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Differentially Rotating Disks of Dust: Arbitrary Rotation Law

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In this paper, solutions to the Ernst equation are investigated that depend on two real analytic functions defined on the interval [0,1]. These solutions are introduced by a suitable limiting process of Bäcklund transformations applied to seed solutions of the Weyl class. It turns out that this class of solutions contains the general relativistic gravitational field of an arbitrary differentially rotating disk of dust, for which a continuous transition to some Newtonian disk exists. It will be shown how for given boundary conditions (i.e. proper surface mass density *or* angular velocity of the disk) the gravitational field can be approximated in terms of the above solutions. Furthermore, particular examples will be discussed, including disks with a realistic profile for the angular velocity and more exotic disks possessing two spatially separated ergoregions.

KEY WORDS: Ernst equation, disk.

1. INTRODUCTION

Differentially rotating disks of dust have already been studied by Ansorg and Meinel [1]. They considered the class of hyperelliptic solutions to the Ernst equation introduced by Meinel and Neugebauer [2], see also [3–6]. These hyperelliptic solutions depend on a number of complex parameters and a real potential function. Ansorg and Meinel concentrated on the case in which *one* complex parameter can be prescribed. They determined the real potential function in order to satisfy a particular boundary condition valid for all disks of dust. To generate their solutions, they used Neugebauer's and Meinel's rigorous solution [7, 8, 9] to the boundary value problem of a rigidly rotating disk of dust which also belongs to the hyperelliptic class.

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A subclass of Ansorg's and Meinel's solutions is made up of Bäcklund transforms of seed solutions of the Weyl class.[2] Solutions of this type are of particular interest since their mathematical structure is much simpler than that of the more general hyperelliptic solutions.

With this in mind, the following questions arise:

- Is it possible to find solutions corresponding to more general differentially rotating disks of dust by increasing the number of prescribed complex parameters?
- If so, is there a rapidly converging method for approximating arbitrary differentially rotating disks of dust with given boundary conditions (i.e. proper mass density *or* angular velocity)?
- Is it perhaps possible to construct such a method by restriction to the much simpler solutions of the Bäcklund type?

To answer these questions, the paper is organized as follows. In the first section the metric tensor, Ernst equation, and boundary conditions are introduced and the class of solutions of the Bäcklund type is represented. As will be discussed in the second section, the properties of these solutions can be used to obtain more general solutions by a suitable limiting process. Since these more general solutions depend on two real analytic functions defined on the interval [0, 1], a rapidly converging numerical scheme to satisfy arbitrary boundary conditions for disks of dust can be created. This is described in the third section. Finally, the fourth section contains particular examples of differentially rotating disks of dust, including disks with a realistic profile for the angular velocity and more exotic disks possessing two spatially separated ergoregions.

Units are used in which the velocity of light as well as Newton's constant of gravitation are equal to 1.

1.1. Metric Tensor, Ernst Equation, and Boundary Conditions

The metric tensor for axisymmetric stationary and asymptotically flat space-times reads as follows in Weyl-Papapetrou-coordinates (ρ, ζ, φ, t):

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a\, d\varphi)^2.$$

For this line element, the vacuum field equations are equivalent to a single complex equation—the so-called Ernst equation [22, 23]

[2] The construction of solutions to the Ernst equation by means of Bäcklund transformations belongs to the powerful analytic methods developed by several authors [10–20]. For a detailed introduction see [21].

$$(\mathcal{R}f)\Delta f = (\nabla f)^2,$$

$$\Delta = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2}, \qquad \nabla = \left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\zeta}\right), \tag{1}$$

where the Ernst potential f is given by

$$f = e^{2U} + \mathrm{i}b \quad \text{with} \quad b_{,\zeta} = \frac{e^{4U}}{\rho}\, a_{,\rho}, \quad b_{,\rho} = -\frac{e^{4U}}{\rho}\, a_{,\zeta}. \tag{2}$$

The remaining function k can be calculated from the Ernst potential f by a line integration:

$$\frac{k_{,\rho}}{\rho} = (U_{,\rho})^2 - (U_{,\zeta})^2 + \frac{1}{4}\, e^{-4U}[(b_{,\rho})^2 - (b_{,\zeta})^2]$$

$$\frac{k_{,\zeta}}{\rho} = 2U_{,\rho}U_{,\zeta} + \frac{1}{2}\, e^{-4U}b_{,\rho}b_{,\zeta}.$$

To obtain the boundary conditions for differentially rotating disks of dust, one has to consider the field equations for an energy-momentum-tensor

$$T^{ik} = \epsilon u^i u^k = \sigma_p(\rho)e^{U-k}\delta(\zeta)u^i u^k,$$

where ϵ and σ_p stand for the energy-density and the invariant (proper) surface mass-density, respectively, δ is the usual Dirac delta-distribution, and u^i denotes the four-velocity of the dust material.[3]

Integration of the corresponding field equations from the lower to the upper side of the disk (with coordinate radius ρ_0) yields the conditions (see [24], pp. 81–83)

$$2\pi\sigma_p = e^{U-k}(U_{,\zeta} + \tfrac{1}{2}Q) \tag{3}$$

$$e^{4U}Q^2 + Q(e^{4U})_{,\zeta} + (b_{,\rho})^2 = 0 \tag{4}$$

for $\zeta = 0^+$, $0 \le \rho \le \rho_0$ and

$$Q = -\rho e^{-4U}[b_{,\rho}b_{,\zeta} + (e^{2U})_{,\rho}(e^{2U})_{,\zeta}]. \tag{5}$$

[3] u^i has only φ- and t-components.

Note that boundary condition (4) for the Ernst potential f does not involve the surface mass-density σ_p. This condition comes from the nature of the material the disk is made of. Therefore, equation (4) will be referred to as the *dust-condition*. Instead of prescribing the proper surface mass-density σ_p [which leads to the boundary condition (3)] one can alternatively assume a given angular velocity $\Omega = \Omega(\rho) = u^\varphi/u^t$ of the disk which results in the boundary condition ($\zeta = 0^+$, $0 \le \rho \le \rho_0$):

$$\Omega = \frac{Q}{a_{,\zeta} - aQ}. \tag{6}$$

The following requirements due to symmetry conditions and asymptotical flatness complete the set of boundary conditions:

- Regularity at the rotation axis is guaranteed by

$$\frac{\partial f}{\partial \rho}(0, \zeta) = 0.$$

- At infinity asymptotical flatness is realized by $U \longrightarrow 0$ and $a \longrightarrow 0$. For the potential b this has the consequence $b \longrightarrow b_\infty = \text{const}$. Without loss of generality, this constant can be set to 0, i.e. $f \longrightarrow 1$ at infinity.
- Finally, reflectional symmetry with respect to the plane $\zeta = 0$ is assumed, i.e. $f(\rho, -\zeta) = \overline{f(\rho, \zeta)}$ (with a bar denoting complex conjugation).

1.2. Solutions of the Bäcklund Type

For a given integer $q \ge 1$, a set $\{Y_1, \ldots, Y_q\} = \{Y_\nu\}_q$[4] of complex parameters, and a real analytic function g defined on the interval [0, 1], the following expression

[4] In the following, the notation $\{Y_1, \ldots, Y_q\}$ will be abbreviated by $\{Y_\nu\}_q$.

$$
f = f_0 \frac{\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
-1 & \alpha_1\lambda_1 & \alpha_1^*\lambda_1^* & \alpha_2\lambda_2 & \alpha_2^*\lambda_2^* & \cdots & \alpha_q\lambda_q & \alpha_q^*\lambda_q^* \\
1 & \lambda_1^2 & (\lambda_1^*)^2 & \lambda_2^2 & (\lambda_2^*)^2 & \cdots & \lambda_q^2 & (\lambda_q^*)^2 \\
-1 & \alpha_1\lambda_1^3 & \alpha_1^*(\lambda_1^*)^3 & \alpha_2\lambda_2^3 & \alpha_2^*(\lambda_2^*)^3 & \cdots & \alpha_q\lambda_q^3 & \alpha_q^*(\lambda_q^*)^3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_1^{2q} & (\lambda_1^*)^{2q} & \lambda_2^{2q} & (\lambda_2^*)^{2q} & \cdots & \lambda_q^{2q} & (\lambda_q^*)^{2q}
\end{vmatrix}}{\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & \alpha_1\lambda_1 & \alpha_1^*\lambda_1^* & \alpha_2\lambda_2 & \alpha_2^*\lambda_2^* & \cdots & \alpha_q\lambda_q & \alpha_q^*\lambda_q^* \\
1 & \lambda_1^2 & (\lambda_1^*)^2 & \lambda_2^2 & (\lambda_2^*)^2 & \cdots & \lambda_q^2 & (\lambda_q^*)^2 \\
1 & \alpha_1\lambda_1^3 & \alpha_1^*(\lambda_1^*)^3 & \alpha_2\lambda_2^3 & \alpha_2^*(\lambda_2^*)^3 & \cdots & \alpha_q\lambda_q^3 & \alpha_q^*(\lambda_q^*)^3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
1 & \lambda_1^{2q} & (\lambda_1^*)^{2q} & \lambda_2^{2q} & (\lambda_2^*)^{2q} & \cdots & \lambda_1^{2q} & (\lambda_q^*)^{2q}
\end{vmatrix}} \tag{7}
$$

with (a bar denotes complex conjugation)

- $f_0 = \exp\left(-\int_{-1}^{1} \frac{(-1)^q g(x^2)dx}{Z_D}\right), Z_D = \sqrt{(\mathrm{i}x - \zeta/\rho_0)^2 + (\rho/\rho_0)^2}, \mathcal{R}(Z_D) < 0$
- $\lambda_\nu = \sqrt{\frac{Y_\nu - \mathrm{i}\bar{z}}{Y_\nu + \mathrm{i}z}}, \quad z = \frac{1}{\rho_0}(\rho + \mathrm{i}\zeta), \quad \lambda_\nu^*\bar{\lambda}_\nu = 1$
- $\alpha_\nu = \frac{1-\gamma_\nu}{1+\gamma_\nu}, \quad \gamma_\nu = \exp\left(\lambda_\nu(Y_\nu + \mathrm{i}z)\int_{-1}^{1}\frac{(-1)^q g(x^2)dx}{(\mathrm{i}x - Y_\nu)Z_D}\right), \quad \alpha_\nu^*\bar{\alpha}_\nu = 1$

satisfies the Ernst equation. With the additional requirement that for each parameter Y_ν there is also a parameter Y_μ with $Y_\nu = -\overline{Y}_\mu$, reflectional symmetry, $f(\rho, -\zeta) = \overline{f(\rho, \zeta)}$, is ensured.[5] Moreover, the parameters Y_ν are assumed to lie outside the imaginary interval $[-\mathrm{i}, \mathrm{i}]$.

The above Ernst potential $f = f(\rho/\rho_0, \zeta/\rho_0; \{Y_\nu\}_q; g)$ is obtained by a Bäcklund transformation applied to the real seed solution f_0, see [16]. On the other hand, as demonstrated in appendix A, it can be constructed from the hyperelliptic solutions by a suitable limiting process (see also [4]). The particular ansatz chosen for the seed solution f_0 guarantees a resulting Ernst potential

[5] Hence, the set $\{\mathrm{i}Y_\nu\}_q$ consists of real parameters and/or pairs of complex conjugate parameters.

which corresponds to a disk-like source of the gravitational field (see also section 1.2 of [1]).

Furthermore, f does not possess singularities at $(\rho, \zeta) = \rho_0(|\mathcal{I}[Y_\nu]|, -\mathcal{R}[Y_\nu])$. This is due to the fact that $\alpha_\nu\lambda_\nu$ is a function of λ_ν^2, and this means that f does not behave like a square root function near the critical points $(\rho, \zeta) = \rho_0(|\mathcal{I}[Y_\nu]|, -\mathcal{R}[Y_\nu])$, but rather like a rational function. Now, in the whole area of physically interesting solutions that will be treated in the subsequent sections, each zero of the denominator is cancelled by a corresponding zero of the numerator in (7) such that the resulting gravitational field is regular outside the disk.

The real function g that enters the Ernst potential is assumed to be analytic on [0, 1] in order to guarantee an analytic behaviour of the angular velocity Ω for all $\rho \in [0, \rho_0]$. Moreover, the additional requirement

$$g(1) = 0$$

leads to a surface mass density σ_p of the form

$$\sigma_p(\rho) = \sigma_0 \psi_p[(\rho/\rho_0)^2]\sqrt{1 - (\rho/\rho_0)^2}$$
$$[\text{with } \psi_p \text{ analytic in } [0,1], \psi_p(0) = 1] \tag{8}$$

and therefore ensures that σ_p vanishes at the rim of the disk.

In this article the question as to whether the above expression for the Ernst potential is sufficiently general to approximate arbitrary differentially rotating disks of dust is investigated. Of particular interest is a rapidly converging method to perform this approximation. To this end, the set $\{Y_\nu\}_q$ of complex parameters will be translated into an analytic function

$$\xi : [0, 1] \longrightarrow \mathbb{R}.$$

Thus the Ernst potential will depend on two real analytic functions defined on [0, 1]:

$$f = f(\rho/\rho_0, \zeta/\rho_0; \xi; g),$$

which eventually proves to be sufficient to satisfy both the *dust condition* (4) and the boundary condition (3) [or alternatively (6)]. The *rapid* and *accurate* approximation can be realized since both g and ξ are analytic on [0, 1] and thus permit elegant expansions in terms of Chebyshev polynomials.

2. GENERALIZATION OF THE BÄCKLUND TYPE SOLUTIONS BY A LIMITING PROCESS

As demonstrated in [1] for the Bäcklund type solutions with $q = 1$, the *dust condition* (4) can be satisfied by an appropriate choice of the function g if the complex parameters Y_ν are prescribed. To fulfil a second boundary condition, (3) or (6), the set $\{Y_\nu\}_q$ of these parameters has to be translated into a real analytic function ξ. To this end, consider the following equalities for the above solutions $f = f(\{Y_\nu\}_q;\, g)$[6] which are proved in appendix B:

$$f[\{Y_1, \dots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \dots, Y_{q-2}\}; g] \qquad \text{if } Y_{q-1} = -Y_q \in \mathbb{R} \tag{9}$$

$$f[\{Y_1, \dots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \dots, Y_{q-2}\}; g] \qquad \text{if } Y_{q-1} = \overline{Y}_q \tag{10}$$

$$\lim_{t \to \infty} f[\{Y_1, \dots, Y_{q-1}, \mathrm{i}t\}; g] = f[\{Y_1, \dots, Y_{q-1}\}; g] \qquad \text{if } t \in \mathbb{R} \tag{11}$$

$$\lim_{Y_q \to \infty} f[\{Y_1, \dots, Y_{q-2}, Y_{q-1}, Y_q\}; g] = f[\{Y_1, \dots, Y_{q-2}\}; g] \qquad \text{if } Y_{q-1} = -\overline{Y}_q. \tag{12}$$

In order to find an approximation scheme, the desired function $\xi = \xi(\{Y_\nu\}_q)$ is supposed to be invariant under the modifications (9–12) of the set $\{Y_\nu\}_q$ that do not effect the Ernst potential. This property will be necessary to solve the boundary conditions uniquely.

It is realized by the real analytic function

$$\xi(x^2; \{Y_\nu\}_q) = \frac{1}{x} \ln\left[\prod_{\nu=1}^{q} \frac{\mathrm{i}Y_\nu - x}{\mathrm{i}Y_\nu + x}\right], \qquad x \in [-1, 1], \tag{13}$$

which can be proved by considering that for each parameter Y_ν there is also a parameter Y_μ with $Y_\nu = -\overline{Y}_\mu$, and that, moreover, the parameters Y_ν do not lie on the imaginary interval $[-\mathrm{i}, \mathrm{i}]$.

The set $\mathcal{X}$ of all functions $\xi = \xi(x^2;\, \{Y_\nu\}_q)$, $q \in \mathbb{N}$, which are defined by (13) forms a dense subset of the set $\mathcal{A}$ of all real analytic functions on [0, 1]. Now, for a given function g, each $\xi \in \mathcal{X}$ is mapped by (7) onto a uniquely defined Ernst potential $f \in \mathcal{E}$[7]:

[6] In the following the Ernst potentials f given by (7) are considered as complex functions depending on the set $\{Y_\nu\}_q$ of complex parameters and on g.

[7] Here, $\mathcal{E}$ denotes the set of all Ernst potentials corresponding to disk-like sources.

$$\Phi_g : \mathcal{X} \longrightarrow \mathcal{E}, \quad \Phi_g(\xi) = f(\{Y_\nu\}_q; g), \tag{14}$$

where the set $\{Y_\nu\}_q$ results from ξ by (13).

In the following, it is assumed that this mapping Φ_g can be extended to form a continuous function defined on $\mathcal{A}$.[8] Then, given the two real functions g and ξ, defined and analytic on the interval [0, 1], the Ernst potential

$$f(\xi; g) = \lim_{q \to \infty} f(\{Y_\nu^{(q)}\}_q; g)$$

exists and is independent of the particular choice of the sequence $\{\{Y_\nu^{(q)}\}_q\}_{q=q_0}^{\infty}$ which serves to represent ξ by

$$\xi(x^2) = \frac{1}{x} \lim_{q \to \infty} \ln\left[\prod_{\nu=1}^{q} \frac{\mathrm{i}Y_\nu^{(q)} - x}{\mathrm{i}Y_\nu^{(q)} + x}\right] \qquad \text{for} \quad x \in [-1, 1].$$

This provides the groundwork for the approximation scheme that will be developed in the next section. The treatment additionally assumes that the boundary conditions (3) and (4) [or (4) and (6)] interpreted as functions of g and ξ are invertible. The accurate and rapid convergence of the numerical methods justifies this assumption although a rigorous proof cannot be given.

3. AN APPROXIMATION SCHEME FOR ARBITRARY DIFFERENTIALLY ROTATING DISKS OF DUST

It is now possible to attack general boundary value problems for differentially rotating disks of dust. With the above generalized solutions $f = f(\xi; g)$ the boundary conditions [see formulas (3–6, 8)] become a problem of inversion to determine g and ξ from σ_p or Ω:

$$\begin{aligned}
&(A) \quad S(g; \xi) = \{e^{U-k}[U_{,\zeta} + \tfrac{1}{2}Q]/[\sigma_0\sqrt{1-(\rho/\rho_0)^2}]\}(\xi; g) \doteq 2\pi\psi_p \quad \text{or} \\
&(A') \quad O(g; \xi) = \{Q/[\Omega(0)(a_{,\zeta} - aQ)]\}(\xi; g) \doteq \Omega/\Omega(0) = \Omega^* \\
&(B) \quad D(g; \xi) = \{\rho_0^2[Q^2e^{4U} + Q(e^{4U})_{,\zeta} + (b_{,\rho})^2]\}(\xi; g) \doteq 0, \quad g(1) \doteq 0
\end{aligned} \tag{15}$$

This inversion problem is tackled in the following manner:

1. The only way to treat the complicated system (15) numerically seems

[8] The mathematical aspects of this assumption will be discussed in Section 5.

to be by restricting it to a finite, discretized version and solving this by means of a Newton–Raphson method.

2. For this method, a good initial guess for the solution is needed. As shown in appendix C.1, there exists a representation of the functions g and ξ in terms of σ_p or Ω in the Newtonian regime $\epsilon \ll 1$ where $\epsilon = M^2/J$ and the gravitational mass M and the total angular momentum J are given by

$$\begin{aligned} M &= 2\int_S (T_{ab} - \tfrac{1}{2}T g_{ab}) n^a \xi^b dV \\ J &= -\int_S T_{ab} n^a \eta^b dV, \quad T_{ab} = g_{ab} T^{ab}. \end{aligned} \tag{16}$$

(S is the spacelike hypersurface $t = \mathit{constant}$ with the unit future-pointing normal vector n^a; the Killingvectors ξ^a and η^a correspond to stationarity and axisymmetry, respectively.)

3. This motivates the following finite version which results from expansions of (15) in terms of Chebyshev-polynomials $T_j(\tau) = \cos[j \arccos(\tau)]$:

$$F_j(v_k) \doteq 0 \quad (1 \le j, k \le N_1 + N_2 - 1):$$

- $F_j = D_j \quad (1 \le j \le N_1 - 1), \quad F_{N_1} = \epsilon(g_m; \xi_n) - \epsilon,$

$$F_{N_1+j-1} = S_j - 2\pi\psi_j \quad \text{or} \quad F_{N_1+j-1} = O_j - \Omega_j^* \quad (2 \le j \le N_2),$$

$$v_k = g_{k+1} \quad (1 \le k \le N_1 - 1), \quad v_{N_1+k-1} = \xi_k \quad (1 \le k \le N_2)$$

- $g(x^2) \approx \displaystyle\sum_{j=1}^{N_1} g_j T_{j-1}(2x^2 - 1) - \tfrac{1}{2} g_1, \quad g(1) \doteq 0 \Rightarrow g_1 = -2\sum_{j=2}^{N_1} g_j$

- $\xi(x^2) \approx \displaystyle\sum_{j=1}^{N_2} \xi_j T_{j-1}(2x^2 - 1) - \tfrac{1}{2}\xi_1$

- $\psi_p(x^2) \approx \displaystyle\sum_{j=1}^{N_2} \psi_j T_{j-1}(2x^2 - 1) - \tfrac{1}{2}\psi_1,$

$$\psi_p(0) \doteq 1 \Rightarrow \psi_1 = 2\sum_{j=2}^{N_2} (-1)^j \psi_j + 2$$

- $\Omega^*[(\rho/\rho_0)^2] = \Omega(\rho)/\Omega(0)$:

$$\Omega^*(x^2) \approx \sum_{j=1}^{N_2} \Omega_j^* T_{j-1}(2x^2-1) - \tfrac{1}{2}\Omega_1^*,$$

$$\Omega^*(0) \doteq 1 \Rightarrow \Omega_1^* = 2\sum_{j=2}^{N_2} (-1)^j \Omega_j^* + 2$$

- $S(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_2} S_j(g_m; \xi_n) T_{j-1}(2x^2-1) - \tfrac{1}{2} S_1(g_m; \xi_n)$

- $O(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_2} O_j(g_m; \xi_n) T_{j-1}(2x^2-1) - \tfrac{1}{2} O_1(g_m; \xi_n)$

- $D(x^2 = \rho^2/\rho_0^2; g; \xi) \approx \sum_{j=1}^{N_1-1} D_j(g_m; \xi_n) T_{j-1}(2x^2-1) - \tfrac{1}{2} D_1(g_m; \xi_n)$

[The function $\epsilon(g_m;\ \xi_n) = M^2/J$ is determined using (16) for the above functions g and ξ.]

4. For the above system, the boundary values are assumed to be given in the form of the ψ_k's or Ω_k^*'s ($k = 2, \ldots, N_2$). Moreover, some $\epsilon \ll 1$ has to be prescribed. Then, good initial v_k's come from the Newtonian expansion. The Newton–Raphson method improves the v_k's and yields a very accurate solution to (15) for the chosen small ϵ. Now, this solution serves as the initial estimate for the v_k's belonging to a marginally increased value for ϵ. Again, the Newton–Raphson method improves the solution, and one continues in this manner until this procedure ceases to converge. This occurs for some finite value ϵ_0, at the latest for $\epsilon = 1$. A further discussion of this limit is given below.
5. A rather technical detail is the retranslation of the ξ_j into a set $\{Y_\nu\}_g$ which then gives a satisfactory approximation of ξ in terms of (13). There are many ways to do this. Here, the following one has been chosen.

   One rewrites equation (13) in the equivalent form

$$\exp[x\xi(x^2; \{Y_\nu\}_q)] = \prod_{\nu=1}^{q} \frac{\mathrm{i}Y_\nu - x}{\mathrm{i}Y_\nu + x} = \frac{P_q(-x)}{P_q(x)} \quad \text{with} \quad P_q(x) = \sum_{\nu=0}^{q} b_\nu x^\nu.$$

The coefficients b_ν of the polynomial P_q can be determined by evaluating the left hand side at q arbitrary different points $x_\mu \in [0, 1]$[9] and solving the following linear system:

$$\exp[x_\mu \xi(x_\mu^2; \{Y_\nu\}_q)] \sum_{\nu=0}^{q} b_\nu x_\mu^\nu = \sum_{\nu=0}^{q} b_\nu (-x_\mu)^\nu$$

The zeros of P_q determine the Y_ν.

The above scheme has been performed for many different prescribed surface mass densities and angular velocities. This provides strong evidence for the conjecture that, in this manner, all Newtonian disks can be extended into the relativistic regime. It has been found that the value for ϵ_0, the limiting parameter for the convergence of this scheme, depends on the chosen profile for ψ_p (or equivalently for Ω^*). It is illustrated in appendix C.2, how the Ernst potential always tends to the extreme Kerr solution [25] as $\epsilon \longrightarrow 1$. This supports a conjecture by Bardeen and Wagoner [26]. But $\epsilon_0 = 1$ does not hold for all given surface mass densities. Even in the Newtonian regime there are surface mass densities for which a realistic physical disk cannot be found since the corresponding angular velocity would become imaginary. If one chooses a profile for σ_p not very different from these, then the Newtonian limit still might exist, but some $\epsilon_0 < 1$ turns up, beyond which the method does not converge. In the case of prescribed angular velocity, the situation is similar. Here, for any sequence $f = f(g_\epsilon; \xi_\epsilon)$ the angular velocity Ω^* tends for all $x^2 \in [0, 1]$ to 1 as $\epsilon \longrightarrow 1$. So, each nonuniform rotation law will lead to some $\epsilon_0 < 1$ (see Section 4 for examples).

The above expansions in terms of Chebyshev-polynomials allow a very accurate representation with only a small number of coefficients. However, the retranslation of ξ (see the above point 5) leads to functions that are not especially well suited for an approximation. In particular, if the boundary condition ψ_p is chosen to be close to those for which there is no Newtonian disk, then the accuracy cannot be driven particularly high by the computer program used, although the method in principle allows arbitrary approximation (see Section 4.2).

For ψ_i's sufficiently far away from those critical ones, the accuracy obtained was very high. By choosing appropriate values for N_1 and N_2 one can always achieve extremly good agreement with the *dust condition* (4) (12 digits and beyond) which ensures a realistic physical interpretation of the solution. The accuracy to which the second boundary condition, (3) or (6), can be satisfied, depends on the parameter ϵ. It is usually around 8 digits in the weak relativistic regime, and falls as ϵ increases, but is still around 4 digits as ϵ tends to ϵ_0. These

[9] Here, zeros of Chebyshev-polynomials have been used.

values arose for $N_1 = 30$, $N_2 = 12$, and typical ψ_p's (like ψ_p's depending linearly on x^2) and Ω^*'s (e.g. the realistic one considered in Section 4.1). The number q of the parameters Y_ν by which ξ is represented, was chosen to be between 20 and 30 (independently of N_2).

What remains to be discussed is the regularity of the Ernst potentials that were obtained. For a few of the solutions, the functions e^{2U} and b were plotted over the coordinates ρ and ζ. Moreover, the agreement of the alternative representations of M and J, as given by the behaviour of the Ernst potential at infinity

$$U = -\frac{M}{r} + O(r^{-2}), \quad b = -2J\,\frac{\cos\theta}{r^2} + O(r^{-3}), \quad (r = \sqrt{\rho^2 + \zeta^2}, \zeta = r\cos\theta)$$

with the results from formulas (16) yields good confirmation of the regularity. This agreement was checked for all solutions that were calculated.

4. REPRESENTATIVE EXAMPLES

From the numerous solutions obtained, three particular sets of differentially rotating disks are discussed in more detail. The first one is an example of disks revolving with a realistic rotation law. The second set illustrates the break down of the numerical method for a specially prescribed surface mass density σ_p at some $\epsilon_0 < 1$. On the other hand it is demonstrated that, for the same σ_p, regular solutions can be found in the highly relativistic regime. Finally, the third example concerns the occurrence of a second ergoregion for a particular series of disks and, moreover, the gradual merging of the two spatially separated ergoregions as ϵ increases.

The deviations between the boundary values obtained for particular numerical solutions and the given boundary conditions are listed in tables. The quantities Δ_D, Δ_Ω, and Δ_σ therein are defined by

$$\begin{aligned}
\Delta_D &= \max_{x^2 \in [0,1]} |D_{\mathrm{obt}}(x^2; g; \xi)| \\
\Delta_\Omega &= \max_{x^2 \in [0,1]} |\Omega^*_{\mathrm{obt}}(x^2) - \Omega^*_{\mathrm{giv}}(x^2)| \\
\Delta_\sigma &= \max_{x^2 \in [0,1]} |\psi_p^{\mathrm{obt}}(x^2) - \psi_p^{\mathrm{giv}}(x^2)|,
\end{aligned}$$

where the indices 'obt' and 'giv' refer to obtained and given quantities, respectively. Moreover, by letters (a), ..., (e), special examples are marked, for which illustrative graphs have been made. Here, curves drawn in the same line style belong to the same solution. The graphs show the dimensionless quantities $\rho_0\sigma_p$ and $\rho_0\Omega$ as well as g and ξ plotted against the normalized radial coordinate ρ/ρ_0 and x, respectively.

4.1. Disks Possessing a Realistic Rotation Law

As motivated by observations in astrophysics the rotation law of a galaxy is often modelled by an equation of the form (see [27])

$$\Omega(\rho) = \frac{\Omega(0)}{\sqrt{1+\rho^2/\rho_1^2}}. \tag{17}$$

Here, the parameter ρ_1 varies for different galaxies. In the following series of solutions illustrated in Figure 1, $\rho_1 = 0.7\rho_0$ has been chosen. As described in Section 3, there is a limiting parameter $\epsilon_0 \approx 0.935$, for which the numerical method ceases to converge.

4.2. Disks with a Critical Surface Mass Density

For the following sequence of solutions, a surface mass density of the form

$$\sigma_p(\rho) = \sigma_0\left(1 - 3\,\frac{\rho^2}{\rho_0^2} + \beta\,\frac{\rho^4}{\rho_0^4}\right)\sqrt{1 - \frac{\rho^2}{\rho_0^2}} \tag{18}$$

has been assumed.

It turns out that for $\beta > \beta_N \approx 7$ no Newtonian disks with a real angular velocity can be found. On the other hand, for $\beta = 5.5$, all relativistic solutions for $0 \le \epsilon \le 1$ exist. The table and graphs of Figure 2 refer to the case $\beta = 6$. Starting here from the Newtonian solution, one soon recognizes a first limiting parameter $\epsilon_0 \approx 0.60$ for which the method breaks down. However, by coming from solutions with $\beta = 5.5$ and ϵ close to 1, it is possible to create highly relativistic solutions with $\beta = 6$. In fact, there is another limiting parameter, $\epsilon_1 \approx 0.97$, above which the solutions with $\beta = 6$ exist once again. Due to the nearness to the critical surface mass density (for $\beta = \beta_N$), the accuracy obtained for the boundary condition (3) is not very high.

4.3. Disks Possessing Spatially Separated Ergoregions

The particular set of disks depicted in Figure 3 demonstrates the occurrence of a second ergoregion.[10] These solutions do not satisfy a specially prescribed

[10] An ergoregion is a portion of the (ρ, ζ)-space within which the function e^{2U} is negative.

	(a)		(b)		(c)	(d)		(e)			
$\varepsilon = M^2/J$	0.5	0.65	0.72	0.79	0.86	0.9	0.92	0.93	0.9336	0.9344	0.9345
$\Delta_D \cdot 10^{12}$	0.6	0.6	4	2	0.9	40	20	100	50	80	3000
$\Delta_\Omega \cdot 10^6$	0.02	0.04	0.09	0.7	6	30	60	120	170	180	200

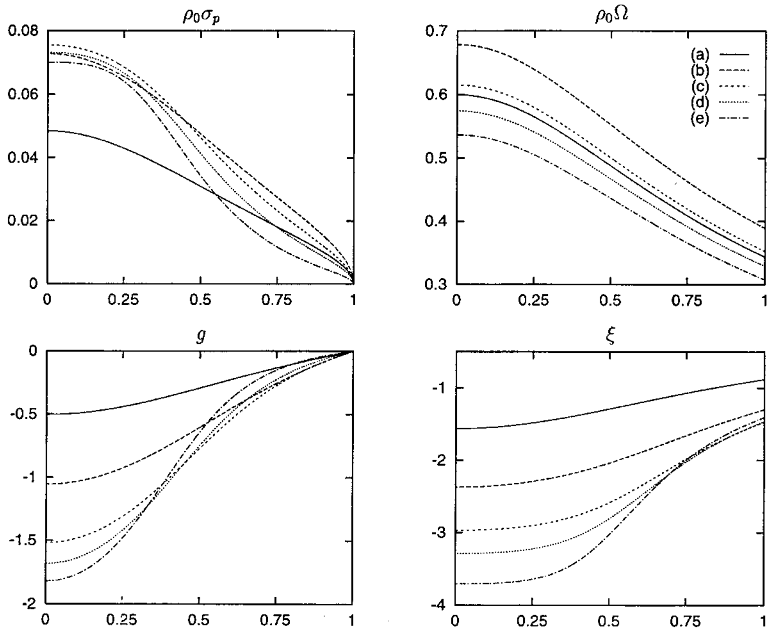

Figure 1. Disks possessing the rotation law (17) with $\rho_1 = 0.7\rho_0$ ($N_1 = 30$, $N_2 = 12$).

boundary condition (3) or (6), but have been constructed in the following manner as intermediate solutions. If one investigates solutions with surface mass densities similar to those of (18), one recognizes two minima for e^{2U} (taken as a function of ρ, $0 \le \rho \le \rho_0$, $\zeta = 0$), say at ρ_a and $\rho_b > \rho_a$. Now, for a particular choice of σ_p it is possible to get $e^{2U}(\rho_a) > 0$ and $e^{2U}(\rho_b) < 0$, whilst by another choice one can achieve $e^{2U}(\rho_a) < 0$ and $e^{2U}(\rho_b) > 0$. This makes clear, that disks with spatially separated ergoregions can be constructed by interpolating between these solutions. For the chosen example, there is only a narrow interval (ϵ_a, ϵ_b) for which the two separated ergoregions occur. As can be seen from Figure 3, after creation of the second ergoregion at $\epsilon_a \approx 0.8403$, both ergoregions grow as ϵ increases. Eventually, at $\epsilon_b \approx 0.8415$, the ergoregions merge into one ergoregion.

	(a)	(b)	(c)	(d)					(e)
$\varepsilon = M^2/J$	0.1	0.2	0.3	0.4	0.5	0.56	0.974	0.984	0.994
$\Delta_D \cdot 10^{10}$	0.4	0.4	0.2	1	0.7	0.3	7	2	1
$\Delta_\sigma \cdot 10^3$	3	3	4	5	8	14	17	7	3

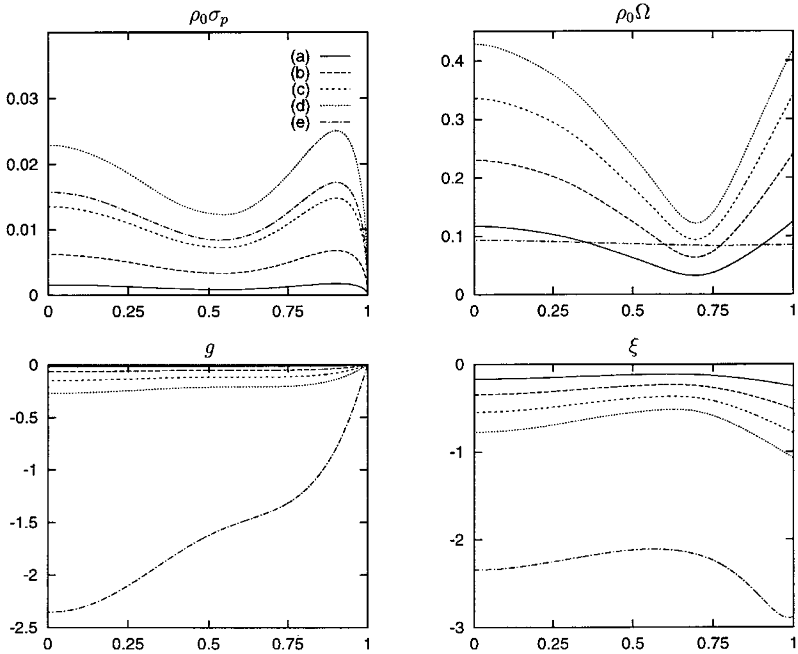

Figure 2. Disks possessing the surface mass density (18) with $\beta = 6$ ($N_1 = 30$, $N_2 = 12$).

5. DISCUSSION OF MATHEMATICAL ASPECTS

As already mentioned in Section 2, the assumption that the function Φ_g introduced in (14) can be extended to form a continuous mapping defined on $\mathcal{A}$, lies at the heart of the above numerical methods. Although this assumption seems to be intuitive, it is not trivial. Consider the following example:

For any analytic function $\psi : [0, 1] \longrightarrow \mathbb{R}$ one finds the equality:[11]

[11] To verify this formula one simply expands the logarithms in the form $\ln(1 + \epsilon) = \epsilon + O(\epsilon^2)$ and notes that the resulting sum tends to the Riemann integral of the right hand side.

	(a)	(b)	(c)	(d)	(e)
$\varepsilon = M^2/J$	0.84038	0.84054	0.84079	0.84120	0.84162
$\Delta_D \cdot 10^{12}$	3	2	2	5	2

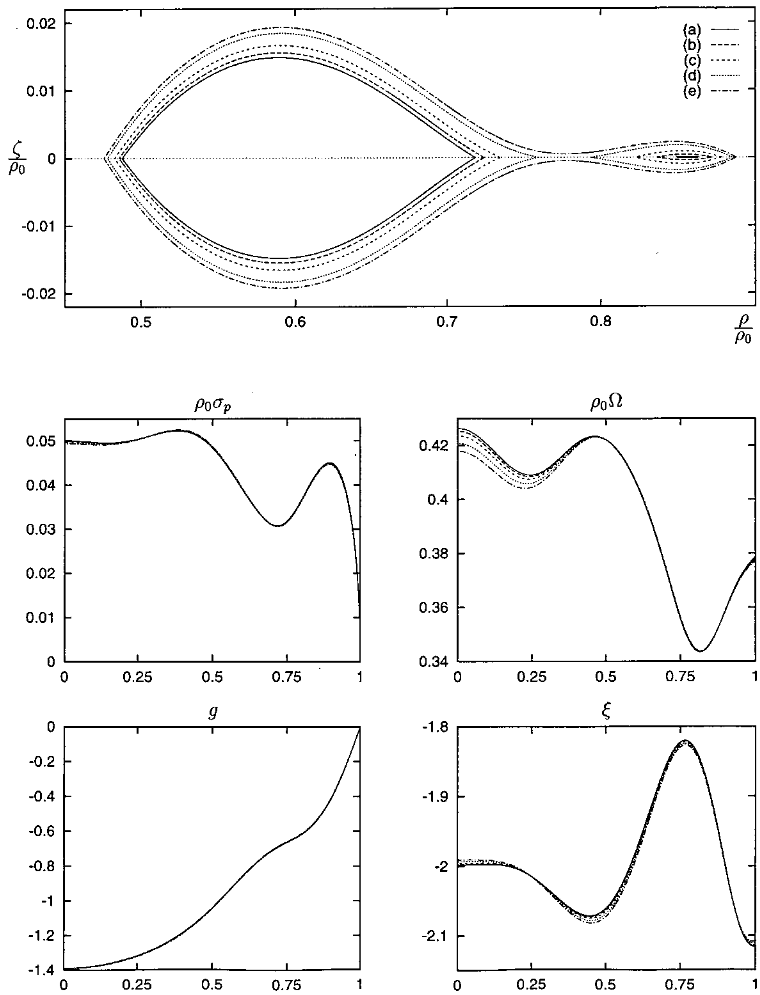

Figure 3. Example for a series of disks possessing spatially separated ergoregions. In the uppermost picture, the rims of the ergoregions in the $(\rho/\rho_0,\ \zeta/\rho_0)$-space are to be seen ($N_1 = 40$, $N_2 = 9$).

$$\lim_{q \to \infty} \sum_{\nu=1}^{q} \ln\left[1 + \frac{1}{q}\,\psi\left(\frac{\nu}{q}\right)\right] = \int_0^1 \psi(t)dt.$$

From this it follows that

$$2\int_0^1 \phi(t)dt = \frac{1}{x} \lim_{q \to \infty} \sum_{\nu=1}^{q} \ln \frac{q + x\phi(\nu/q)}{q - x\phi(\nu/q)} \qquad \text{with } \psi(t) = \pm x\phi(t).$$

Hence, the function $\xi(x^2) \equiv 2$ can be represented by any sequence of the form

$$Y_\nu^{(q)} = \mathrm{i}\,\frac{q}{\phi(\nu/q)} \quad \text{with} \quad \int_0^1 \phi(t)dt = 1.$$

Since these sequences might be quite different from each other, it is rather surprising that all of them approximate the same Ernst potential given by (7). But this follows from the above assumption.

This already indicates the difficulties which are connected with a rigorous proof of this assumption because the Ernst potential is only given in terms of the set $\{Y_\nu\}_q$ and not directly in terms of ξ.

A further conjecture is strongly confirmed by extensive numerical investigations:

For the hyperelliptic class of solutions represented by (19) in appendix A, the functions ξ and g are given by

$$\xi(x^2) = \frac{1}{2x} \ln\left[\prod_{\nu=1}^{p} \frac{\mathrm{i}X_\nu - x}{\mathrm{i}X_\nu + x}\right]$$

$$g(x^2) = \mathrm{sign}\left(\prod_{\nu=1}^{p} X_\nu\right) A_g(x^2)h(x^2),$$

$$A_g(x^2) = \sqrt{\prod_{\nu=1}^{p} (\mathrm{i}x - X_\nu)(\mathrm{i}x - \overline{X}_\nu)}, \quad A_g(x^2) > 0.$$

In particular, in this formulation, the solution for the Neugebauer–Meinel-disk [7, 8, 9] assumes the form $f = f(\xi; g)$ where

$$\xi(x^2) = \frac{1}{2x} \ln \frac{x^2 - C_1(\mu)x + C_2(\mu)}{x^2 + C_1(\mu)x + C_2(\mu)},$$

$$C_1(\mu) = \sqrt{2[1 + C_2(\mu)]}, \quad C_2(\mu) = \frac{1}{\mu}\sqrt{1 + \mu^2},$$

$$g(x^2) = -\frac{1}{\pi} \operatorname{arsinh}[\mu(1 - x^2)],$$

and the parameter μ, $0 < \mu < \mu_0 = 4.62966184\ldots$, *is related to the angular velocity by*

$$\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}, \qquad V_0 = U(\rho = 0, \zeta = 0).$$

As already mentioned, a direct proof of the above assumptions promises to be very complicated. But there might be an alternative proof which relies on relating a general solution of the Ernst equation to the solution of a so-called Riemann–Hilbert problem, see [18, 21, 28, 29]. In this treatment, an appropriately introduced matrix function, from which the Ernst potential can be extracted, is supposed to be regular on a two-sheeted Riemann surface of genus zero except for some given curve, where it possesses a well-defined jump behaviour. The freedom of two jump functions defined on this curve corresponds to the freedom to choose ξ and g. Now, if one succeeds in finding a particular formulation of a Riemann–Hilbert problem in which ξ and g are involved, then the final solution for f proves to depend only on ξ (and g) and not on a particular global representation in terms of $\{Y_\nu\}_q$. This deserves further investigation.

There is very strong numerical evidence for the validity of both assumptions. For various functions ξ (and functions g), different representations $\{Y_\nu\}_q$ have been seen to approximate the same Ernst potential. In particular, the approximation of the Neugebauer–Meinel-solution in terms of Bäcklund solutions was carried out to give an agreement up to the 12th digit with the hyperelliptic solution, which confirms both assumptions.

APPENDIX A. THE TRANSITION FROM THE HYPERELLIPTIC SOLUTIONS TO THE BÄCKLUND TYPE SOLUTIONS

In this Section the Bäklund type solutions are derived from the hyperelliptic class. The latter is assumed to be given in the form represented in [1][12] for an even integer $p \geq 2$:

[12] The parameters K_ν, the upper integration limits $K^{(\nu)}$, and the integration variable K have to be replaced by their 'normalized' values $X_\nu = K_\nu/\rho_0$, $X^{(\nu)} = K^{(\nu)}/\rho_0$, and $X = K/\rho_0$, respectively.

$$f = \exp\left(\sum_{\nu=1}^{p} \int_{X_\nu}^{X^{(\nu)}} \frac{X^p dX}{V(X)} - u_p \right) \tag{19}$$

- $V(X) = \sqrt{(X + \mathrm{i}z)(X - \mathrm{i}\bar{z}) \prod_{\nu=1}^{p} (X - X_\nu)(X - \overline{X}_\nu)}, \quad z = \frac{1}{\rho_0} (\rho + \mathrm{i}\zeta)$

- $\sum_{\nu=1}^{p} \int_{X_\nu}^{X^{(\nu)}} \frac{X^j dX}{V(X)} = u_j, \qquad 0 \le j < p$ (20)

- $u_j = \int_{-1}^{1} \frac{(\mathrm{i}x)^j h(x^2) dx}{Z_D}, \quad 0 \le j \le p, \quad h : [0, 1) \longrightarrow \mathbb{R},$ analytic,

  Z_D as defined in (7)

The set $\{\mathrm{i}X_\nu\}_p$ consists of arbitrary real parameters and/or pairs of complex conjugate parameters (in order to guarantee reflectional symmetry). The (z-dependent) values for the $X^{(\nu)}$ as well as the integration paths on a two-sheeted Riemann surface result from the Jacobian inversion problem (20).

The transition to the Bäcklund type solutions (7) can be obtained in the limit $\epsilon \longrightarrow 0$ by the following assumptions:

- $p = 2q$
- $X_{2\nu-1} = Y_\nu + \epsilon\beta_\nu, \quad X_{2\nu} = Y_\nu \ (1 \le \nu \le q), \quad \{\beta_\nu\}_q$ arbitrary
- $g(x^2) = (-1)^q h(x^2) A(\mathrm{i}x), \quad A(X) = \prod_{\nu=1}^{q} (X - Y_\nu)\,(X - \overline{Y}_\nu).$

to this end, the above expression for f is rewritten in the equivalent form:

$$f = \exp\left[\sum_{\nu=1}^{q} \left(\int_{X_{2\nu-1}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)} + \int_{X_{2\nu}}^{X^{(2\nu)}} \frac{A(X)dX}{V(X)} \right) - \int_{-1}^{1} \frac{(-1)^q g(x^2)dx}{Z_D} \right]$$

The Jacobian inversion problem (20) reads as follows in a similarly rewritten form ($1 \le \mu \le q$):

$(1 \leq \mu \leq q)$:

$$\bullet \sum_{\nu=1}^{q} \left(\int_{X_{2\nu-1}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)(X-Y_\mu)} + \int_{X_{2\nu}}^{X^{(2\nu)}} \frac{A(X)dX}{V(X)(X-Y_\mu)} \right)$$

$$= \int_{-1}^{1} \frac{(-1)^q g(x^2)dx}{(\mathrm{i}x - Y_\mu)Z_D}$$

$$\bullet \sum_{\nu=1}^{q} \left(\int_{X_{2\nu-1}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)(X-\overline{Y}_\mu)} + \int_{X_{2\nu}}^{X^{(2\nu)}} \frac{A(X)dX}{V(X)(X-\overline{Y}_\mu)} \right)$$

$$= \int_{-1}^{1} \frac{(-1)^q g(x^2)dx}{(\mathrm{i}x - \overline{Y}_\mu)Z_D}$$

Furthermore

$$\int_{X_{2\nu-1}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)(X-Y)} + \int_{X_{2\nu}}^{X^{(2\nu)}} \frac{A(X)dX}{V(X)(X-Y)}$$

$$= -\int_{X_{2\nu}}^{X_{2\nu-1}} \frac{A(X)dX}{V(X)(X-Y)} + \int_{X^{(2\nu)}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)(X-Y)}$$

with $X^{(2\nu)}$ now lying in the other sheet of the Riemann surface.

In the limit $\epsilon \longrightarrow 0$, one obtains

$$\lim_{\epsilon \to 0} \int_{X_{2\nu}}^{X_{2\nu-1}} \frac{A(X)dX}{V(X)(X-Y)} = \begin{cases} \pm \pi \mathrm{i} \delta_{\mu\nu} / [\lambda_\mu (Y_\mu + \mathrm{i}z)] & \text{for} \quad Y = Y_\mu \\ 0 \quad \text{for} \quad Y = \overline{Y}_\mu \end{cases}$$

with $\delta_{\mu\nu}$ being the usual Kronecker symbol and λ_μ as defined in (7).

The second term amounts to

$$\lim_{\epsilon \to 0} \int_{X^{(2\nu)}}^{X^{(2\nu-1)}} \frac{A(X)dX}{V(X)(X-Y)}$$

$$= \int_{X^{(2\nu)}}^{X^{2\nu-1)}} \frac{dX}{(X-Y)\sqrt{(X+\mathrm{i}z)(X-\mathrm{i}\overline{z})}}$$

$$= \frac{1}{\lambda(Y)(Y+\mathrm{i}z)} \ln \left(\frac{[\lambda(X^{(2\nu-1)}) - \lambda(Y)][\lambda(X^{(2\nu)}) + \lambda(Y)]}{[\lambda(X^{(2\nu-1)}) + \lambda(Y)][\lambda(X^{(2\nu)}) - \lambda(Y)]} \right),$$

where for evaluation of the second integral the substitution

$$\lambda = \lambda(X) = \sqrt{\frac{X - \mathrm{i}\bar{z}}{X + \mathrm{i}z}}$$

has been used.

Hence, the Jacobian inversion problem reads as follows in the limit $\epsilon \longrightarrow 0$:

$$\bullet \prod_{\nu=1}^{q} \frac{[\lambda(X^{(2\nu-1)}) - \lambda_\mu][\lambda(X^{(2\nu)}) + \lambda_\mu]}{[\lambda(X^{(2\nu-1)} + \lambda_\mu][\lambda(X^{(2\nu)}) - \lambda_\mu]} = -\gamma_\mu \tag{21}$$

$$\bullet \prod_{\nu=1}^{q} \frac{[\lambda(X^{(2\nu-1)}) - \lambda_\mu^*][\lambda(X^{(2\nu)}) + \lambda_\mu^*]}{[\lambda(X^{(2\nu-1)}) + \lambda_\mu^*][\lambda(X^{(2\nu)}) - \lambda_\mu^*]} = \bar{\gamma}_\mu \tag{22}$$

and in an analogous manner

$$f = f_0 \prod_{\nu=1}^{p} \frac{[\lambda(X^{(2\nu-1)}) + 1][\lambda(X^{(2\nu)}) - 1]}{[\lambda(X^{(2\nu-1)}) - 1][\lambda(X^{(2\nu)}) + 1]} \tag{23}$$

[with γ_μ, λ_μ^* and f_0 as defined in (7)].

Instead of evaluating the quantities $\lambda(X^{(\nu)})$, $(1 \le \nu \le 2q)$, the coefficients b_ν and $c_\nu (1 \le \nu \le q)$ of the polynomial

$$P(\lambda) = \prod_{\nu=1}^{p} [\lambda - \lambda(X^{(2\nu-1)})][\lambda + \lambda(X^{(2\nu)})]$$

$$= \lambda^{2q} + \lambda \sum_{\nu=1}^{q} b_\nu \lambda^{2\nu-2} + \sum_{\nu=1}^{q} c_\nu \lambda^{2\nu-2} \tag{24}$$

are determined. Since

$$\frac{P(\lambda_\mu)}{P(-\lambda_\mu)} = -\gamma_\mu, \quad \frac{P(\lambda_\mu^*)}{P(-\lambda_\mu^*)} = \bar{\gamma}_\mu, \quad f = f_0 \frac{P(-1)}{P(1)}, \tag{25}$$

the following system of linear equations for the quantities b_ν, c_ν, $P(1)$, and $P(-1)$ emerges:

$$\bullet \sum_{\nu=1}^{q} [b_\nu \alpha_\mu \lambda_\mu^{2\nu-1} + c_\nu \lambda_\mu^{2\nu-2}] = -\lambda_\mu^{2q},$$

$$\bullet \sum_{\nu=1}^{q} [b_\nu \alpha_\mu^* (\lambda_\mu^*)^{2\nu-1} + c_\nu (\lambda_\mu^*)^{2\nu-2}] = -(\lambda_\mu^*)^{2q} \tag{26}$$

$$\bullet \sum_{\nu=1}^{q} (b_\nu - c_\nu) + P(-1) = 1$$

$$\bullet \sum_{\nu=1}^{q} (b_\nu + c_\nu) - P(1) = -1,$$

with α_μ and α_μ^* as defined in (7).

Finally, if the solution of this linear system for $P(\pm 1)$ is expressed by means of Cramer's rule, the desired form (7) of the Bäcklund type is obtained.

APPENDIX B. INVARIANCE PROPERTIES OF THE ERNST POTENTIAL

For the proof of the properties (9–12), the Ernst potential (7) is reformulated by

$$f(\{Y_\nu\}_q; g) = f_0 \, \frac{D(-1; \{Y_\nu\}_q; g)}{D(1; \{Y_\nu\}_q; g)} \tag{27}$$

with

$$\bullet D(\lambda; \{Y_\nu\}_q; g) = \begin{vmatrix} a_1 & (a_1 x_1) & \cdots & (a_1 x_1^{q-1}) & 1 & x_1 & \cdots & x_1^q \\ a_2 & (a_2 x_2) & \cdots & (a_2 x_2^{q-1}) & 1 & x_2 & \cdots & x_2^q \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2q+1} & (a_{2q+1} x_{2q+1}) & \cdots & (a_{2q+1} x_{2q+1}^{q-1}) & 1 & x_{2q+1} & \cdots & x_{2q+1}^q \end{vmatrix}$$

$$\bullet a_1 = \lambda, \quad a_{2\nu} = \alpha_\nu \lambda_\nu, \quad a_{2\nu+1} = \alpha_\nu^* \lambda_\nu^*,$$

$$\bullet x_1 = \lambda^2, \quad x_{2\nu} = \lambda_\nu^2, \quad x_{2\nu+1} = (\lambda_\nu^*)^2.$$

The above expression for $D(\lambda;\ \{Y_\nu\}_q;\ g)$ is a Vandermonde-like determinant. These determinants have been studied in detail by Steudel, Meinel and Neugebauer [30]. By their *reduction formula* [(8) of [30]], D assumes the form:

$$
\begin{aligned}
&D(\lambda; \{Y_\nu\}_q; g) \\
&\quad = \mathcal{V}_{q,q+1}(a_r; b_r | x_r) \qquad [\text{with } b_r = 1 \text{ for } r = 1 \ldots (2q+1)] \\
&\quad = \sum_P \epsilon_P \left(\prod_{j=1}^{q} a_{r(j)} \right) \mathcal{V}_q[x_{r(1)}, \ldots, x_{r(q)}] \mathcal{V}_{q+1}[x_{r(q+1)}, \ldots, x_{r(2q+1)}]
\end{aligned}
$$

where

- the sum runs over all permutations $P = [r(1), \ldots, r(2q+1)]$ of $(1, 2, \ldots, 2q+1)$
  with $r(k) < r(j)$ for $k < j < q$ as well as for $q \leq k < j$
- $\epsilon_P = \begin{cases} +1 & \text{for } P \text{ even} \\ -1 & \text{for } P \text{ odd} \end{cases}$
- the Vandermonde determinants are given by
  $$\mathcal{V}_N[x_1, \ldots, x_N] = \prod_{k>j} (x_k - x_j).$$

In this formulation the following properties can be proved:
(A) If $x_{2q+1} = x_{2q}$ then

$$
\begin{aligned}
&D(\lambda; \{Y_\nu\}_q; g) \\
&\quad = (-1)^q (a_{2q} - a_{2q+1}) \left[\prod_{j=1}^{2q-1} (x_{2q} - x_j) \right] D(\lambda; \{Y_\nu\}_{q-1}; -g)
\end{aligned}
$$

(B) If $x_{2q} = 1 + \kappa\epsilon + O(\epsilon^2)$, $x_{2q+1} = 1 - \kappa\epsilon + O(\epsilon^2)$, and $(a_{2q}a_{2q+1}) = 1 + O(\epsilon)$, then

$$
\begin{aligned}
&D(\mp 1; \{Y_\nu\}_q; g) \\
&\quad = \kappa\epsilon \left[\prod_{j=2}^{2q-1} (1 - x_j) \right] (a_{2q} + a_{2q+1} \pm 2) D(\mp 1; \{Y_\nu\}_{q-1}; g) + O(\epsilon^2).
\end{aligned}
$$

With (A) the equalities (9) and (10) can be derived whilst (B) serves to confirm (11) and (12). In order to prove (A) consider the following groups of permutations separately:

$$\bullet\; P_1 : r(q-1) = 2q, r(q) = 2q+1$$
$$P_2 : r(2q) = 2q, r(2q+1) = 2q+1$$
$$P_3 : r(q) = 2q, r(2q+1) = 2q+1$$
$$P_4 : r(q) = 2q+1, r(2q+1) = 2q$$

For $x_{2q+1} = x_{2q}$, all terms belonging to P_1 and P_2 vanish while all terms belonging to P_3 and P_4 possess a common factor, $[a_{2q}\prod_{j=1}^{2q-1}(x_{2q} - x_j)]$ and $[a_{2q+1}\prod_{j=1}^{2q-1}(x_{2q} - x_j)]$, respectively. After reordering (from which the factor $(-1)^q$ results), (A) is easily obtained.

The proof for (B) works similarly. Now, eight groups of permutations have to be considered separately:

$$P_{1a} : r(1) = 1, r(q-1) = 2q, r(q) = 2q+1$$
$$P_{1b} : r(q+1) = 1, r(2q) = 2q, r(2q+1) = 2q+1$$
$$P_{2a} : r(q) = 2q, r(q+1) = 1, r(2q+1) = 2q+1$$
$$P_{2b} : r(1) = 1, r(q) = 2q, r(2q+1) = 2q+1$$
$$P_{3a} : r(q) = 2q+1, r(q+1) = 1, r(2q+1) = 2q$$
$$P_{3b} : r(1) = 1, r(q) = 2q+1, r(2q+1) = 2q$$
$$P_{4a} : r(q-1) = 2q, r(q) = 2q+1, r(q+1) = 1$$
$$P_{4b} : r(1) = 1, r(2q) = 2q, r(2q+1) = 2q+1$$

All terms of permutations with a coinciding first index can be combined to give:[13]

$$\{P_{1a}, P_{1b}\} \Rightarrow O(\epsilon^3)$$
$$\{P_{2a}, P_{2b}\} \Rightarrow a_{2q}F + O(\epsilon^2)$$
$$\{P_{3a}, P_{3b}\} \Rightarrow a_{2q+1}F + O(\epsilon^2)$$
$$\{P_{4a}, P_{4b}\} \Rightarrow \pm 2F + O(\epsilon^2)$$

with

$$F = (-1)^{q+1}\kappa\,\epsilon \prod_{j=2}^{2q-1}(1 - x_j)D(\pm 1; \{Y_\nu\}_{q-1}; -g).$$

[13] Here the requirements $a_1 = \mp 1$, $x_1 = 1$ are necessary. Additionally, for P_{4a} and P_{4b}, the constraint $a_{2q}a_{2q+1} = 1 + O(\epsilon)$ is needed.

APPENDIX C. NEWTONIAN AND ULTRARELATIVISTIC LIMITS

C.1. The Newtonian Limit

In the limit of small functions g and ξ, i.e.

$$g(x^2) = \epsilon_g g_0(x^2) + O(\epsilon_g^2), \qquad \xi(x^2) = \epsilon_\xi \xi_0(x^2) + O(\epsilon_\xi^2),$$

the Ernst potential $f = f(\xi;\, g)$ as introduced in Section 2 is given by

$$f(\xi; g) = 1 - \epsilon_g \int_{-1}^{1} \frac{g_0(x^2)dx}{Z_D} - \mathrm{i}\epsilon_g\epsilon_\xi \int_{-1}^{1} \frac{(\mathrm{i}x)g_0(x^2)\xi_0(x^2)dx}{Z_D} + O(\epsilon_g^2) + O(\epsilon_g\epsilon_\xi^2). \tag{28}$$

In this section, the above property will be proved and the functions g_0 and ξ_0 will be derived as they result from the Newtonian expansion of the boundary conditions.

C.1.1. The Ernst Potential for Small Functions g and ξ

Due to the assumption that the function Φ_g introduced in (14) can be extended to form a continuous mapping defined on $\mathcal{A}$ (see Sections 2 and 5), the representation of ξ in terms of $\{Y_\nu\}_q$ can be chosen arbitrarily. Here, the following set $\{Y_\nu\}_q$ is used:

- $q = 4r$
- $\left\{\begin{array}{ll} Y_{4\nu-3} = Z_\nu(1 + \epsilon_\xi z_\nu), & Y_{4\nu-2} = -\overline{Y}_{4\nu-3} \\ Y_{4\nu-1} = \overline{Z}_\nu(1 - \epsilon_\xi z_\nu), & Y_{4\nu} = -\overline{Y}_{4\nu-1} \end{array}\right\}, \begin{array}{ll} \mathcal{R}(Z_\nu) \neq 0, & z_\nu \in \mathbb{R} \\ (\nu = 1 \cdots r) & \end{array}$

Then, it follows from (13) that $\xi(x^2) = \epsilon_\xi \xi_0(x^2) + O(\epsilon_\xi^2)$ with

$$\xi_0(x^2) = -4\mathrm{i} \sum_{\nu=1}^{r} \frac{z_\nu(Z_\nu - \overline{Z}_\nu)(x^2 - Z_\nu\overline{Z}_\nu)}{(x^2 + Z_\nu^2)(x^2 + \overline{Z}_\nu^2)}.$$

To evaluate the Ernst potential in this limit, the formulation (21–26) in appendix A is used and the following steps are performed:

1. At first, it turns out that in the limit $\epsilon_\xi \longrightarrow 0$ the coefficients b_ν of the polynomial (24) vanish. This can be seen by considering the solution to linear system (26).

$$b_\nu = \frac{D_\nu}{D}:$$

- $D = \begin{vmatrix} a_2 & \cdots & (a_2 x_2^{q-1}) & 1 & x_2 & \cdots & x_2^{q-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2q+1} & \cdots & (a_{2q+1} x_{2q+1}^{q-1}) & 1 & x_{2q+1} & \cdots & x_{2q+1}^{q-1} \end{vmatrix}$
- $a_{2\eta} = \alpha_\eta \lambda_\eta, \quad a_{2\eta+1} = \alpha_\eta^* \lambda_\eta^*, \quad x_{2\eta} = \lambda_\eta^2, \quad x_{2\eta+1} = (\lambda_\eta^*)^2$
- D_ν is derived from D by replacing the ν-th column by the vector

$$\{-x_2^q, \ldots, -x_{2q+1}^q\}.$$

For $1 \le \nu \le q$, D_ν can be expanded in terms of Vandermonde determinants

$$\mathcal{V}_{q+1}(x_{r(1)}, \ldots, x_{r(q+1)}), \quad r(\eta) \in \{2, \ldots, 2q{+}1\}, \quad r(\eta) < r(\mu) \text{ for } \eta < \mu.$$

In the limit $\epsilon_\xi \longrightarrow 0$, any set $\{x_{r(\eta)}\}_{q+1}$ contains at most q different values, and therefore all D_ν vanish. On the other hand, D remains finite (here only Vandermonde determinants $\mathcal{V}_q$ are involved), and hence all b_ν tend to zero.

2. Thus, with any zero $\tilde{\lambda}_\nu$ of the Polynomial (24), $(-\tilde{\lambda}_\nu)$ also becomes a zero as $\epsilon_\xi \longrightarrow 0$. This set of zeros is ordered in the following way:

$$\{\lambda(X^{(1)}), -\lambda(X^{(2)}), \ldots, \lambda(X^{(2q-1)}), -\lambda(X^{(2q)})\} = \{\tilde{\lambda}_1, -\tilde{\lambda}_1, \ldots, \tilde{\lambda}_q, -\tilde{\lambda}_q\},$$

Suppose there is a λ_μ different from all zeros:

$$\lambda_\mu \neq \lambda(X^{(2\nu-1)}) = \lambda(X^{(2\nu)}) \quad \text{and} \quad \lambda_\mu \neq -\lambda(X^{(2\nu-1)}) \quad \text{for all } \nu = 1 \ldots q.$$

Then, since $\gamma_\mu \neq -1$ for small g, (21) cannot be satisfied.

3. This gives rise to the following ansatz ($\nu = 1 \ldots q$):

$$\lambda^2(X^{(2\nu-1)}) = \lambda_\nu^2 + \epsilon_\xi \kappa_{2\nu-1} + O(\epsilon_\xi^2), \qquad \lambda^2(X^{(2\nu)}) = \lambda_\nu^2 + \epsilon_\xi \kappa_{2\nu} + O(\epsilon_\xi^2),$$

by which the system (21/22) can easily be solved to get the set $\{\kappa_\nu\}_{2q}$.

4. Finally, if $g(x^2) = \epsilon_g g_0(x^2) + O(\epsilon_g^2)$ is considered, then (28) follows from (23) by inserting the values obtained for $\{\lambda(X^{(\nu)})\}_{2q}$.

C.1.2. The Functions g_0 and ξ_0 as Resulting from the Boundary Conditions

For any family of Ernst potentials $f = f(g_\epsilon;\ \xi_\epsilon)$ describing a sequence of differentially rotating disks of dust with the parameter $\epsilon = M^2/J$ [M and J as defined in (16)], the following expansion is valid (see [24], pp. 83–89):

$$f = 1 + e_2(\rho,\zeta)\epsilon^2 + \mathrm{i}b_3(\rho,\zeta)\epsilon^3 + O(\epsilon^4).$$

By comparison with (28) one gets

- $\epsilon_g = \epsilon^2, \quad \epsilon_\xi = \epsilon,$
- $e_2(\rho,\zeta) = -\displaystyle\int_{-1}^{1} \frac{g_0(x^2)dx}{Z_D}, \quad b_3(\rho,\zeta) = -\displaystyle\int_{-1}^{1} \frac{(\mathrm{i}x)g_0(x^2)\xi_0(x^2)dx}{Z_D}.$

If the boundary conditions,

- $\sigma_p(\rho) = \sigma_0\psi_2[(\rho/\rho_0)^2]\sqrt{1-(\rho/\rho_0)^2}\epsilon^2 + O(\epsilon^4) \quad (\text{with } \psi_2(0) = 1) \quad \text{or}$
- $\Omega(\rho) = \Omega_0\Omega_1[(\rho/\rho_0)^2]\epsilon + O(\epsilon^3) \quad (\text{with } \Omega_1(0) = 1),$

are given, then it follows from equations (3–6) that

- $(e_2)_{,\zeta} = 4\pi\sigma_0\psi_2\sqrt{1-(\rho/\rho_0)^2} \quad \text{or} \quad (e_2)_{,\rho} = 2\Omega_0^2\Omega_1^2\rho \quad \text{and}$
- $(b_3)_{,\rho} = 2\rho\Omega_0\Omega_1(e_2)_{,\zeta}.$

By expressing e_2 and b_3 in terms of g_0 and ξ_0 in these equations, one gets Abelian integral equations for ξ_0 and g_0. Their solutions read as follows:

$$g_0(x^2) = -4\sigma_0(1-x^2)\int_0^{\pi/2} (\sin^2\phi)\psi_2(\cos^2\phi + x^2\sin^2\phi)d\phi$$

$$g_0(x^2)\xi_0(x^2) = 8\sigma_0\Omega_0(1-x^2)\int_0^{\pi/2} (\sin^2\phi)\tilde{\Omega}_l(\cos^2\phi + x^2\sin^2\phi)d\phi$$

$$[\text{with } \tilde{\Omega}_1(x^2) = \Omega_1(x^2)\psi_2(x^2)].$$

Note that only *one* of the functions ψ_2 and Ω_1 can be prescribed since both represent different boundary conditions of the same Newtonian potential e_2. Likewise, the constants σ_0 and Ω_0^2 depend on each other. Moreover, these constants in terms of ψ_2 and Ω_1 are prescribed by the equation $\epsilon = M^2/J$.

C.2. The Ultrarelativistic Limit

It is difficult to relate the functions g and ξ of an Ernst potential $f = f(g;\xi)$ to its physical properties like M and J. Nevertheless, if a sequence $f(g_\epsilon;\xi_\epsilon)$ can be extended to arbitrary values $\epsilon < 1$, then, in the limit $\epsilon \longrightarrow 1$, the universal solution of an extreme Kerr black hole is reached. It is illustrated how this limit results from the form (7) of the Ernst potential.

If the limit $\rho_0 \longrightarrow 0$ is considered for finite values of $r = \sqrt{\rho^2 + \zeta^2}$, then by using the formulation (27) one gets (with $\zeta = r\cos\theta$):

$$f = \left(1 - \frac{\rho_0}{r}\int_{-1}^{1}(-1)^q g(x^2)dx + O(\rho_0^2)\right) \cdot \left[\frac{E_1 r + \rho_0[E_3\cos\theta - (-1)^q E_2]}{E_1 r + \rho_0[E_3\cos\theta + (-1)^q E_2]} + O(\rho_0^2)\right].$$

The E_j do not depend on ρ and ζ but on g and ξ. In particular:

- $E_1 = \begin{vmatrix} b_1 & (b_1 Z_1) & \cdots & (b_1 Z_q^{q-1}) & 1 & Z_1 & \cdots & Z_1^{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2q} & (b_{2q} Z_{2q}) & \cdots & (b_{2q} Z_{2q}^{q-1}) & 1 & Z_{2q} & \cdots & Z_{2q}^{q-1} \end{vmatrix}$
- $E_2 = \begin{vmatrix} b_1 & (b_1 Z_2) & \cdots & (b_1 Z_1^{q-2}) & 1 & Z_1 & \cdots & Z_1^{q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{2q} & (b_{2q} Z_{2q}) & \cdots & (b_{2q} Z_{2q}^{q-2}) & 1 & Z_{2q} & \cdots & Z_{2q}^{q} \end{vmatrix}$
- $b_{2\nu-1} = -\tanh\left[\frac{1}{2}\int_{-1}^{1}\frac{(-1)^q g(x^2)dx}{\mathrm{i}x - Y_\nu}\right], \qquad b_{2\nu}\overline{b}_{2\nu-1} = 1$
- $Z_{2\nu-1} = Y_\nu, \qquad Z_{2\nu} = \overline{Y}_\nu.$

Clearly, if $E_1 \neq 0$ then $\lim_{\rho_0 \to 0} f = 1$. The Ernst potential passes to an ultrarelativistic limit if E_1 and ρ_0 tend simultaneously to zero such that[14]

[14] It can be shown that $E_1^2 \in \mathbb{R}$. Hence, the ultrarelativistic limit for the family $f(g_\epsilon;\xi_\epsilon)$ is performed when some function $E_a = E_a(g_\epsilon;\xi_\epsilon) = E_b(\epsilon)E_1^2(\epsilon)$, which is independent of the representation $\{Y_\nu\}_q$, vanishes.

$$\Omega_U = \lim_{\rho_0 \to 0} \frac{(-1)^q E_1}{2E_2\rho_0}$$

exists. Then one gets

$$f = \frac{2\Omega_U r + E_4 \cos\theta - 1}{2\Omega_U r + E_4 \cos\theta + 1}.$$

The only Ernst potential of this form which is asymptotically flat and regular for $r > 0$ is the extreme Kerr solution. The constant Ω_U is then real and describes the 'angular velocity of the horizon'. Moreover, $J = 1/(4\Omega_U^2) = M^2$, and hence $\epsilon = 1$.

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