

Gravitational field and equations of motion of spinning compact binaries to 2.5 post-Newtonian order

Hideyuki Tagoshi

Department of Earth and Space Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Akira Ohashi

Department of Fundamental Science, FIHS, Kyoto University, Kyoto 606-8501, Japan

Benjamin J. Owen

*Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, California 91125
and Albert Einstein Institut (Max Planck Institut für Gravitationsphysik), Am Mühlenberg 1, 14476 Golm, Germany
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We derive spin-orbit coupling effects on the gravitational field and equations of motion of compact binaries in the 2.5 post-Newtonian approximation to general relativity, one PN order beyond where spin effects first appear. Our method is based on that of Blanchet, Faye, and Ponsot, who use a post-Newtonian metric valid for general (continuous) fluids and represent pointlike compact objects with a δ -function stress-energy tensor, regularizing divergent terms by taking the Hadamard finite part. To obtain post-Newtonian spin effects, we use a different δ -function stress-energy tensor introduced by Bailey and Israel. In a future paper we will use the 2.5PN equations of motion for spinning bodies to derive the gravitational-wave luminosity and phase evolution of binary inspirals, which will be useful in constructing matched filters for signal analysis. The gravitational field derived here may help in posing initial data for numerical evolutions of binary black hole mergers.

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I. INTRODUCTION

Gravitational waves from coalescing compact binaries are the most promising candidate for detection by near-future, ground-based laser interferometers such as LIGO, VIRGO, GEO600, and TAMA [1]. Detection of signals and estimation of signal parameters in noisy data require detailed modeling of the *inspiral* phase of the waveform [2] and—depending on the mass of the system—some knowledge of the *merger* phase [3]. The matched filtering techniques used to analyze the inspiral waveform require that the model, or template, match the real signal to less than one cycle out of the roughly 10^3 – 10^4 in the detectable band. This precision requires high-order post-Newtonian calculations of the equations of motion and gravitational-wave luminosity. The late inspiral and merger stages of coalescence are not amenable to post-Newtonian approximation methods and require numerical evolution of binary black-hole spacetimes. However, a post-Newtonian approximation of the gravitational field may help in posing initial data for such numerical evolutions [4].

The standard method for modeling inspiral waveforms from binaries of arbitrary mass ratio is the post-Newtonian expansion in powers of the binary's orbital velocity v/c and gravitational potential $GM/rc^2 \sim (v/c)^2$. There are two nearly equivalent approaches to calculating gravitational waves at high Post-Newtonian order: one developed by Blanchet, Damour, and Iyer (BDI) [5–7] and one by Will and Wiseman [8] based on previous work by Epstein, Wagoner, and Will [9]. The gravitational waveforms and luminosity are expanded in time derivatives of radiative multipoles, which are expressed as integrals of the matter source and gravitational field. The radiative multipoles are combined with the equations of motion to obtain explicit expressions in terms of the source masses, positions, and velocities, which can be converted to gauge-invariant frequencies observed at infinity. Under the assumption that the bodies can be treated as pointlike particles characterized only by their masses, the matter source (stress-energy tensor) is given as a δ -function. In combination with a regularization scheme for infinite self-field effects, this source greatly simplifies the field integrals compared to a detailed fluid body calculation.

It is standard to count post-Newtonian (PN) orders in powers of $(v/c)^2$ beyond the Newtonian result for the equation of motion and the quadrupole formula for the gravitational waves. For the case of nonspinning bodies, the equations of motion have been known to 2.5PN order for decades [10–13], and the gravitational-wave luminosity also has been evaluated recently to 2.5PN order [7]. At the moment the 3PN calculations of the equations of motion for nonspinning bodies are nearing completion (see [14,15] and references therein).

The post-Newtonian expansion of gravitational waves from spinning bodies has not been carried as far as for nonspinning bodies—almost exclusively the literature contains calculations of leading-order effects. There are several

kinds of spin effects: Spin causes precession of the orbital plane of a binary, changes the orbital frequency, affects the gravitational-wave luminosity, and modifies the amplitudes of the gravitational waveforms. Spin effects can be further divided into spin-orbit coupling (involving a single spin) and spin-spin interactions.¹ The leading-order terms in the equations of motion were derived in the 1950s by Papapetrou [16], and in precession in the 1970s by Barker and O’Connell [17]. These results have been re-derived by several methods [11,18–22]. Kidder, Will, and Wiseman [23] in the 1990s evaluated the leading-order (1.5PN spin-orbit and 2PN spin-spin) terms in the gravitational-wave luminosity (not including precession). Kidder [24] completed that work with evaluation of the leading-order (1PN spin-orbit and 1.5PN spin-spin) terms in the waveform amplitudes. He, along with Apostolatos *et al.* [25], also considered the effects of precession on the waveforms. All of these calculations were carried out only to leading post-Newtonian order in each spin effect, due in large part to the unwieldiness of calculations representing spinning bodies with a fluid stress-energy tensor of nonzero compact support.

However, it is possible to make a “spinning particle” approximation to simplify the calculations. Mino *et al.* [26,27] used a δ -function stress-energy tensor, a compact reformulation of the work of Dixon [28], to represent a spinning test mass in orbit around a massive spinning black hole to high post-Newtonian order in the limit of small mass ratio. Cho [29] used a similar stress-energy tensor due to Bailey and Israel [30] to re-derive the lowest-order results of Kidder, Will, and Wiseman [23]. Recently we [31] made the first calculation in full (arbitrary mass ratio) post-Newtonian theory of a non-leading order effect, a 2PN spin-orbit contribution to the waveform amplitudes, using the stress-energy tensor of Mino *et al.* [26]. Our goal is to eventually obtain the lowest-order unknown spin effect in the phase evolution of the waveforms, a 2.5PN spin-orbit term. The new term requires evaluation of the radiative multipoles at relative 1PN order, i.e. 1PN beyond the leading-order 1.5PN effect, a task which can be done in a straightforward manner using the techniques of Ref. [31]. However there is also a contribution from the 2.5PN spin-orbit terms in the equations of motion, which we must first calculate.

Recently Blanchet, Faye and Ponsot (BFP) [32] derived the equations of motion for nonspinning objects using a new approach, based on the post-Newtonian fluid metric used in wave generation calculations combined with a δ -function source and the use of the Hadamard finite part [13,33] to regularize the resulting divergent integrals. In this paper we generalize the BFP calculation to include spin, thereby obtaining the missing 2.5PN spin-orbit term in the equations of motion. This term, together with the well-known 2PN spin-spin term and the 2PN quadrupole term [34,35] completes the equations of motion of two finite bodies to 2.5PN order. The calculation is similar to our previous work [31], except that we now use the stress-energy tensor of Bailey and Israel [30] due to its advantages in deriving the regularized equations of motion and precession.

We have made some effort to check that our spinning particle approximation is actually a good model of a spinning compact object. Of course, any δ -function approximation of an extended body yields nonsensical results near the body, but we only require a good approximation to the field some distance outside the body and to quantities obtained by volume integrals of the field or its source. (The use of our field results as initial data for numerical binary black hole simulations must be supplemented by an additional prescription near the bodies, such as matched asymptotic expansion [4].) Following the lead of Ref. [33], we give some consistency arguments. First, we show that the one-body limit of our result for the gravitational field reproduces the Kerr metric up to the post-Newtonian order considered. Second, we verify that the equations of motion and precession we derive from a regularized version of the stress-energy conservation law satisfy the harmonic gauge condition assumed at the beginning. Third, in our approach we reproduce the well-known 1.5PN spin-orbit terms in the equations of motion. Finally, we verify that the new 2.5PN terms we derive are Lorentz invariant and reduce (in the test mass limit) to the results of black-hole perturbation theory.

This paper is organized as follows. In Sec. II, we describe our spinning particle approximation, including the stress-energy tensor, regularization scheme, and derivation of the regularized equations of motion and precession. In Sec. III, we use the post-Newtonian expansion to write needed quantities in terms of a set of post-Newtonian potentials. Section IV is a description of the calculation of the post-Newtonian potentials in terms of properties of the bodies. The (lengthy) results for the potentials are relegated to Appendix A. In Sec. V, we present the new 2.5PN terms in the equations of motion. In Sec. VI, we summarize our results and describe problems of future interest. In Appendix B, we check our result for the circular-orbit frequency by comparing it to results for spinning test particle in orbit around a Kerr black hole. In Appendix C we provide a brief summary of the post-Newtonian check for Lorentz invariance.

Throughout this paper, we use units such that Newton’s gravitational constant and the speed of light equal unity. The brackets $()$ and $[\]$ on tensor indices indicate symmetrization and antisymmetrization respectively:

¹At higher post-Newtonian orders it is also possible to have three-body, three-spin interactions; but these have not been considered yet.

$$\Phi_{(\alpha\beta)} = \frac{1}{2}(\Phi_{\alpha\beta} + \Phi_{\beta\alpha}), \quad \Phi_{[\alpha\beta]} = \frac{1}{2}(\Phi_{\alpha\beta} - \Phi_{\beta\alpha}) \quad (1.1)$$

Greek indices run from 0 to 3, and Latin indices from 1 to 3.

II. SPINNING PARTICLE APPROXIMATION

In this section we describe how we simplify fluid-body calculations by approximating spinning compact objects as pointlike particles. We use the Bailey-Israel δ -function stress-energy tensor and the Hadamard finite-part regularization scheme for the gravitational field of a point source. Conservation of the stress-energy tensor and vanishing of its antisymmetric part yield the Papapetrou equations of motion and precession in a form where the regularization is unambiguous.

A. Stress-energy tensor

The particle-like stress-energy tensor of Bailey and Israel [30] can be divided into monopole, spin, and spin antisymmetric parts:

$$T^{\alpha\beta} = T_{(M)}^{\alpha\beta} + T_{(S)}^{\alpha\beta} + T_{(SA)}^{\alpha\beta}. \quad (2.1)$$

These parts are expressed as integrals over the trajectory $y(\tau)$ of the particle as

$$T_{(M)}^{\alpha\beta}(x) = \int d\tau p^\alpha(\tau) u^\beta(\tau) \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g}} \quad (2.2a)$$

$$T_{(S)}^{\alpha\beta}(x) = -\nabla_\gamma \int d\tau \hat{S}^{\gamma(\alpha}(\tau) u^{\beta)}(\tau) \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g}}, \quad (2.2b)$$

$$T_{(SA)}^{\alpha\beta}(x) = -\frac{1}{2} \nabla_\gamma \int d\tau \hat{S}^{\alpha\beta}(\tau) u^\gamma(\tau) \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g}}. \quad (2.2c)$$

Here $u^\mu \equiv dy^\mu/d\tau$, $p^\mu(\tau)$ is the linear momentum of the particle, and $\hat{S}^{\alpha\beta}(\tau)$ is an antisymmetric tensor representing spin angular momentum. However, the trajectory parameter τ is generally not quite the proper time of the body (see below). To represent multiple bodies, we label each particle and its associated quantities by A (y_A^μ , etc.) and sum all parts of Eq. (2.2) over A .

When treating spinning compact bodies as point particles, we introduce spurious degrees of freedom which must be fixed. A finite spinning body has a center of mass (which must be defined carefully in post-Newtonian theory, see e.g. Ref. [23]). However, in the limit as the body is shrunk to a point particle, the trajectory of the particle does not necessarily correspond to the worldline of the finite spinning body's center of mass. The ambiguity in precisely what the trajectory represents (or equivalently in defining the center of mass) is fixed by imposing a *spin supplementary condition*. We choose

$$\hat{S}^{\mu\nu} p_\nu = 0, \quad (2.3)$$

but note that other choices are common in the literature (see the discussion by Kidder [24]). These other choices of spin supplementary condition lead to equations of motion that appear different but are physically identical, a common source of confusion. The ambiguity in the definition of the spin also implies that τ is not precisely the proper time along the body worldline (although τ reduces to the proper time in the Newtonian limit). Thus $u^\mu(\tau)$, the tangent vector to the trajectory $y^\mu(\tau)$, is not precisely the four-velocity of the body. We define

$$\hat{u}^\mu \equiv p^\mu/m, \quad (2.4)$$

where m is the Schwarzschild mass of the body. Holding the linear momentum constant along the worldline implies

$$g_{\mu\nu} \hat{u}^\mu \hat{u}^\nu = -1. \quad (2.5)$$

We also introduce a spin vector four-vector S^μ . The spin supplementary condition (2.3) allows the natural definition

$$\hat{S}^{\alpha\beta} = \frac{-1}{\sqrt{-g}} \varepsilon^{\alpha\beta\mu\nu} \hat{u}_\mu S_\nu, \quad (2.6)$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric symbol with $\varepsilon^{0123} = 1$. Even after demanding that $S^\mu S_\mu$ be conserved along the trajectory and that S^i reduce to the Newtonian spin in the appropriate limit, S^0 is still undefined. We fix this remaining degree of freedom by imposing the condition

$$S^\mu p_\mu = 0. \quad (2.7)$$

Note that the literature also contains different conventions for this condition (see e.g. [20]).

B. Regularization procedure

In deriving the equations of motion, we need to evaluate the gravitational field and associated quantities at the locations of the bodies. However, as usual when dealing with pointlike sources in field theory, the quantities we need diverge if we naively integrate the (unphysical) δ -function stress-energy tensor. Thus we must augment our method with a regularization procedure to remove the infinite self-field effects while preserving the correct physical terms. In this paper, we adopt the Hadamard finite part [13,32,33,36].

In our calculations we encounter a class of functions F which admit, when the field point \mathbf{x} approaches one of the source points ($r_1 = |\mathbf{x} - \mathbf{y}_1| \rightarrow 0$), an expansion of the form

$$F(\mathbf{x}) = \sum_{-k_0 \leq k \leq 0} r_1^k f_k(\mathbf{n}_1) + O(r_1), \quad (2.8)$$

where k is a set of integers and $\mathbf{n}_1 = (\mathbf{x} - \mathbf{y}_1)/r_1$. We define the effective value of the function F at the position of body 1 as the Hadamard finite part,²

$$(F)_1 \equiv F(\mathbf{y}_1, \mathbf{y}_1) \equiv \int \frac{d\Omega(\mathbf{n}_1)}{4\pi} f_0(\mathbf{n}_1, \mathbf{y}_1). \quad (2.9)$$

This effective value also holds for integrals of F with a three-dimensional δ -function,

$$\int d^3\mathbf{x} F(\mathbf{x}, \mathbf{y}_1) \delta(\mathbf{x} - \mathbf{y}_1) \equiv (F)_1. \quad (2.10)$$

We introduce a symbol $(\)_A$ to denote the Hadamard finite part at $x = y_A$ of the function within parentheses $(\)$.

When we calculate the post-Newtonian potentials, encounter integrals of the form

$$\int d^3x F(x) T^{\alpha\beta}. \quad (2.11)$$

Each part of this integral can be evaluated by means of Hadamard's regularization as

$$\int d^3x F(x) T_{(M)}^{\alpha\beta}(x) = \sum_A \frac{m_A}{u_A^0} \left(\frac{F}{\sqrt{-g}} \right)_A, \quad (2.12a)$$

$$\begin{aligned} & \int d^3x F(x) T_{(S)}^{\alpha\beta}(x) \\ &= \sum_A \left\{ \left[\hat{S}_A^{\gamma(\alpha} v_A^{\beta)} \frac{\partial_\gamma F}{\sqrt{-g}} \right]_A - \partial_t \left(\hat{S}_A^{0(\alpha} v_A^{\beta)} \frac{F}{\sqrt{-g}} \right)_A \right. \\ & \quad \left. - \left[\left(\Gamma_{\gamma\delta}^\gamma \hat{S}_A^{\delta(\alpha} v_A^{\beta)} + \hat{S}_A^{\gamma(\alpha} \Gamma_{\gamma\delta}^\beta) v_A^\delta \right) \frac{F}{\sqrt{-g}} \right]_A \right\}, \end{aligned} \quad (2.12b)$$

²A more sophisticated treatment of the Hadamard finite part is used in some higher-order calculations [15]. However, this version, which is based on that of BFP, is sufficient for our purposes.

$$\begin{aligned}
& \int d^3x F(x) T_{(SA)}^{\alpha\beta}(x) \\
&= \frac{1}{2} \sum_A \left\{ \left(\hat{S}_A^{\alpha\beta} v_A^\gamma \frac{\partial_\gamma F}{\sqrt{-g}} \right)_A - \partial_t \left(\hat{S}_A^{\alpha\beta} \frac{F}{\sqrt{-g}} \right)_A \right. \\
&\quad - \left[\left(\Gamma_{\mu\nu}^\alpha \hat{S}_A^{\nu\beta} v_A^\mu + \Gamma_{\mu\nu}^\beta \hat{S}_A^{\alpha\nu} v_A^\mu + \Gamma_{\mu\nu}^\mu \hat{S}_A^{\alpha\beta} v_A^\nu \right) \right. \\
&\quad \left. \left. \times \frac{F}{\sqrt{-g}} \right]_A \right\}. \tag{2.12c}
\end{aligned}$$

C. Equations of motion and precession

The equations of motion and precession can be derived from the regularized stress-energy tensor. However, the regularization does not commute with some derivatives and multiplications, and moreover our version of it takes place on slices of constant t rather than in the rest frame of each body. Therefore at each step we must make some consistency checks.

We consider the following form of the conservation of the stress-energy tensor,

$$\partial_\beta (\sqrt{-g} T^{\alpha\beta}) + \Gamma_{\beta\mu}^\alpha \sqrt{-g} T^{\mu\beta} = 0. \tag{2.13}$$

By using the Bailey-Israel stress-energy tensor (2.2) and integrating over a three-volume D_A containing only body A , we obtain the equations of motion as

$$\frac{d}{dt} p_A^\alpha + (\Gamma_{\mu\nu}^\alpha)_A p_A^\mu v_A^\nu + \frac{1}{2} (R_{\beta\mu\nu}^\alpha)_A v_A^\beta \hat{S}_A^{\mu\nu} = 0, \tag{2.14}$$

where $v_A^\mu = u_A^\mu / u_A^0$. (If we used Dixon's stress-energy tensor as before [31], we would have both the equations of motion and of precession entangled in this expression, and it would be difficult to separate them.) By performing the same integral with the free index covariant, we obtain

$$\frac{d}{dt} p_{A\alpha} - (\Gamma_{\alpha\nu}^\mu)_A p_{A\mu} v_A^\nu + \frac{1}{2} (R_{\alpha\beta\mu\nu})_A v_A^\beta \hat{S}_A^{\mu\nu} = 0. \tag{2.15}$$

It can be shown by direct calculation that (2.14) and (2.15) are equivalent up to 1PN order for non-spinning terms and 2.5PN order for spinning terms, which is a useful consistency check of our calculations. Direct calculations also show that Eqs. (2.14) and (2.15) are Lorentz invariant up to 1PN order for non-spinning terms and 2.5PN order for spin-orbit terms.

To derive the regularized equations of precession from the Bailey-Israel stress-energy tensor, we demand that

$$\int_{D_A} d^3x \sqrt{-g} T^{[\alpha\beta]} = 0. \tag{2.16}$$

[Strictly speaking, we require that $h^{[\alpha\beta]}$ vanish, but to the order we need this condition reduces to Eq. (2.16).] It is straightforward to derive the spin precession equations from this volume integral as

$$\frac{d}{dt} \hat{S}_A^{\alpha\beta} + (\Gamma_{\mu\nu}^\alpha)_A \hat{S}_A^{\mu\beta} v_A^\nu + (\Gamma_{\mu\nu}^\beta)_A \hat{S}_A^{\alpha\mu} v_A^\nu + 2p_A^{[\alpha} v_A^{\beta]} = 0. \tag{2.17}$$

In the same way, we can also derive the spin precession equation with covariant indices as

$$\begin{aligned}
& \frac{d}{dt} \left(\hat{S}_{A\alpha\beta} \right)_A + (\Gamma_{\alpha\gamma}^\mu)_A \hat{S}_{A\mu\beta} v_A^\gamma + \left(\Gamma_{\beta\gamma}^\mu \right)_A \hat{S}_{A\alpha\mu} v_A^\gamma \\
& \quad + 2p_{A[\alpha} v_{A\beta]} = 0. \tag{2.18}
\end{aligned}$$

As we shall see in later sections, (2.17) and (2.18) are equivalent, at least to the order we need.

Using Eqs. (2.14) and (2.17), we can also derive the relation

$$p_A^\alpha - m_A u_A^\alpha = -\frac{1}{2m_A} (R_{\beta\nu\rho\sigma})_A S_A^{\alpha\beta} u_A^\nu S_A^{\rho\sigma}. \tag{2.19}$$

Equation (2.19) tells us that if we consider only spin-orbit effects we can neglect the difference between \hat{u}_A^μ and u_A^μ .³ Therefore in the rest of this paper we set

$$\hat{u}_A^\mu = u_A^\mu = \frac{dy^\mu}{d\tau} \quad (2.20)$$

and set τ to be the proper time of the body.

III. THE POST-NEWTONIAN EXPANSION

We introduce a parameter ε of the order of a characteristic velocity v . Since we consider bound systems, we can use the virial theorem to put $v^2 = O(M/r)$ for the total mass M and (harmonic coordinate) separation r . To isolate spin effects, we also introduce a parameter χ which is also dimensionless and (at most) of order unity for compact objects. This χ is of order (spin)/(mass)² for some combination of the bodies. Terms of $O(\chi)$ correspond to spin-orbit effects, spin-spin effects are $O(\chi^2)$, etc. We use the shorthand $O(m, n)$ to denote terms of order ε^m and $\chi\varepsilon^n$, and $O(m)$ to denote terms simply of order ε^m or $\chi\varepsilon^m$ (depending on the context). In this paper we neglect terms of $O(\chi^2)$ and higher.

We will expand numerous quantities in ε and χ , but first let us determine what orders are needed. In Eq. (2.14), the Newtonian force is $O(\varepsilon^4)$; thus the 2.5PN spin-orbit terms are $O(\chi\varepsilon^9)$. By expanding the entire expression in terms of the metric tensor, we find that we need g_{00} to $O(4, 7)$; g_{i0} to $O(5, 6)$; and g_{ij} to $O(4, 5)$. By integrating different terms in the stress-energy tensor (2.2) over 3-volume and substituting $S \sim \chi m^2$, we find

$$\begin{aligned} |T^{00}/T_{(M)}^{00}| &= O(0, 3), \\ |T^{i0}/T_{(M)}^{00}| &= O(1, 2), \\ |T^{ij}/T_{(M)}^{00}| &= O(2, 3). \end{aligned} \quad (3.1)$$

We can insert this order counting into the metric below.

The Einstein equations can be written in harmonic coordinates as

$$\square h^{\mu\nu} = 16\pi |g| T^{\mu\nu} + \Lambda^{\mu\nu}(h), \quad (3.2)$$

where

$$h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu} \quad (3.3)$$

and the harmonic gauge condition is

$$h^{\mu\nu}{}_{,\nu} = 0. \quad (3.4)$$

Here $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and $\Lambda^{\mu\nu}(h)$ represents the non-linear terms in the Einstein equations.

It is convenient to define the densities

$$\sigma \equiv T^{00} + T^{ii}, \quad (3.5a)$$

$$\sigma_i \equiv T^{i0}, \quad (3.5b)$$

$$\sigma_{ij} \equiv T^{ij}, \quad (3.5c)$$

and the retarded potentials [32]

³By counting orders with the post-Newtonian metric used in Sec. III, we see that this difference does not matter until the 3PN spin-spin terms in the equations of motion and the 2PN spin-spin terms in the equations of precession.

$$V = \square_R^{-1} \{-4\pi G\sigma\} , \quad (3.6a)$$

$$V_i = \square_R^{-1} \{-4\pi G\sigma_i\} , \quad (3.6b)$$

$$\hat{W}_{ij} = \square_R^{-1} \{-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V\} , \quad (3.6c)$$

$$\hat{R}_i = \square_R^{-1} \{-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V \partial_i V_k - \frac{3}{2}\partial_t V \partial_i V\} , \quad (3.6d)$$

$$\hat{X} = \square_R^{-1} \{-4\pi G V \sigma_{ii} + 2V_i \partial_t \partial_i V + V \partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i + \hat{W}_{ij} \partial_{ij}^2 V\} , \quad (3.6e)$$

where

$$\square_R^{-1} \{-4\pi f(\mathbf{x}, t)\} \equiv \int \frac{d^3 \mathbf{z}}{|\mathbf{x} - \mathbf{z}|} f(\mathbf{z}, t - |\mathbf{x} - \mathbf{z}|). \quad (3.7)$$

Inserting the order counting of Eq. (3.1), we find that the potentials have the following post-Newtonian orders:

$$\begin{aligned} V &= O(2, 5), & V_i &= O(3, 4), & \hat{W}_{ij} &= O(4, 5), \\ \hat{R}_i &= O(5, 6), & \hat{X} &= O(6, 7). \end{aligned} \quad (3.8)$$

To the order we require, the solution to the Einstein equations (3.2) is given by

$$g_{00} = -1 + 2V - 2V^2 + 8\hat{X} + O(6, 8), \quad (3.9a)$$

$$g_{i0} = -4V_i - 8\hat{R}_i + O(7, 8), \quad (3.9b)$$

$$g_{ij} = \delta_{ij} (1 + 2V + 2V^2) + 4\hat{W}_{ij} + O(5, 6), \quad (3.9c)$$

$$\sqrt{-g} = 1 + 2V + 4V^2 + 2\hat{W}_{kk} + O(6, 7). \quad (3.9d)$$

The harmonic coordinate condition (3.4) reduces to the following identities between the potentials:

$$-4\partial_t V_i - 4\partial \left(\hat{W}_{ij} - \frac{1}{2}\delta_{ij}\hat{W}_{kk} \right) = O(5, 6), \quad (3.10a)$$

$$\begin{aligned} \partial_t \left(V + \frac{1}{2}\hat{W}_{kk} + 2V^2 \right) + \partial_i \left[V_i + 2\hat{R}_i + 2V V_i \right] \\ = O(6, 7). \end{aligned} \quad (3.10b)$$

(Actually, these are redundant and we can use only the former.)

The (regularized) post-Newtonian metric (3.9) allows us to relate the relativistic source parameters to the Newtonian ones. From here onward, indices on the right-hand sides of expressions are raised and lowered with the Cartesian metric δ_{ij} and are written up or down merely for convenience. From Eq. (2.5) we obtain

$$\begin{aligned} u_A^0 &= 1 + \frac{1}{2}v_A^2 + (V)_A + \frac{3}{8}v_A^4 + \frac{5}{2}v_A^2(V)_A - 4v_A^i(V_i)_A \\ &\quad + \frac{1}{2}(V^2)_A + O(6, 7). \end{aligned} \quad (3.11)$$

From the definition (2.7) of S^0 we obtain

$$S_A^0 = S_A^i \left[v_A^i - 4(V_i)_A + 4v_A^i(V)_A + O(5) \right]. \quad (3.12)$$

We introduce for convenience the shorthand $S_A^{ij} \equiv \varepsilon^{ijk} S_A^k$ (where ε^{ijk} is the three-dimensional antisymmetric symbol), and by combining Eqs. (2.3) and (2.6) we obtain

$$\begin{aligned} \hat{S}_A^{i0} &= S_A^{ij} \left\{ v_A^j \left[1 + \frac{1}{2}v_A^2 + 3(V)_A \right] - 4(V_j)_A \right. \\ &\quad \left. + O(5) \right\}, \end{aligned} \quad (3.13a)$$

$$\begin{aligned} \hat{S}_A^{ij} &= S_A^{ij} \left[1 + \frac{1}{2}v_A^2 - (V)_A \right] - \varepsilon_{ijk} v_A^k (v_A S_A) \\ &\quad + m_A^2 O(\chi \varepsilon^4). \end{aligned} \quad (3.13b)$$

Here we have also introduced the shorthand for the (Cartesian) scalar product of two vectors $(ab) \equiv a^i b^i$.

IV. CALCULATION OF THE POTENTIALS

We now describe the calculation of the potentials to the needed orders. The results are lengthy and are displayed in Appendix A. Two of the potentials (V and V_i) are needed to relative 1PN order, i.e. to $O(2)$ beyond the leading spin and non-spin terms, while the rest are only needed to the lowest order at which they appear. To obtain the explicit results in terms of the bodies' coordinates, velocities, masses, and spins, we must substitute lower-order results for the equations of motion and precession as well as for V and V_i .

The lower-order quantities we need are easily obtained. The lowest-order potentials are simple to evaluate as

$$V = \frac{m_1}{r_1} + \frac{2S_1^{ij}n_1^i v_1^j}{r_1^2} + 1 \leftrightarrow 2 + O(4, 7), \quad (4.1a)$$

$$V_i = \frac{m_1}{r_1} v_1^i + \frac{n_1^k}{2r_1^2} S_1^{ki} + 1 \leftrightarrow 2 + O(5, 6), \quad (4.1b)$$

where $r_A = |\mathbf{x} - \mathbf{y}_A|$ and $n_A^i = (x^i - y_A^i)/r_A$. The lower-order terms in the equations of motion are well known. Alternatively they can be calculated by inserting (4.1) and the results of Sec. III into Eq. (2.14) or (2.15) as

$$\begin{aligned} \frac{dv_1^i}{dt} = & -\frac{m_2}{r_{12}^2} \left(n_{12}^i \left\{ 1 + \left[-\frac{5m_1}{r_{12}} - \frac{4m_2}{r_{12}} + (v_1)^2 + 2(v_2)^2 \right. \right. \right. \\ & \left. \left. \left. - 4(v_1 v_2) - \frac{3}{2}(n_{12} v_2)^2 \right\} - v_{12}^i [4(n_{12} v_1) \right. \right. \\ & \left. \left. - 3(n_{12} v_2)] - \left(\frac{S_1^{ik}}{m_1} + \frac{2S_2^{ik}}{m_2} \right) \frac{v_{12}^j}{r_{12}} (3n_{12}^k n_{12}^j - \delta^{kj}) \right. \right. \\ & \left. \left. - 2 \left(\frac{S_1^{kj}}{m_1} + \frac{S_2^{kj}}{m_2} \right) \frac{v_{12}^j}{r_{12}} (3n_{12}^k n_{12}^i - \delta^{ki}) \right. \right. \\ & \left. \left. + O(4, 5) \right), \end{aligned} \quad (4.2)$$

with $\mathbf{n}_{12} = |\mathbf{y}_1 - \mathbf{y}_2|/r_{12}$ and $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. The leading order terms in the equations of precession can be calculated by the same procedure using Eqs. (2.17) and (2.18) as

$$\begin{aligned} \frac{dS_1^i}{dt} = & \frac{m_2}{r_{12}^2} \{ S_1^i (n_{12} v_{12}) + 2n_{12}^i [(S_1 v_2) - (S_1 v_1)] \\ & + (S_1 n_{12})(v_1^i - 2v_2^i) + S_1 \times O(4) \}. \end{aligned} \quad (4.3)$$

The equations of motion and precession for body 2 are given by exchanging labels 1 and 2 in Eqs. (4.2) and (4.3). Note that the effect of spin precession first appears at 1PN order, in the sense that $dS/dt = (S v)/r \times O(2)$.

The spin precession equations (4.3) are different from the usual form of the geodesic precession [20,37]

$$\frac{d\hat{S}_1^i}{dt} = \frac{m_2}{r_{12}^2} \varepsilon_{ijk} S_1^{\hat{k}} \varepsilon_{jpq} (2v_2^p - \frac{3}{2}v_1^p) n_{12}^q. \quad (4.4)$$

This is due to differing definitions of spin. Equation (4.4) holds for spins defined in the local asymptotic rest frame of each body such that $S_A^{\hat{i}} S_A^{\hat{i}}$ is constant along the trajectory. However, for our definition of the spin vector this does not hold true (beyond leading order). The spin definitions are related by

$$S_1^{\hat{i}} = \left(1 + \frac{m_2}{r_{12}} \right) S_1^i - \frac{1}{2} v_1^i (S_1 v_1) + S_1 \times O(4). \quad (4.5)$$

The compact parts of the potentials (integrals with compact support) are straightforward to evaluate by a retardation expansion of the integrands of Eqs. (3.6a)–(3.6e) and substitution of the lower-order equations of motion (4.2) and precession (4.3). We integrate only the symmetric part of the stress-energy tensor. The equations of precession (4.3) ensure that $T_{(SA)}^{\alpha\beta}$ does not contribute the metric. It can be verified term by term that $T_{(SA)}^{\alpha\beta}$ does not contribute to the individual potentials either, to the order considered here.

We also encounter what are generally called quadratic terms (integrals of products of two potentials or their derivatives) which do not have compact support. We use the distributional derivative

$$\partial_{ij}^2 \left(\frac{1}{r_1} \right) = \frac{3n_1^i n_1^j - \delta^{ij}}{r_1^3} - \frac{4\pi}{3} \delta^{ij} \delta(\mathbf{x} - \mathbf{y}_1), \quad (4.6)$$

which is sufficient to our desired order [32], change time derivatives to spatial derivatives by

$$\partial_t \left(\frac{1}{r_1} \right) = v_1^i \partial_{1i} \left(\frac{1}{r_1} \right), \quad (4.7)$$

where $\partial_{1i} \equiv \partial/\partial y_1^i$, use $\partial_i(1/r_1) = -\partial_{1i}(1/r_1)$, and move the operator ∂_{1i} outside the integral sign. We also make frequent use of the identity

$$\Delta^{-1} \left(\frac{1}{r_1 r_2} \right) = \ln(r_1 + r_2 + r_{12}) \quad (4.8)$$

and its derivatives, where Δ here represents the Laplacian.

We check the potentials in two ways. First, we verify that they satisfy the identities (3.10b) equivalent to the harmonic gauge condition. Next, we verify that the metric reduces to Kerr in the appropriate limit. By using the potentials in Appendix A and setting $m_2 = 0$, $v_2^i = 0$, $S_2^{ij} = 0$, and $v_1^i = 0$, we obtain the metric

$$g_{00} = -1 + \frac{2m_1}{r_1} - \frac{2m_1^2}{r_1^2} + O(6, 8), \quad (4.9a)$$

$$g_{0i} = -\frac{2}{r_1^2} n_1^k S_1^{ki} + \frac{2m_1}{r_1^3} n_1^k S_1^{ki} + O(6, 7), \quad (4.9b)$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{2m_1}{r_1} + \frac{m_1^2}{r_1^2} \right) + \frac{m_1^2}{r_1^2} n_1^i n_1^j + O(5, 6). \quad (4.9c)$$

This is equivalent to the Kerr metric in harmonic coordinates truncated at the indicated post-Newtonian orders. Thus we have some evidence that our spinning particle approximation produces physically sensible results.

V. EQUATIONS OF MOTION AT 2.5PN ORDER

Using the post-Newtonian potentials derived in the previous section, we derive the spin-orbit interaction terms in the equations of motion at 2.5PN order. Direct calculation shows that it does not matter whether we use (2.14) or (2.15).

A. Body-centered form

We can write the equations of motion in the Newtonian-like form

$$\frac{d}{dt} \mathcal{P}_1^i = \mathcal{F}_1^i. \quad (5.1)$$

The post-Newtonian “linear momentum” is given by

$$\begin{aligned} \mathcal{P}_1^i = m_1 v_1^i & \left[1 + \frac{1}{2} v_1^2 + (V)_1 + \frac{3}{8} v_1^4 + \frac{5}{2} v_1^2 (V)_1 \right. \\ & \left. + \frac{1}{2} (V^2)_1 - 4v_1^j (V_j)_1 + O(6, 7) \right]. \end{aligned} \quad (5.2)$$

The “force” has two components, the “frame force”

$$\begin{aligned} \mathcal{F}_{1(F)}^i &= -m_1 (\Gamma^i{}_{\mu\nu})_1 v_1^\mu v_1^\nu u^0 \\ &= m_1 u_1^0 \left\{ V_{,i} (1 + v_1^2 - 4V) + 4\dot{V}_i (1 - 2V) + 4V_i (\dot{V} + 2v_1^j V_{,j}) + 8\dot{R}_i + 4\hat{X}_{,i} + 8V_j V_{j,i} - 4V_{,j} \hat{W}_{ij} - 2v_1^i (\dot{V} \right. \\ & \quad \left. + v_1^j V_{,j}) + 4v_1^j [2V_{[i,j]} (1 - 2V) + 4\hat{R}_{[i,j]} - \dot{W}_{ij}] + 2v_1^j v_1^k (\hat{W}_{jk,i} - 2\hat{W}_{ij,k}) \right\}_1 + O(7, 10) \end{aligned} \quad (5.3)$$

and the spin-curvature coupling force

$$\begin{aligned}
\mathcal{F}_{1(SC)}^i &= -\frac{m_1}{2} (R^i{}_{\beta\mu\nu})_1 v_1^\beta \hat{S}_1^{\mu\nu} \\
&= m_1 \left\{ \hat{S}_1^{i0} \left(\ddot{V} + V_{,j} V_{,j} + v_1^j \dot{V}_{,j} \right) + \hat{S}_1^{ij} \left[\dot{V}_{,j} + 4V_{,k} V_{[j,k]} + \dot{V} V_{,j} + v_1^j V_{,k} V_{,k} + v_1^k (V_{,jk} - V_{,j} V_{,k}) \right] + \hat{S}_1^{j0} \right. \\
&\quad \times \left[V_{,ij} (1 - 4V) + 4\dot{V}_{(i,j)} - 3V_{,i} V_{,j} - v_1^j \dot{V}_{,i} + 4v_1^k V_{[i,k]j} \right] + \hat{S}_1^{jk} \left[2V_{,ik} (1 - 2V) + 2\dot{W}_{ij,k} + 4\hat{R}_{j,ik} \right. \\
&\quad \left. \left. - 8V_{,(i} V_{j),k} + v_1^k (V_{,ij} - V_{,i} V_{,j}) + 2v_1^l (\hat{W}_{ij,kl} + \hat{W}_{kl,ij}) \right] \right\} + O(7, 10). \tag{5.4}
\end{aligned}$$

Note that our conventions are slightly different from those of BFP.

By inserting the explicit formulas of the post-Newtonian potentials, given in Appendix A and by performing the Hadamard regularization, we obtain the equations of motion in terms of body masses, spins, positions, and velocities. The equations of motion of body 1 can be expressed as

$$\begin{aligned}
\frac{dv_1^i}{dt} &= \frac{m_2}{r_{12}^2} \left[A_0^i(\mathbf{y}_1 - \mathbf{y}_2) + A_1^i(\mathbf{y}_1 - \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2) \right. \\
&\quad + B_{1.5}^i(\mathbf{y}_1 - \mathbf{y}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{S}_1, \mathbf{S}_2) \\
&\quad + B_2^i(\mathbf{y}_1 - \mathbf{y}_2, \mathbf{S}_1, \mathbf{S}_2) \\
&\quad \left. + B_{2.5}^i(\mathbf{y}_1 - \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2, \mathbf{S}_1, \mathbf{S}_2) + O(4, 6) \right]. \tag{5.5}
\end{aligned}$$

Here A_0^i and A_1^i are respectively the Newtonian and 1PN forces independent of spin, given by

$$A_0^i = -n_{12}^i, \tag{5.6}$$

$$\begin{aligned}
A_1^i &= n_{12}^i \left[5 \frac{m_1}{r_{12}} + 4 \frac{m_2}{r_{12}} - v_1^2 - 2v_2^2 + 4(v_1 v_2) \right. \\
&\quad \left. + \frac{3}{2} (n_{12} v_2)^2 \right] + v_{12}^i [4(n_{12} v_1) - 3(n_{12} v_2)]. \tag{5.7}
\end{aligned}$$

The leading-order (1.5PN) spin-orbit interaction force is given by

$$\begin{aligned}
B_{1.5}^i &= \left(\frac{S_1^{ik}}{m_1} + 2 \frac{S_2^{ik}}{m_2} \right) v_{12}^j (-\delta^{kj} + 3n_{12}^k n_{12}^j) \\
&\quad + 2 \left(\frac{S_1^{kj}}{m_1} + \frac{S_2^{kj}}{m_2} \right) v_{12}^j (-\delta^{ki} + 3n_{12}^k n_{12}^i). \tag{5.8}
\end{aligned}$$

The leading-order (2PN) spin-spin interaction force is given by

$$\begin{aligned}
B_2^i &= -\frac{3}{r_{12}^2} \left[\mathbf{n}_{12} \frac{(S_1 S_2)}{m_1 m_2} + \frac{\mathbf{S}_1}{m_1} \frac{(n_{12} S_2)}{m_2} + \frac{\mathbf{S}_2}{m_2} \frac{(n_{12} S_1)}{m_1} \right. \\
&\quad \left. - 5 \mathbf{n}_{12} \frac{(n_{12} S_1)}{m_1} \frac{(n_{12} S_2)}{m_2} \right]. \tag{5.9}
\end{aligned}$$

All of these terms are well known and can be found in references such as Damour [21].

The new term, the 2.5PN (1PN beyond leading order) spin-orbit coupling term, is given by

$$\begin{aligned}
B_{2.5}^i &= \frac{n^j S_1^{ji}}{m_1 r} \left[\left(\frac{14m_1}{r} + \frac{9m_2}{r} \right) (n v_{12}) + \frac{15}{2} (n v_{12})(n v_2)^2 - 3(n v_2)(v_{12} v_2) \right] + \frac{v_{12}^j S_1^{ji}}{m_1 r} \left[-\frac{14m_1}{r} - \frac{9m_2}{r} - \frac{15}{2} (n v_2)^2 \right. \\
&\quad + 3(n v_1)(n v_2) - 3(v_{12} v_2) \left. \right] + \frac{n^j S_2^{ji}}{m_2 r} \left[\left(\frac{35m_1}{2r} + \frac{16m_2}{r} \right) (n v_{12}) - \frac{2m_1}{r} (n v_2) + 15(n v_{12})(n v_2)^2 \right. \\
&\quad + 6(n v_{12})(v_{12} v_2) - 6(n v_2)(v_{12} v_2) \left. \right] + \frac{v_{12}^j S_2^{ji}}{m_2 r} \left[-\frac{23m_1}{2r} - \frac{12m_2}{r} - 6(n v_2)^2 - 4(v_{12} v_2) \right] + \frac{n^i}{r} \left\{ \frac{S_1^{jk}}{m_1} n^j v_{12}^k \right. \\
&\quad \left. \times \left[-\frac{26m_1}{r} - \frac{18m_2}{r} - 15(n v_2)^2 - 6(v_{12} v_2) \right] + \frac{S_2^{jk}}{m_2} n^j v_{12}^k \left[-\frac{49m_1}{2r} - \frac{20m_2}{r} - 15(n v_2)^2 - 6(v_{12} v_2) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& -12\varepsilon^{jkl}n^jv_1^kv_2^l\frac{(v_1S_1)}{m_1}\Big\} + \frac{v_{12}^i}{r}\Big\{\frac{S_1^{jk}}{m_1}[-6(nv_1)n^jv_{12}^k] + \frac{S_2^{jk}}{m_2}[-6(nv_1)n^jv_{12}^k]\Big\} + \frac{v_1^i}{r}\Big\{\frac{S_1^{jk}}{m_1}[3(nv_{12})n^jv_1^k \\
& -3v_1^jv_2^k] + \frac{S_2^{jk}}{m_2}[6(nv_{12})n^jv_2^k - 4v_1^jv_2^k]\Big\} + \frac{\varepsilon^{ijk}}{r}\Big\{-n^jv_{12}^k\frac{4m_1}{r}\frac{(nS_2)}{m_2} - 3n^jv_1^k(nv_{12})\frac{(v_1S_1)}{m_1} \\
& -6n^jv_2^k(nv_{12})\frac{(v_1S_1)}{m_1} + 7v_1^jv_2^k\frac{(v_1S_1)}{m_1}\Big\}. \tag{5.10}
\end{aligned}$$

B. Center-of-mass form

Here we express the equations of motion in terms of the bodies' relative coordinate, in the center-of-mass frame. We define the mass parameters $M = m_1 + m_2$, $\eta = m_1m_2/M^2$, and $\Delta = (m_1 - m_2)/M$; and the dimensionless, symmetrized spin parameters [8]

$$\boldsymbol{\chi}_s = \frac{1}{2}\left(\frac{\mathbf{S}_1}{m_1^2} + \frac{\mathbf{S}_2}{m_2^2}\right), \tag{5.11a}$$

$$\boldsymbol{\chi}_a = \frac{1}{2}\left(\frac{\mathbf{S}_1}{m_1^2} - \frac{\mathbf{S}_2}{m_2^2}\right). \tag{5.11b}$$

It is also convenient to introduce the shorthand

$$\chi_s^{ij} = \frac{1}{2}\left(\frac{S_1^{ij}}{m_1^2} + \frac{S_2^{ij}}{m_2^2}\right), \tag{5.12a}$$

$$\chi_a^{ij} = \frac{1}{2}\left(\frac{S_1^{ij}}{m_1^2} - \frac{S_2^{ij}}{m_2^2}\right). \tag{5.12b}$$

The relation between the body coordinates $\mathbf{y}_1, \mathbf{y}_2$, and the relative coordinate is

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{x}\left[\frac{m_2}{M} + \frac{1}{2}\eta\Delta\left(v^2 - \frac{M}{r}\right)\right] - M\eta\mathbf{v} \times (\boldsymbol{\chi}_a + \Delta\boldsymbol{\chi}_s), \\
\mathbf{y}_2 &= \mathbf{x}\left[-\frac{m_1}{M} + \frac{1}{2}\eta\Delta\left(v^2 - \frac{M}{r}\right)\right] - M\eta\mathbf{v} \times (\boldsymbol{\chi}_a + \Delta\boldsymbol{\chi}_s). \tag{5.13}
\end{aligned}$$

The last terms in the above equation are 1.5PN order and contribute to the 2.5PN relative acceleration through corrections to A_1^i . From now on we drop the subscript 12 on r, \mathbf{n} , and \mathbf{v} .

We define the relative acceleration as

$$\mathbf{a} \equiv \frac{d\mathbf{v}_1}{dt} - \frac{d\mathbf{v}_2}{dt}, \tag{5.14}$$

and we express the equations of motion as

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{PN} + \mathbf{a}_{SO} + \mathbf{a}_{SS} + \mathbf{a}_{PNSO} + \mathcal{O}(6, 8). \tag{5.15}$$

The Newtonian acceleration \mathbf{a}_N is given by

$$\mathbf{a}_N = -\frac{M}{r^2}\mathbf{n}. \tag{5.16}$$

The 1PN acceleration \mathbf{a}_{PN} is given by

$$\begin{aligned}
\mathbf{a}_{PN} &= -\frac{M}{r^2}\Big\{\mathbf{n}\left[(1 + 3\eta)v^2 - 2(2 + \eta)\frac{M}{r} - \frac{3}{2}\eta(nv)\right] \\
& - 2(2 - \eta)(nv)\mathbf{v}\Big\}. \tag{5.17}
\end{aligned}$$

The 1.5PN spin-orbit acceleration \mathbf{a}_{SO} is given by

$$\begin{aligned}
\mathbf{a}_{SO} = & \frac{M^2}{r^3} \{6\mathbf{n}(\mathbf{n} \times \mathbf{v}) \cdot (\boldsymbol{\chi}_s + \Delta\boldsymbol{\chi}_a) \\
& -2\mathbf{v} \times [(2-\eta)\boldsymbol{\chi}_s + 2\Delta\boldsymbol{\chi}_a] \\
& +6(nv)\mathbf{n} \times [(1-\eta)\boldsymbol{\chi}_s + \Delta\boldsymbol{\chi}_a]\}.
\end{aligned} \tag{5.18}$$

The 2PN spin-spin acceleration \mathbf{a}_{SS} is given by

$$\begin{aligned}
\mathbf{a}_{SS} = & -\frac{M^3}{r^4} 3\eta \{ \mathbf{n} [|\boldsymbol{\chi}_s|^2 - |\boldsymbol{\chi}_a|^2 - 5(n\chi_s)^2 + 5(n\chi_a)^2] \\
& +2[\boldsymbol{\chi}_s(n_{12}\boldsymbol{\chi}_s) - \boldsymbol{\chi}_a(n\chi_a)] \}.
\end{aligned} \tag{5.19}$$

The new 2.5PN spin-orbit acceleration $\mathbf{a}_{PN\text{SO}}$ (i.e. the post-Newtonian correction to \mathbf{a}_{SO}) is given by

$$\begin{aligned}
a_{PN\text{SO}}^i = & \frac{M^2}{2r^3} \left(\chi_s^{ji} \left\{ n^j(nv) \left[(32-15\eta+8\eta^2) \frac{M}{r} + 15\eta(1+2\eta)(nv)^2 - 6\eta(3+2\eta)v^2 \right] + v^j \left[3(-8+\eta) \frac{M}{r} \right. \right. \right. \\
& -3\eta(5+2\eta)(nv)^2 + 14\eta v^2 \left. \left. \left. \right\} + \Delta\chi_a^{ji} \left\{ n^j(nv) \left[(32+13\eta) \frac{M}{r} + 15\eta(nv)^2 - 18\eta v^2 \right] + v^j \left[-3(8+3\eta) \frac{M}{r} \right. \right. \right. \\
& -15\eta(nv)^2 + 14\eta v^2 \left. \left. \left. \right\} + n^i n^j v^k \left\{ \chi_s^{jk} \left[(-40-13\eta-16\eta^2) \frac{M}{r} - 30\eta(nv)^2 + 24\eta v^2 \right] + \Delta\chi_a^{jk} \right. \right. \\
& \times \left[(-40-21\eta) \frac{M}{r} - 30\eta(nv)^2 + 24\eta v^2 \right] \left. \left. \right\} - 6v^i n^j v^k (nv) [(2-3\eta-2\eta^2)\chi_s^{jk} + (2-\eta)\Delta\chi_a^{jk}] + (\mathbf{n} \times \mathbf{v})^i \right. \\
& \left. \left. \left. \left\{ -8\eta[(n\chi_s) + \Delta(n\chi_a)] \frac{M}{r} - 6\eta(nv)[(1+2\eta)(v\chi_s) - \Delta(v\chi_a)] \right\} \right) \right).
\end{aligned} \tag{5.20}$$

C. Quasicircular orbits

When the spins are aligned or anti-aligned with the orbital angular momentum, the spin vectors and the orbital angular momentum vector do not precess. In this case, a quasicircular orbit exists. We define $\chi_a = \pm|\boldsymbol{\chi}_a|$ and $\chi_s = \pm|\boldsymbol{\chi}_s|$. The signs of χ_a and χ_s are positive when the vectors are aligned with the angular momentum axis ($\mathbf{n} \times \mathbf{v}$) and negative when antialigned. For quasicircular orbits, the equations of motion take the form

$$\mathbf{a} = -\omega^2 \mathbf{x}, \tag{5.21}$$

where ω is the orbital angular frequency. Using the identity $v = \omega r$, we can eliminate v from the equations of motion and express ω as an expansion in M/r ,

$$\begin{aligned}
\omega^2 = & \frac{M}{r^3} \left\{ 1 - (3-\eta) \frac{M}{r} - 2[(1+\eta)\chi_s + \Delta\chi_a] \left(\frac{M}{r} \right)^{3/2} \right. \\
& +3\eta \left(\frac{M}{r} \right)^2 [(\chi_s)^2 - (\chi_a)^2] + [(9-3\eta+\eta^2)\chi_s \\
& \left. + (9-6\eta)\Delta\chi_a] \left(\frac{M}{r} \right)^{5/2} + O(4,7) \right\}.
\end{aligned} \tag{5.22}$$

In Appendix B, we show that Eq. (5.22) agrees in the limit $\eta \rightarrow 0$ with the motion of a test particle around a Kerr black hole to 2.5PN order.

The new terms in Eq. (5.22) are proportional to $M/r^3(M/r)^{5/2}$: The $O(\eta^2)$ term and parts of the $O(\eta)$ terms have not been derived before. Their magnitude indicates that 2.5PN spin-orbit contributions change the circular orbit frequency by an amount of less than about two percent near the innermost stable orbit for a system of maximally spinning black holes. However, the importance of this effect on the phase evolution of the orbit, and thus on the effectiveness of matched filtering for the gravitational-wave signal, cannot be evaluated until the gravitational-wave luminosity is known.

VI. SUMMARY

We have calculated the 2.5PN spin-orbit effects on the gravitational field and equations of motion of compact binaries by approximating the compact bodies as spinning point particles. Although the formal basis of this approximation remains uncertain, we have demonstrated that it is reasonable in that it reproduces some physical behavior of the Kerr spacetime to this post-Newtonian order, as well as lower-order, previously known terms in the equations of motion.

Using the techniques of this paper, it is a straightforward though lengthy task to calculate the equations of precession to 2PN order and the gravitational-wave luminosity (needed to construct templates) to 2.5PN order. We plan to do this in the future for circular orbits. The 2.5PN gravitational field given by combining our potentials with the spinless potentials given by BFP could serve toward providing initial data for the numerical evolution of spinning binary black-hole mergers, if augmented by some scheme for approximating the field near and inside the black holes [4].

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APPENDIX A: EXPLICIT POST-NEWTONIAN POTENTIALS

In this appendix, we present explicit expressions (in terms only of the masses, spins, positions, and velocities) for the post-Newtonian potentials necessary to obtain the equations of motion and the metric away from the bodies. We denote spin and monopole parts of quantities with subscript (S) and (M), respectively: e.g., $V_{(S)}$ and $V_{(M)}$. All expressions are fully reduced using the equations of motion (4.2) and precession (4.3). The monopole terms in the potentials are given to higher order by BFP, but for convenience we reproduce the pieces needed in our calculation.

The relevant potentials are

$$V_{(M)} = \frac{m_1}{r_1} + m_1 \left[\frac{2v_1^2}{r_1} - \frac{(n_1 v_1)^2}{2r_1} - \frac{m_2 r_1}{4r_{12}^3} - \frac{5m_2}{4r_1 r_{12}} + \frac{m_2 r_2^2}{4r_1 r_{12}^3} \right] + 1 \leftrightarrow 2 + O(5), \quad (\text{A1a})$$

$$\begin{aligned} V_{(S)} = & S_1^{ij} \left(n_1^i v_1^j \left[\frac{2}{r_1^2} + \frac{2v_1^2}{r_1^2} - \frac{3}{r_1^2} (n_1 v_1)^2 - \frac{2m_2}{r_1^2 r_{12}} + \frac{m_2}{r_1 r_{12}^2} (n_1 n_{12}) + \frac{5}{2} \frac{m_2}{r_{12}^3} \right] - n_1^i v_2^j \left(\frac{4m_2}{r_1^2 r_{12}} + \frac{5}{2} \frac{m_2}{r_{12}^3} \right) + n_1^i n_{12}^j \left[\frac{2m_2}{r_1 r_{12}^2} \right. \right. \\ & \times (n_1 v_1) - \left. \frac{9m_2}{2r_{12}^3} (n_{12} v_{12}) \right] + n_{12}^i v_1^j \left\{ \frac{7m_2}{r_1 r_{12}^2} - \frac{3m_2}{r_{12}^3} [(n_1 n_{12}) - (n_2 n_{12})] \right\} - n_{12}^i v_2^j \left\{ \frac{4m_2}{r_1 r_{12}^2} - \frac{3m_2}{r_{12}^3} [(n_1 n_{12}) \right. \\ & \left. \left. - (n_2 n_{12})] \right\} - 2n_{12}^i v_{12}^j \frac{m_2}{r_2 r_{12}^2} - 2n_2^i v_{12}^j \frac{m_2}{r_{12}^3} + 3n_2^i n_{12}^j (n_{12} v_{12}) \frac{m_2}{r_{12}^3} \right) + 1 \leftrightarrow 2 + O(8), \quad (\text{A1b}) \end{aligned}$$

$$\begin{aligned} V_{i(M)} = & \frac{m_1 v_1^i}{r_1} \left[1 + v_1^2 - \frac{1}{2} (n_1 v_1)^2 \right] + m_1 m_2 \left[\frac{(n_1 v_1)}{r_{12}^2} n_{12}^i + \frac{3}{2} \frac{r_1}{r_{12}^3} n_{12}^i (n_{12} v_{12}) + \frac{1}{2} \frac{r_1}{r_{12}^3} v_2^i \right] \\ & + \frac{m_1 m_2}{4} v_1^i \left(-\frac{3r_1}{r_{12}^3} - \frac{5}{r_1 r_{12}} + \frac{r_2^2}{r_1 r_{12}^3} \right) + 1 \leftrightarrow 2 + O(6), \quad (\text{A1c}) \end{aligned}$$

$$\begin{aligned} V_{i(S)} = & S_1^{ji} \left\{ n_1^j \left[\frac{1}{2r_1^2} + \frac{v_1^2}{2r_1^2} - \frac{3m_2}{2r_1^2 r_{12}} - \frac{3}{4r_1^2} (n_1 v_1)^2 + \frac{m_2}{4r_1 r_{12}^2} (n_1 n_{12}) \right] + n_{12}^j \frac{3m_2}{4r_1 r_{12}^2} \right\} + v_1^i S_1^{jk} \frac{n_1^j v_1^k}{2r_1^2} + \varepsilon^{ijk} n_1^j v_1^k \frac{(v_1 S_1)}{2r_1^2} \\ & + 1 \leftrightarrow 2 + O(7), \quad (\text{A1d}) \end{aligned}$$

$$\hat{W}_{ij(M)} = \delta^{ij} \left(-\frac{m_1 v_1^2}{r_1} - \frac{m_1^2}{4r_1^2} + \frac{m_1 m_2}{r_{12}s} \right) + \frac{m_1 v_1^i v_1^j}{r_1} + \frac{m_1^2 n_1^i n_1^j}{4r_1^2} + m_1 m_2 \left[\frac{1}{s^2} \left(n_1^{(i} n_2^{j)} + 2n_1^{(i} n_{12}^{j)} \right) - n_{12}^i n_{12}^j \left(\frac{1}{s^2} + \frac{1}{r_{12}s} \right) \right] + 1 \leftrightarrow 2 + O(5), \quad (\text{A1e})$$

$$\hat{W}_{ij(S)} = \left(S_1^{k(i} v_1^{j)} - \delta_{ij} S_1^{kl} v_1^l \right) \frac{n_1^k}{r_1^2} + 1 \leftrightarrow 2 + O(6), \quad (\text{A1f})$$

$$\hat{R}_{i(M)} = m_1 m_2 n_{12}^i \left[-\frac{(n_{12} v_1)}{2s} \left(\frac{1}{s} + \frac{1}{r_{12}} \right) - \frac{2(n_2 v_1)}{s^2} + \frac{3(n_2 v_2)}{2s^2} \right] + n_1^i \left\{ \frac{m_1^2 (n_1 v_1)}{8r_1^2} + \frac{m_1 m_2}{s^2} \left[2(n_{12} v_1) - \frac{3}{2}(n_{12} v_2) + 2(n_2 v_1) - \frac{3}{2}(n_2 v_2) \right] \right\} + v_1^i \left[-\frac{m_1^2}{8r_1^2} + m_1 m_2 \left(\frac{1}{r_1 r_{12}} + \frac{1}{2r_{12}s} \right) \right] - v_2^i \frac{m_1 m_2}{r_1 r_{12}} + 1 \leftrightarrow 2 + O(6), \quad (\text{A1g})$$

$$\hat{R}_{i(S)} = S_1^{ji} \left[n_1^j \left(-\frac{m_1}{4r_1^3} + \frac{m_2}{2r_1^2 r_{12}} + \frac{m_2}{r_{12}s} \right) + n_{12}^j \left(-\frac{m_2}{2r_1 r_{12}^2} + \frac{m_2}{2r_{12}^2 r_2} + \frac{m_2}{r_1 s^2} \right) + n_2^j \left(\frac{m_2}{r_1 s^2} + \frac{m_2}{r_{12} s^2} \right) \right] + n_1^i S_1^{jk} \left[n_1^j \times (n_{12}^k + n_2^k) \left(\frac{m_2}{r_1 s^2} + \frac{2m_2}{s^3} \right) - 2n_{12}^j n_2^k \frac{m_2}{s^3} \right] + n_{12}^i S_1^{jk} \left[(n_1^j + n_2^j) n_{12}^k \left(-\frac{m_2}{r_{12} s^2} - \frac{2m_2}{s^3} \right) - 2n_1^j n_2^k \frac{m_2}{s^3} \right] + 1 \leftrightarrow 2 + O(7), \quad (\text{A1h})$$

$$\hat{X}_{(S)} = S_1^{ij} \left\{ n_1^i v_1^j \left(-\frac{m_1}{2r_1^3} + \frac{m_2}{r_1^2 r_{12}} + \frac{m_2}{r_{12}s^2} + \frac{m_2}{r_2 s^2} \right) + n_{12}^i v_1^j \left(-\frac{m_2}{r_1 r_{12}^2} + \frac{m_2}{r_{12}^2 r_2} - \frac{m_2}{r_2 s^2} \right) + n_2^i v_1^j \frac{m_2}{r_{12} s^2} + n_1^i v_2^j \left(\frac{m_2}{r_{12} s^2} - \frac{m_2}{r_2 s^2} \right) + n_{12}^i v_2^j \left(\frac{2m_2}{r_1 s^2} + \frac{m_2}{r_2 s^2} \right) + n_2^i v_2^j \left(\frac{2m_2}{r_1 s^2} + \frac{m_2}{r_{12} s^2} \right) + n_1^i n_{12}^j \left[(n_{12} v_1) \left(-\frac{m_2}{r_{12} s^2} - \frac{2m_2}{s^3} \right) - 2(n_2 v_{12}) \frac{m_2}{s^3} + (n_1 v_2) \left(\frac{2m_2}{r_1 s^2} + \frac{4m_2}{s^3} \right) + (n_{12} v_2) \left(-\frac{m_2}{r_1 s^2} - \frac{2m_2}{s^3} \right) \right] + n_1^i n_2^j \left[(n_1 v_2) \left(\frac{2m_2}{r_1 s^2} + \frac{4m_2}{s^3} \right) - 2(n_{12} v_1) \frac{m_2}{s^3} + (n_2 v_{12}) \left(-\frac{m_2}{r_2 s^2} - \frac{2m_2}{s^3} \right) - 2(n_{12} v_2) \frac{m_2}{s^3} \right] + n_{12}^i n_2^j \left[(n_{12} v_1) \left(\frac{m_2}{r_{12} s^2} + \frac{2m_2}{s^3} \right) + (n_2 v_{12}) \left(\frac{m_2}{r_2 s^2} + \frac{2m_2}{s^3} \right) + (n_{12} v_2) \left(\frac{m_2}{r_{12} s^2} + \frac{2m_2}{s^3} \right) - 4(n_1 v_2) \frac{m_2}{s^3} \right] \right\} + 1 \leftrightarrow 2 + O(8), \quad (\text{A1i})$$

where $s = r_1 + r_2 + r_{12}$. These potentials, inserted into the post-Newtonian metric (3.9), give the spin-orbit interaction terms in the gravitational field to 2.5PN order away from the bodies.

Using the Hadamard finite part, we find the values of the spin parts of the potentials at body 1 to be

$$(V)_{1(S)} = \frac{1}{r_{12}^2} \left(-S_1^{pq} \frac{m_2}{r_{12}} n_{12}^p v_{12}^q + S_2^{pq} n_{12}^p \left\{ \left[2 - \frac{21m_1}{2r_{12}} + 2v_2^2 - 3(n_{12} v_2)^2 \right] v_2^q + \frac{m_1}{2r_{12}} v_1^q \right\} \right) + O(8), \quad (\text{A2a})$$

$$(V_i)_{1(S)} = \frac{1}{r_{12}^2} \left\{ S_2^{pi} n_{12}^p \left[\frac{1}{2} - \frac{5m_1}{2r_{12}} + \frac{1}{2}(v_2)^2 - \frac{3}{4}(n_{12} v_2)^2 \right] + \frac{1}{2} v_2^i S_2^{pq} n_{12}^p v_2^q + \frac{1}{2} \epsilon^{ipq} n_{12}^p v_2^q (v_2 S_2) \right\} + O(7), \quad (\text{A2b})$$

$$\left(\hat{W}_{ij} \right)_{1(S)} = \frac{1}{r_{12}^2} \left(S_2^{p(i} v_2^{j)} n_{12}^p - \delta^{ij} S_2^{pq} n_{12}^p v_2^q \right) + O(6), \quad (\text{A2c})$$

$$\begin{aligned} \left(\hat{R}_i\right)_{1(S)} &= \frac{1}{r_{12}^2} n_{12}^p \left[S_1^{pi} \frac{m_2}{4r_{12}} + S_2^{pi} \left(\frac{m_1}{r_{12}} - \frac{m_2}{4r_{12}} \right) \right] \\ &+ O(7), \end{aligned} \tag{A2d}$$

$$\begin{aligned} \left(\hat{X}\right)_{1(S)} &= \frac{1}{r_{12}^2} \left\{ S_1^{pq} n_{12}^p \left(v_1^q - \frac{1}{2} v_2^q \right) \frac{m_2}{r_{12}} + S_2^{pq} n_{12}^p \right. \\ &\times \left. \left[\frac{m_1}{4r_{12}} v_1^q + \left(\frac{7m_1}{4r_{12}} - \frac{m_2}{2r_{12}} \right) v_2^q \right] \right\} \\ &+ O(8). \end{aligned} \tag{A2e}$$

APPENDIX B: COMPARISON WITH KERR ORBITAL FREQUENCY

We perform one check of our equations of motion by comparing the orbital frequency (5.22) in the test-mass limit to the orbital frequency of a spinning test particle in a circular orbit around a Kerr black hole derived by Tanaka *et al.* [27]

The transformation from Boyer-Lindquist coordinates $(t_{BL}, r_{BL}, \theta_{BL}, \phi_{BL})$ to harmonic coordinates (t_H, x_H, y_H, z_H) was found by Cook and Scheel [38] to be

$$t_H = t_{BL} + \frac{r_+^2 + a^2}{r_+ - r_-} \ln \left| \frac{r_{BL} - r_+}{r_{BL} - r_-} \right|, \tag{B1a}$$

$$x_H + iy_H = (r_{BL} - M + ia) e^{i\phi} \sin \theta_{BL}, \tag{B1b}$$

$$z_H = (r_{BL} - M) \cos \theta_{BL}, \tag{B1c}$$

where

$$\phi = \phi_{BL} + \frac{a}{r_+ - r_-} \ln \left| \frac{r_{BL} - r_+}{r_{BL} - r_-} \right|, \tag{B2}$$

M and a are the Kerr mass and spin parameters, and $r_{\pm} = M \pm \sqrt{M^2 - a^2}$. The angular frequency in Boyer-Lindquist coordinates is the same as in harmonic coordinates, $\omega = d\phi_{BL}/dt_{BL} = d\phi_H/dt_H$, where $\phi_H = \tan^{-1}(y_H/x_H)$. The transformation of the radial coordinate can be written as

$$r_H = r_{BL} - M + O(a^2), \tag{B3}$$

which is sufficiently accurate for spin-orbit effects.

The orbital angular frequency ω of a test particle in a circular orbit in the Kerr spacetime with spin perpendicular to the plane of the orbit is given by Eq. (4.26) of Ref. [27]. That equation can be written as

$$\omega = \frac{M_{BH}^{1/2}}{r_{BL}^{3/2}} \left[1 - \left(q + \frac{3}{2} \frac{m_p}{M_{BH}} s_{\perp} \right) \left(\frac{M_{BH}}{r_{BL}} \right)^{3/2} + O(6) \right]. \tag{B4}$$

Here $q = a/M$, s_{\perp} is the magnitude of the spin of the particle divided by the square of its mass, m_p is the particle's mass, and M_{BH} is the (much greater) black hole's mass. Note that the effect of the particle's spin s_{\perp} is already $O(m_p/M_{BH})$ smaller than that of the black hole.⁴ The parameter s_{\perp} is explicitly written in terms of the coordinate components of the spin vector as

$$s_{\perp} = \frac{S_p^{\theta}}{m_p^2} \sqrt{r_{BL}^2 + a^2 \cos^2 \theta_{BL}}, \tag{B5}$$

where S_p^{θ} is the θ_{BL} -component of the spin vector of particle. Since we consider the equatorial plane, $\theta = \pi/2$. Using the coordinate transformation (B1), we find that

⁴The coordinate transformation (B3) is modified by the presence of the particle. However the modification appears at $O(m_p/M_{BH})$ in the spinless terms and at $O(m_p/M_{BH})^2$ in the spin terms; thus it does not affect the comparison here.

$$s_{\perp} = \left(1 + \frac{M_{BH}}{r_H}\right) \frac{S_2^z}{m_2^2}, \quad (\text{B6})$$

where S_2^z is the z -component of the spin vector of the particle in harmonic coordinate and $m_2 = m_p$ is the particle's mass.

Using above formulas, we write the angular frequency of a test particle in harmonic coordinates as

$$\begin{aligned} \omega^2 = & \left(\frac{M_{BH}}{r_H^3}\right) \left[1 - 3\left(\frac{M_{BH}}{r_H}\right) - \left(2q + 3\frac{m_2}{M_{BH}}s_2\right)\right. \\ & \times \left(\frac{M_{BH}}{r_H}\right)^{3/2} + 6\left(\frac{M_{BH}}{r_H}\right)^2 + \left(9q + \frac{21}{2}\frac{m_2}{M_{BH}}s_2\right) \\ & \left. \times \left(\frac{M_{BH}}{r_H}\right)^{5/2} + O(4, 6)\right], \end{aligned} \quad (\text{B7})$$

where $s_2 \equiv S_2^z/m_2^2$. By taking our post-Newtonian result (5.22) for ω^2 , setting body 1 to be the Kerr black hole and body 2 to be the particle, taking the test-mass limit

$$\begin{aligned} \eta & \approx \frac{m_2}{M_{BH}}, \\ \Delta & \approx 1 - 2\frac{m_2}{M_{BH}}, \\ \chi_s & = \frac{1}{2}(q + s_2), \\ \chi_a & = \frac{1}{2}(q - s_2), \end{aligned}$$

we obtain the same result. Thus our post-Newtonian calculation agrees (in the test-mass limit) with the previous results of black-hole perturbation theory.

APPENDIX C: LORENTZ INVARIANCE OF THE EQUATIONS OF MOTION

In this appendix, we briefly describe the Lorentz invariance of the equations of motion (5.5). It is well known that the 2PN equations of motion, including spin-orbit and spin-spin terms, are Lorentz invariant [21]. Therefore we concentrate on the new 2.5PN spin-orbit terms.

Consider the case where the coordinates $(t', x^{i'})$ are moving relative to (t, x^i) with constant velocity U^i . The Lorentz transform of a vector A is defined as

$$A^\mu = \Lambda^\mu{}_{\nu'} A^{\nu'}, \quad (\text{C1})$$

where

$$\Lambda^0{}_0 = \gamma, \quad (\text{C2a})$$

$$\Lambda^0{}_{i'} = \Lambda^i{}_{0'} = \gamma U^i, \quad (\text{C2b})$$

$$\Lambda^i{}_{j'} = \delta^{ij} + \frac{U^i U^j}{U^2} (\gamma - 1), \quad (\text{C2c})$$

and $\gamma = 1/\sqrt{1 - U^2}$. We expand the Lorentz transform (C1) in powers of $U = |U^i|$, assuming that U is of the same order as v_A so that the slow-motion assumption underlying the post-Newtonian expansion is still valid.

We also account for the fact that the time slices in the primed coordinates intersect the bodies' world lines at different points than in the unprimed coordinates. Without loss of generality, we can take the time slice in the unprimed coordinates to be $t = 0$ and the primed time slice to be $t' = 0$. Then it follows from the inverse Lorentz transformation that quantities evaluated at body A at $t = 0$ are evaluated at

$$t' = t'_A = y_A^{0'}(t = 0) = -U^k y_A^k(t = 0). \quad (\text{C3})$$

When transforming the equations of motion to the primed coordinates, we must evaluate the primed vectors at $t' = 0$ rather than $t' = t'_A$. Therefore we expand the transformed equations of motion in powers of the small quantity t'_A .

Specifically, the position y_1^i of body 1 on the time slice $t = 0$ is related to the position $y_1^{i'}$ of body 1 on the slice $t' = 0$ by

$$y_1^i \simeq y_1^{i'} - \frac{1}{2}U^i U^j y_1^{j'} - v_1^{i'} U^j y_1^{j'}, \quad (\text{C4})$$

where $v_1^{i'} = dy_1^{i'}/dt'$. Here and below, (unprimed) vectors on the left-hand sides of equations are evaluated at $t = 0$ while (primed) vectors on right-hand sides are evaluated at $t' = 0$. By transforming the four-velocity and four-acceleration, we find that

$$v_1^i \simeq U^i \left[1 - \frac{1}{2}(Uv_1') \right] + v_1^{i'} \left[1 - \frac{1}{2}U^2 - (Uv_1') \right] + t_1' \frac{dv_1^{i'}}{dt'}, \quad (\text{C5})$$

$$\begin{aligned} \frac{dv_1^i}{dt} \simeq & [1 - U^2 - 2(Uv_1')] \frac{dv_1^{i'}}{dt'} + t_1' \frac{d^2 v_1^{i'}}{dt'^2} - v_1^{i'} U^k \frac{dv_1^{k'}}{dt'} \\ & - \frac{1}{2}U^i U^k \frac{dv_1^{k'}}{dt'}, \end{aligned} \quad (\text{C6})$$

where $r_{12}' = |\mathbf{y}_1' - \mathbf{y}_2'|$, $n_{12}^{i'} = (y_1^{i'} - y_2^{i'})/r_{12}'$, and $t_1' = \Lambda_\nu^0(-U)y_1^\nu$. In the same way, by considering the Lorentz transform and time slicing of the spin variables and the spin supplementary condition, we find that

$$S_1^i \simeq S_1^{i'} + \frac{1}{2}U^i (US_1') + U^{i'} (S_1' v_1'), \quad (\text{C7})$$

$$\begin{aligned} S_1^{ij} \simeq & S_1^{i'j'} \left[1 - \frac{1}{2}U^2 - (v_1' U) \right] + \varepsilon_{ijk} v_1^{k'} (S_1' U) \\ & + \varepsilon_{ijk} U^k [(v_1' S_1') + (US_1')] - U^{[i'} S_1^{j']k'} U^k \\ & - 2U^{[i} S_1^{j']k'} v_1^{k'}. \end{aligned} \quad (\text{C8})$$

The transformations of the variables of body 2 are given by exchanging subscripts 1 and 2 in the above formulas.

We insert these formulas into the equations of motion (5.5)–(5.10), keeping the lowest-order acceleration on the right hand side of Eq. (C6) but eliminating other (post-Newtonian) accelerations with the 1.5PN equations of motion in the primed coordinates. Thus we obtain equations of motion in the primed coordinates which, to 2.5PN order, have the same form as Eq. (5.10). This completes the check of the Lorentz invariance of our 2.5PN equations of motion.

- [1] K. S. Thorne, in *Black Holes and Relativistic Stars*, edited by R. M. Wald (University of Chicago Press, Chicago, 1998), p. 41.
- [2] C. Cutler *et al.*, Phys. Rev. Lett. **70**, 2984 (1993).
- [3] É. É. Flanagan and S. A. Hughes, Phys. Rev. D **57**, 4535 (1998); **57**, 4566 (1998).
- [4] K. Alvi, Phys. Rev. D **61**, 124013 (2000).
- [5] L. Blanchet and T. Damour, Philos. Trans. R. Soc. London **A320**, 379(1986); Phys. Rev. D **37**, 1410 (1988); Ann. Inst. Henri Poincaré **50**, 377 (1989); Phys. Rev. D **46**, 4304 (1992).
- [6] T. Damour and B.R. Iyer, Ann. Inst. Henri Poincaré **54**, 115(1991).
- [7] L. Blanchet, Phys. Rev. D **54**, 1417 (1996). See also L. Blanchet, Phys. Rev. D **51**, 2559 (1995); L. Blanchet, T. Damour and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995).
- [8] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996).
- [9] R. Epstein and R. V. Wagoner, Astrophys. J. **197**, 717 (1975); R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976).
- [10] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. **51**, 1220 (1974); **51**, 1598 (1974).
- [11] T. Damour, C. R. Acad. Sci. **294**, Ser. B, 1355 (1982).
- [12] L. P. Grishchuk and S. M. Kopejkin, Sov. Astron. Lett. **9**, 230 (1983); S. M. Kopejkin, Sov. Astron. **29**, 516 (1985).
- [13] G. Schäfer, Ann. Phys. (NY) **161**, 81 (1985); Gen. Rel. Grav. **18**, 255 (1985).

- [14] P. Jaranowski and G. Schäfer, *Phys. Rev. D* **57**, 7242 (1998); *Phys. Rev. D* **60**, 124003 (1999). T. Damour, P. Jaranowski and G. Schäfer, *Phys. Rev. D* **62**, 044024 (2000).
- [15] L. Blanchet and G. Faye, gr-qc/0007051.
- [16] A. Papapetrou, *Proc. R. Soc. (London) Ser. A* **209**, 248 (1951).
- [17] B. M. Barker and R. F. O'Connell, *Gen. Rel. Grav.* **11**, 149 (1979).
- [18] P. D. D'Eath, *Phys. Rev. D* **12**, 2183 (1975).
- [19] J. Ibañez and J. Martin, *Gen. Rel. Grav.* **14**, 439 (1982).
- [20] K. S. Thorne and J. B. Hartle, *Phys. Rev. D* **31**, 1815 (1985).
- [21] For a review of the post-Newtonian equations of motion, see e.g. T. Damour, in *300 Years of Gravitation*, edited by S. W. Hawking and W. Israel, (Cambridge Univ. Press, Cambridge, England, 1987), p. 128.
- [22] C. M. Will, in *Relativistic Cosmology*, edited by M. Sasaki (Universal Academy Press, Tokyo, 1994), p. 83.
- [23] L. E. Kidder, C. M. Will, and A. G. Wiseman, *Phys. Rev. D* **47**, R4183 (1993).
- [24] L. E. Kidder, *Phys. Rev. D* **52**, 821 (1995).
- [25] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, *Phys. Rev. D* **49**, 6274 (1994).
- [26] Y. Mino, M. Shibata, and T. Tanaka, *Phys. Rev. D* **53**, 622 (1996).
- [27] T. Tanaka, Y. Mino, M. Sasaki, and M. Shibata, *Phys. Rev. D* **54**, 3762 (1996).
- [28] W. G. Dixon, in *Isolated Gravitating Systems in General Relativity*, edited by J. Ehlers (North-Holland, Amsterdam, 1979), p. 156.
- [29] H. T. Cho, *Class. Quantum Grav.* **15**, 2465 (1998).
- [30] I. Bailey and W. Israel, *Ann. Phys.* **130**, 188 (1980).
- [31] B. J. Owen, H. Tagoshi, and A. Ohashi, *Phys. Rev. D* **57**, 6168 (1998).
- [32] L. Blanchet, G. Faye, and B. Ponsot, *Phys. Rev. D* **58**, 124002 (1998).
- [33] L. Bel, T. Damour, N. Deruelle, J. Ibañez, and J. Martin, *Gen. Rel. Grav.* **13**, 963 (1981).
- [34] E. Poisson, *Phys. Rev. D* **57**, 5287 (1998).
- [35] A. Ohashi, Ph.D. Thesis, Kyoto University (unpublished, 1998).
- [36] L. Blanchet and G. Faye, gr-qc/0004008.
- [37] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [38] G. B. Cook and M. A. Scheel, *Phys. Rev. D* **56**, 4775 (1997).