pm}, \tag{2.53}$$

$$V_{I_1I_2I_3}^{\sigma\sigma C^{\pm}} = \frac{(2\alpha_3-1)(2\alpha_3-3)}{(k_1+1)(k_2+1)}(k_1^2+k_2^2-(k_3+2)^2-1)t_{I_1I_2I_3}^{\pm}, \tag{2.54}$$

$$W_{I_1I_2I_3}^{\sigma\pm} = -2(k_3+1)\frac{(k_1-1)(k_2-1)}{(k_1+1)(k_2+1)}t_{I_1I_2I_3}^{\pm}, \tag{2.55}$$

$$V_{I_1I_2I_3}^{\sigma s Z^{\pm}} = \frac{2^4(\Sigma+1)(2\alpha_3-1)(k_3+1)}{k_1+1}t_{I_1I_2I_3}^{\pm}. \tag{2.56}$$

Finally the interaction of chiral primaries with symmetric tensors of the 2nd rank are

$$\mathcal{L}_3^{s}(\varphi_{\mu\nu}) = V_{I_1I_2I_3}^{ss\varphi}\left(\nabla^{\mu}s_{I_1}^{r}\nabla^{\nu}s_{I_2}^{r}\varphi_{\mu\nu I_3} - \frac{1}{2}\left(\nabla^{\mu}s_{I_1}^{r}\nabla_{\mu}s_{I_2}^{r} + \frac{1}{2}(m_1^2+m_2^2-\Delta_3)s_{I_1}^{r}s_{I_2}^{r}\right)\varphi_{I_3}\right), \tag{2.57}$$

$$\mathcal{L}_3^{\sigma}(\varphi_{\mu\nu}) = V_{I_1I_2I_3}^{\sigma\sigma\varphi}\left(\nabla^{\mu}\sigma_{I_1}\nabla^{\nu}\sigma_{I_2}\varphi_{\mu\nu I_3} - \frac{1}{2}\left(\nabla^{\mu}\sigma_{I_1}\nabla^{\mu}\sigma_{I_2} + \frac{1}{2}(m_1^2+m_2^2-\Delta_3)\sigma_{I_1}\sigma_{I_2}\right)\varphi_{I_3}\right), \tag{2.58}$$

where

$$V_{I_1I_2I_3}^{ss\varphi} = 2^2(\Sigma+2)\alpha_3 a_{I_1I_2I_3}, \tag{2.59}$$

$$V_{I_1I_2I_3}^{\sigma\sigma\varphi} = \frac{2(\Sigma+2)\alpha_3}{(k_1+1)(k_2+1)}(k_1^2+k_2^2-(k_3+1)^2-1)a_{I_1I_2I_3}. \tag{2.60}$$

Above the summation over I_1, I_2, I_3 and r is assumed and we have defined

$$\Sigma \equiv k_1+k_2+k_3, \quad \alpha_i \equiv \frac{1}{2}(k_l+k_m-k_i), \;\; l\neq m\neq i\neq l, \tag{2.61}$$

and

$$a_{I_1I_2I_3} \equiv \int Y_{I_1}Y_{I_2}Y_{I_3}, \;\; t_{I_1I_2I_3}^{\pm} \equiv \int \nabla^{a}Y_{I_1}Y_{I_2}Y_{aI_3}^{\pm}, \quad p_{I_1I_2I_3}^{\pm} \equiv \int \nabla^{a}Y_{I_1}\nabla^{b}Y_{I_2}Y_{(ab)I_3}^{\pm}. \tag{2.62}$$

To summarize, we have the types of cubic vertices seen in Fig. 1. In particular we see that all possible cubic invariants under $SO(n)\times SO_R(4)$ containing at least two supergravity fields dual to CPOs are present.

B. Cubic couplings at extremality

With the cubic couplings at hand the problem of computing the 3-point correlation functions of two CPOs with an operator associated to another gravity field entering the cubic vertex becomes straightforward. One needs to determine the on-shell value of the corresponding cubic action, which amounts to computing certain integrals over the AdS space, where for the latter problem a well-developed technique is available [9]. Generally the AdS integrals diverge for some ''extremal'' values of conformal dimensions (masses) of the fields involved and this is an indication that the corresponding supergravity coupling should vanish, otherwise the correlation function would be ill-defined [9,13]. For example, the AdS integral corresponding to the 3-point correlation function of scalar fields with conformal dimensions Δ_1, Δ_2 and Δ_3 is ill-defined if $\Delta_1+\Delta_2=\Delta_3$ (or any relation obtained from this by permutation of indices). Inspection shows that the cubic couplings we found do indeed vanish at extremality, i.e., when the accompanying AdS integrals diverge. The only case where this property cannot be seen straightforwardly is for the couplings of scalar fields with vector fields $A_\mu^{\pm}$ and $C_\mu^{\pm}$. Below we present the analysis making the property of vanishing at extremality manifest.

Recall that due to Eq. (2.36) the fields $A_\mu^{\pm}$ and $C_\mu^{\pm}$ do not describe independent degrees of freedom. Regarding, e.g., $A_\mu^{\pm}$ as independent variables we first consider the solution of Eq. (2.37) satisfying

$$P_{k-1}^{\pm}(A^{\pm})_{\mu I} = 0.$$

Then the constraint (2.36) gives $P_{k+3}^{\pm}(C^{\pm})_{\mu I}=0$. Clearly, the last two equations imply the Maxwell equations

$$\nabla^{\nu}F_{\nu\mu I}(A^{\pm}) - (k-1)^2A_{\mu I}^{\pm} = 0, \quad \nabla^{\nu}F_{\nu\mu I}(C^{\pm}) - (k+3)^2C_{\mu I}^{\pm} = 0. \tag{2.63}$$

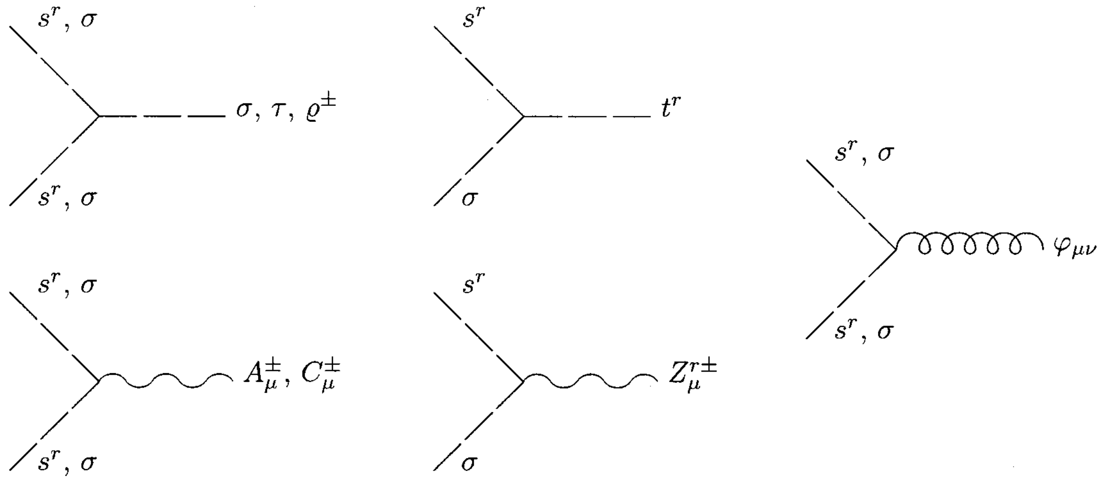

FIG. 1. Cubic vertices containing two supergravity fields dual to CPOs.

Therefore, the masses of the vector fields $A^\pm_{\mu I}$ and $C^\pm_{\mu I}$ are $m_A=k-1$ and $m_C=k+3$. Recalling the formula for the conformal weight Δ_V of an operator dual to a vector field V_μ with mass m in AdS$_{d+1}$ (see, e.g., [4]) we find $\Delta_A=k$ and $\Delta_C=k+4$. It is worthwhile to note that Δ_A has the same conformal dimension as the scalar CPOs. The corresponding CFT operators are the vector CPOs in the spin 2 tower of supermultiplets [19].

The evaluation of the 3-point functions of CPOs with vector fields requires the knowledge of the following AdS integral:

$$\int \frac{d^3\omega}{\omega_0^3} K_{\Delta_1}(\omega,\mathbf{x}_1)\nabla^\mu K_{\Delta_2}(\omega,\mathbf{x}_2)\mathcal{G}_{\mu i\Delta_3}(\omega,\mathbf{x}_3)=\frac{R_{123}}{|\mathbf{x}_{12}|^{\Delta_1+\Delta_2-\Delta_V}|\mathbf{x}_{13}|^{\Delta_1+\Delta_V-\Delta_2}|\mathbf{x}_{23}|^{\Delta_2+\Delta_V-\Delta_1}}\frac{Z_i}{Z}, \tag{2.64}$$

where the coordinate dependence on the right-hand side (RHS) is completely fixed by the conformal symmetry. Here $\mathbf{x}_i$ are the positions of the operators in the correlation function of the boundary CFT, $\mathbf{x}_{ij}=\mathbf{x}_i-\mathbf{x}_j$,

$$Z_i=\frac{(\mathbf{x}_{13})_i}{\mathbf{x}_{13}^2}-\frac{(\mathbf{x}_{23})_i}{\mathbf{x}_{23}^2}, \quad Z^2=Z_iZ_i$$

and $K_\Delta(\omega,\mathbf{x})$, $\mathcal{G}_{\mu\nu\Delta_3}(\omega,\mathbf{x})$ are the scalar and vector bulk-to-boundary propagators, respectively. Applying the inversion method [9] one finds for R_{123} the following answer:

$$R_{123}=\frac{1}{\pi^2}\frac{\Gamma(\frac{1}{2}(\Delta_1+\Delta_2-\Delta_V+1))\Gamma(\frac{1}{2}(\Delta_1+\Delta_V-\Delta_2+1))\Gamma(\frac{1}{2}(\Delta_2+\Delta_V-\Delta_1+1))}{\Gamma(\Delta_1-1)\Gamma(\Delta_2-1)\Gamma(\Delta_V)}\Gamma(\tfrac{1}{2}(\Delta_1+\Delta_2+\Delta_V-1)). \tag{2.65}$$

R_{123} is ill-defined in several cases. First we consider the case when[6]

$$\Delta_1+\Delta_2-\Delta_V+1=0. \tag{2.66}$$

For CPOs with $\Delta=k$ this equation becomes $k_1+k_2-\Delta_V+1=0$ and, therefore, for $C^\pm_\mu$ it reads as

$$k_1+k_2-\Delta_C+1=k_1+k_2-k_3-3=0,$$

i.e., $\alpha_3=3/2$. But the couplings $V^{ssC^\pm}_{I_1I_2I_3}$ and $V^{\sigma\sigma C^\pm}_{I_1I_2I_3}$ [see Eqs. (2.47) and (2.48)] contain the factor $2\alpha_3-3$ and, therefore, vanish.[7] Computing the correlation functions involving the fields $A^\pm_\mu$ a divergence arises when

$$k_1+k_2-\Delta_A+1=k_1+k_2-k_3+1=0,$$

i.e., when $\alpha_3=-1/2$. However, the couplings $V^{ssA^\pm}_{I_1I_2I_3}$ and $V^{\sigma\sigma A^\pm}_{I_1I_2I_3}$ contain the tensors $t^\pm_{I_1I_2I_3}$ that are nonvanishing only if $k_1+k_2\geqslant k_3+1$ (and relations obtained by permutation of the indices). Hence, the divergence is irrelevant since the couplings are zero due to the vanishing of $t^\pm_{I_1I_2I_3}$.

[6] R_{123} is also divergent for $\Delta_1+\Delta_2-\Delta_V+1$ a negative integer, but in that case $t^\pm_{I_1I_2I_3}=0$.

[7] The terms in Eqs. (2.47) and (2.48) proportional to $W^{s\pm}_{I_1I_2I_3}$ and to $W^{\sigma\pm}_{I_1I_2I_3}$ vanish after integrating by parts and taking into account the equations of motion for $A^\pm_\mu$ and $C^\pm_\mu$.

Moreover, R_{123} also diverges when

$$\Delta_1+\Delta_V-\Delta_2+1=0. \tag{2.67}$$

For $C_\mu^\pm$ this gives $k_1+k_3=k_2-5$, i.e., $\alpha_2=-5/2$. On the other hand, nonvanishing of $t^\pm_{I_1I_2I_3}$ requires the inequality $k_1+k_3\geqslant k_2+1$, so that for the case under consideration $t^\pm_{I_1I_2I_3}$ again vanish. For $A_\mu^\pm$ Eq. (2.67) gives $k_1+k_3=k_2-1$, i.e., $\alpha_2=-1/2$, and the couplings vanish by the same reason as for $C_\mu^\pm$.

Equation (2.37) has another solution obeying $P^\mp_{k+1}(A^\pm)_{\mu I}=0$, which we now consider. Perform the shift

$$C^\pm_{\mu I}=C'^\pm_{\mu I}-\frac{k}{k+2}A^\pm_{\mu I}, \tag{2.68}$$

where $A^\pm_{\mu I}$ is not arbitrary, rather it solves $P^\mp_{k+1}(A^\pm)_{\mu I}=0$. Then the linear constraint (2.36) turns into

$$P^\pm_{k+3}(C'^\pm)_{\mu I}+\frac{2}{k+2}P^\mp_{k+1}(A^\pm)_{\mu I}=P^\pm_{k+3}(C'^\pm)_{\mu I}=0.$$

Thus, $C'^\pm_\mu$ decouple from $A^\pm_\mu$. The fields $A^\pm_\mu$ then correspond to operators with $\Delta_A=k+2$. The divergence (2.66) now gives $k_1+k_2=k_3+1$, i.e., $\alpha_3=1/2$. The coupling of two scalars with the vector fields $A^\pm_\mu$ corrected by the shift (2.68) [we again integrate the terms in Eqs. (2.47) and (2.48) proportional to $W^{s\pm}_{I_1I_2I_3}$ and to $W^{\sigma\pm}_{I_1I_2I_3}$ by parts and use the equations of motion for $A^\pm_\mu$ and $C'^\pm_\mu$] reads

$$\overline{V}^{\sigma\sigma A^\pm}_{I_1I_2I_3}\equiv V^{\sigma\sigma A^\pm}_{I_1I_2I_3}+V^{\sigma\sigma C^\pm}_{I_1I_2I_3}+4k_3W^{\sigma\pm}_{I_1I_2I_3} \tag{2.69}$$

and analogously for s^r. The explicit results are given by

$$\overline{V}^{ssA^\pm}_{I_1I_2I_3}=-8(k_3+1)(2\alpha_3-1)t^\pm_{I_1I_2I_3}, \tag{2.70}$$

$$\overline{V}^{\sigma\sigma A^\pm}_{I_1I_2I_3}=-4(k_3+1)(2\alpha_3-1)\frac{(k_1+1)(k_1+k_3)+(k_2+1)(k_2+k_3)-4(k_3+1)}{(k_1+1)(k_2+1)}t^\pm_{I_1I_2I_3} \tag{2.71}$$

and vanish at extremality. The AdS integral is also divergent for Eq. (2.67), i.e., for $\alpha_1=-3/2$. However, in this case $t^\pm_{I_1I_2I_3}$ is zero.

Thus we have shown that all the cubic couplings we determined vanish in the extremal cases.

C. Truncation to the graviton multiplet

The bosonic part of the Lagrangian density for the three-dimensional supergravity based on the $SU(1,1|2)_L\times SU(1,1|2)_R$ supergroup is [20]

$$\mathcal{L}=R+2-\varepsilon^{\mu\nu\rho}\left(A^{ij}_\mu\partial_\nu A^{ji}_\rho+\frac{2}{3}A^{ij}_\mu A^{jk}_\nu A^{ki}_\rho\right)+\varepsilon^{\mu\nu\rho}\left(A'^{ij}_\mu\partial_\nu A'^{ji}_\rho+\frac{2}{3}A'^{ij}_\mu A'^{jk}_\nu A'^{ki}_\rho\right), \tag{2.72}$$

where $A^{ij}_\mu=-A^{ji}_\mu$, $A'^{ij}_\mu=-A'^{ji}_\mu$ are the $SO(3)$ gauge fields and according to our conventions we have set the cosmological constant to -1.

We now demonstrate that the lowest modes of the vector fields $A^\pm_\mu$ obey the first order Chern-Simons equations, although generically the equations of motion are of second order. Thus, we consider the self-interaction of the vector fields $A^\pm_\mu$ and restrict ourselves to the case where two of the three fields, say $A^\pm_{\mu I_2},A^\pm_{\mu I_3}$ come from the massless graviton multiplet, i.e., their equations of motion are

$$P_0(A^\pm)_\mu=\varepsilon_\mu{}^{\nu\rho}\partial_\nu A^\pm_\rho=0\Leftrightarrow\nabla_\mu A^\pm_\nu=\nabla_\nu A^\pm_\mu. \tag{2.73}$$

Then the quadratic corrections to the linear constraint (2.36) can be written as

$$P^\pm_{k_1-1}(A^\pm)_{\mu I_1}+P^\pm_{k_1+3}(C^\pm)_{\mu I_1}=\pm\frac{1}{2}\varepsilon_\mu{}^{\nu\rho}A^\pm_{\nu I_2}A^\pm_{\rho I_3}\int\varepsilon^{abc}Y^\pm_{aI_1}Y^\pm_{bI_2}Y^\pm_{cI_3}. \tag{2.74}$$

Since both vector fields on the RHS transform in the (1,0) of $SU(2)_L\times SU(2)_R$ [or (0,1) respectively], $Y^\pm_{bI_2}Y^\pm_{cI_3}$ transform as

$$(1,0)\otimes(1,0)=(0,0)\oplus(1,0)\oplus(2,0);$$
$$(0,1)\oplus(0,1)=(0,0)\oplus(0,1)\oplus(0,2), \tag{2.75}$$

and therefore the S^3 integral is nonzero only if $k_1=1$. In this case we have

$$P_0(A^\pm)_{\mu I_1}+P^\pm_4(C^\pm)_{\mu I_1}=\pm\frac{1}{2}\varepsilon_\mu{}^{\nu\rho}A^\pm_{\nu I_2}A^\pm_{\rho I_3}\int\varepsilon^{abc}Y^\pm_{aI_1}Y^\pm_{bI_2}Y^\pm_{cI_3}. \tag{2.76}$$

On the other hand, it is easy to show that there is no coupling of $C^\pm_{\mu I}$ with two massless vector fields and therefore it is consistent to set the fields $C^\pm_{\mu I}$ to zero.

Since the S^3 integral is completely antisymmetric in I_1, I_2 and I_3 (and the I_i run from 1 to 3) it is proportional to $\varepsilon^{I_1I_2I_3}$ and can be represented as $\mp 2C^{ij}_{\pm I_1}C^{jk}_{\pm I_2}C^{ki}_{\pm I_3}$, where $C^{ij}_{\pm I}=-C^{ji}_{\pm I}$. Defining

$$A_\mu^{ij}=C_{+I}^{ij}A_\mu^{I+}, \quad A_\mu^{ij}=C_{-I}^{ij}A_\mu^{I-} \tag{2.77}$$

the equation of motion for A_μ^{ij} reads

$$\varepsilon_\mu{}^{\nu\rho}\partial_\nu A_\rho^{ij}=-\varepsilon_\mu{}^{\nu\rho}A_\nu^{ik}A_\rho^{kj} \tag{2.78}$$

and analogously for $A_\mu'^{ij}$. These are precisely the equations of motion following from Eq. (2.72).

Now we address the issue of the consistency of the KK truncation to the sum of two multiplets, one of them naturally the massless graviton multiplet and a second one containing lowest mode scalar CPOs. Surprisingly, all the cubic couplings we computed involving two fields from the sum of the massless graviton multiplet and the special spin-1/2 multiplet[8] and one field belonging to any other multiplet vanish.[9] Recall that the spin-1/2 multiplet contains the scalar modes s^r with $k=1$ and ϕ^{ir} with $k=0$, and spin-1/2 states χ^r [19]. All the operators in the boundary CFT dual to the gravity fields from the spin-1/2 multiplet are either relevant or marginal. Based on the analysis presented here, one cannot exclude that a consistent truncation to the sum ''massless graviton multiplet+special spin-1/2 multiplet'' does exist. Of course, only on the basis of the cubic vertices considered here, this issue cannot be decided. It is worthwhile to note that s^r with $k=1$ correspond in the boundary CFT to the scalar CPOs with the lowest conformal dimension.

Another natural example to consider is the lowest level of the spin 1 $SO(n)$ singlet multiplet, containing σ with $k=2$. Here, however, the consistent truncation is not possible. Indeed, the cubic coupling of two CPOs and one symmetric second rank (massive) tensor

$$V_{I_1I_2I_3}^{\sigma\sigma\varphi}\sim(\Sigma+2)\alpha_3[k_1^2+k_2^2-(k_3+1)^2-1]a_{I_1I_2I_3} \tag{2.79}$$

does not vanish if $k_1=k_2=k_3=2$. Note also that the CFT multiplet dual to the $SO(n)$ singlet discussed above contains irrelevant operators.

[8]Generically the multiplets in the vector representation of $SO(n)$ involve fields with spin 1. However at the lowest level these are absent [19].

[9]For the cubic couplings with vector fields see Sec. II B.

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