A New Transition between Discrete and Continuous Self-Similarity in Critical Gravitational Collapse

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(Dec. 7 2001)

We analyze a bifurcation phenomenon associated with critical gravitational collapse in a family of self-gravitating SU(2) σ-models. As the dimensionless coupling constant decreases, the critical solution changes from discretely self-similar (DSS) to continuously self-similar (CSS). Numerical results provide evidence for a bifurcation which is analogous to a heteroclinic loop bifurcation in dynamical systems, where two fixed points (CSS) collide with a limit cycle (DSS) in phase space as the coupling constant tends to a critical value.

Gravitational collapse at the threshold of black hole formation ties together many of the fundamental issues of general relativity, such as the global aspects of solutions, the structure of singularities arising from regular initial data and the cosmic censorship hypothesis [1]. By fine-tuning initial data for a gravitating massless scalar field to the boundary between eventual dispersal and complete collapse, Choptuik [2] found phenomena reminiscent of criticality associated with phase transitions in statistical physics, such as universality and scaling (e.g. of the black hole mass). Considerable qualitative understanding has been gained by explaining critical collapse in terms of a single unstable mode of the universal critical solution. This solution is understood as an intermediate attractor located in a codimension-one stable hypersurface in phase space, separating data which do or do not form black holes. Critical collapse has by now been studied in a number of matter models – in all of these the physics of the threshold of black hole formation was found to be governed by symmetry: the critical solution exhibits either continuous or discrete self-similarity, or staticity or periodicity in time [3].

In this paper, we study numerically a simple model with a single scalar field in spherical symmetry which exhibits CSS critical behavior at small coupling constants, and DSS critical behavior at large ones. At intermediate coupling constants we observe a competition between CSS and DSS solutions giving rise to a new phenomenon: within an approximately-DSS critical evolution we find several episodes of approximate CSS. Our main focus here is on the interpretation of this observation in terms of an analogy to a heteroclinic loop bifurcation in finite dimensional dynamical systems, at which the limit cycle (DSS) merges with two fixed points (the CSS solution and its negative). Apart from the bifurcation itself, the model also shows other interesting features, such as the existence of a stable (with respect to linear spherical perturbations) self-similar solution for some finite range of the coupling, and a "suppression" effect for the CSS solution in critical searches which we interpret as "shielding" by an apparent horizon.

Our results are based on the direct numerical construction of the CSS and DSS solutions, a linear perturbation analysis of the CSS solutions, and comparison with critical evolutions. The latter are defined by considering a 1-parameter family of initial data $\phi = \phi_p(u_0, r)$, such that (say) for small values of p the field eventually disperses (as determined by a numerical evolution [4]), while for large values of p it eventually forms a black hole (diagnosed by the appearance of an apparent horizon). We use a binary search in p to numerically approximate the critical solution at the threshold of black hole formation. We refer to such fine-tuned numerical solutions as near-critical evolutions, and our results are taken from initial data which are fine-tuned to the same tolerance $\delta p/p < 10^{-14}$.

The self-gravitating SU(2) σ-model [5] under investigation is a wave map from spacetime to the target manifold S^3 with the standard metric. The so called Hedgehog ansatz of spherical symmetry leaves a single matter field $\phi(u, r)$ coupled to gravity:

$$\Box\phi = \frac{\sin(2\phi)}{r^2}\,, \tag{1}$$

where $\Box$ is the spacetime wave operator. Our geometric setup, numerical evolution scheme, and convergence tests are described in a previous paper [4]. In particular, we use retarded Bondi-like coordinates (u, r) with metric functions $\beta(u, r)$ and $V(u, r)$. Suitable combinations of Einstein's equations lead to

$$\beta' = \frac{\eta}{2} r (\phi')^2, \quad V' = e^{2\beta}(1 - 2\eta \sin^2\phi)\,, \tag{2}$$

where prime denotes the derivatives with respect to r, and η the dimensionless coupling constant. The hypersurface equations (2) and the matter field equation (1) suffice to evolve all the dynamical fields V, β, and ϕ.

For vanishing coupling ($\eta = 0$), the theory describes a σ-model on a fixed background. Taking this background as Minkowski space, Bizoń *et al.* [6] and Liebling *et al.* [7] find a CSS critical solution at the threshold of singularity formation. Bizoń [8] and Bizoń and Wasserman [9] have shown that for each $0 \leq \eta < 0.5$, a countably infinite family of CSS solutions exists, indexed by the number of nodes in $\phi(u = \text{constant}, r) - \pi/2$. In the limit $\eta \to \infty$ Liebling [10] finds DSS critical collapse at the threshold of black-hole formation. In Ref. [4] we find that for $\eta \gtrsim 0.2$ the system shows "exact" DSS critical collapse, but for $0.18 \leq \eta \lesssim 0.2$ we see only approximate DSS behavior; furthermore the period Δ exhibits a sharp rise as the coupling decreases from 0.5 to 0.18 (Fig. 1). These results suggest a transition from CSS to DSS critical collapse somewhere in the range $0 < \eta \lesssim 0.18$, which we identify and discuss in the present paper.

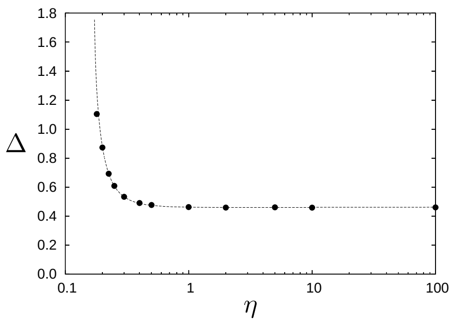

FIG. 1. The DSS echoing period Δ is shown as a function of the coupling constant η, computed from dynamical evolutions (dots) and from direct construction (line). Our dynamical evolution is only approximately DSS at its lowest η value ($\eta = 0.18$), so we can only determine Δ approximately there.

For $0 \leq \eta < 0.5$ we have numerically constructed the CSS solutions discussed in [9] as an ODE-eigenvalue problem. We have also performed a linear stability analysis of the CSS solutions; we find that the first excitation (one node), which we refer to below as "the" CSS solution, has precisely one unstable mode. This CSS solution takes the form $\phi = \pm\phi_{\text{CSS}}(z; u^*_{\text{CSS}})$, where $z = r/(u^*_{\text{CSS}} - u)$, with a single free parameter u^*_{CSS} giving the retarded time of the accumulation point. The sign ambiguity is a consequence of Einstein's equations and the field equation (1) being invariant under $\phi \to -\phi$.

For $\eta \geq 0.1726$ we have also explicitly constructed the "Choptuon" DSS solution via a pseudospectral method following the lines of Gundlach [11] (see [12]). The DSS solution takes the form $\phi = \phi_{\text{DSS}}(\tau, z; u^*) = \phi_{\text{DSS}}(\tau + n\Delta, z; u^*)$, where n is any integer, Δ is the DSS period, z is again given by $r/(u^* - u)$, and $\tau = -\ln(u^* - u)$. As η decreases Δ rises sharply (Fig. 1). Also, a rapidly increasing number of Fourier components is required to accurately represent the Choptuon, and the construction algorithm becomes increasingly ill-conditioned. Below we will give further arguments suggesting that the DSS Choptuon ceases to exist somewhat below the lower limit of our numerical construction.

For very small couplings $\eta \lesssim 0.1$ the stable CSS ground state causes a generic class of initial data to collapse to naked singularities. Here we focus on the transition from CSS to DSS in critical collapse at the threshold of *black hole* formation: We therefore restrict our attention to $\eta \geq 0.1$, where we find only dispersal and black hole (apparent horizon) formation as generic end-states.

For $0.1 \leq \eta \lesssim 0.14$ we find CSS critical collapse, while for large couplings $\eta \gtrsim 0.2$ we have previously found DSS critical collapse [4]. In both ranges we observe scaling of the black hole mass for supercritical initial data and of the maximum central Ricci scalar for subcritical initial data. In the CSS regime $0.1 \leq \eta \lesssim 0.14$ and in the DSS regime for $\eta \gtrsim 0.2$ the critical exponents are approximately constant (within a few percent): $\gamma_{\text{CSS}} \approx 0.18$ and $\gamma_{\text{DSS}} \approx 0.11$.

In the transition regime $0.14 \lesssim \eta \lesssim 0.2$, we find that critical solutions show a new phenomenon which we call "episodic self-similarity": The field configuration closely approximates CSS behavior on large parts of the slice for a finite time, then departs and returns to CSS again. This cycle repeats several times before the evolution either leads to black hole formation or dispersal. We find that $\phi \approx +\phi_{\text{CSS}}$ and $\phi \approx -\phi_{\text{CSS}}$ episodes always alternate. The accumulation times u^*_{CSS} increase from one CSS episode to the next.

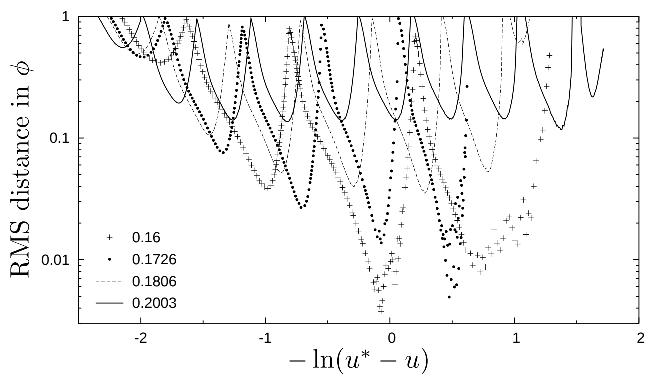

FIG. 2. This figure shows the distance (r^2-weighted RMS difference in ϕ, taken between the origin and the self-similarity horizon, within each near-critical-evolution slice) between the CSS solution and the critical solution, for $\eta = 0.16$, 0.1726, 0.1806, and 0.2003. (The choice of u^*, and thus the horizontal coordinate, is somewhat arbitrary for $\eta = 0.16$.)

In order to study episodic self-similarity quantitatively, we have fitted numerical near-critical evolutions against our explicitly-constructed CSS solutions (fitting the CSS parameter u^*_{CSS} independently at each near-critical-evolution slice). Figure 2 shows these fits for a range of coupling constants. The repeated close approaches of the near-critical evolutions to the CSS solutions are clearly visible; the approaches become closer and closer and the time spent in the neighborhood (i.e. within a given distance) of the CSS solution increases as η is decreased.

In the range where episodic CSS occurs we also observe approximate DSS behavior. It is therefore interesting to compare the near-critical evolution to the explicitly constructed DSS solution.

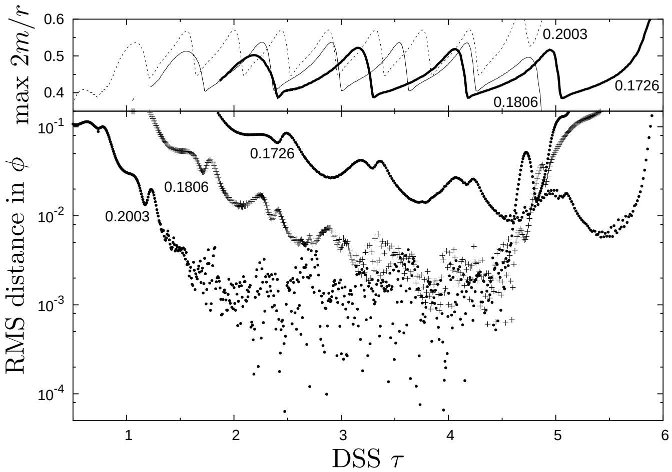

FIG. 3. Best fits between the DSS Choptuon and the critical solution are shown for $\eta = 0.1726$, 0.1806, and 0.2003. The lower subplot shows the distance (same definition as in figure 2) between the DSS Choptuon and the corresponding best-fitting slices of the near-critical evolution, as a function of τ. The upper subplot shows the maximum of $2m/r$ within the same slices of the critical evolution. τ is only defined up to an arbitrary integer multiple of Δ at each coupling constant.

Figure 3 shows fits of numerical near-critical evolutions against the explicitly-constructed DSS solutions (finding best-fitting pairs of slices between the near-critical evolutions and the DSS solutions) for several coupling constants where DSS exists. Notice that as η decreases, the near-critical solution's approach to the DSS solution becomes slower, and the closest approach becomes less close. The time intervals in τ from approaching the Choptuon within an RMS error of ~ 0.1 (which is where the curves in Fig. 3 starts) to the start of the departures are however roughly equal. This and the slow approach account for the (only) approximate DSS behavior of near-critical evolutions already observed in [4] for $0.18 \leq \eta \lesssim 0.2$.

Comparing Figs. 2 and 3 one infers that as the critical evolution is attracted to the DSS solution it comes periodically close to the CSS solution, which implies that the DSS and CSS solutions must themselves be close. Figure 4 shows fits between the CSS solutions and the explicitly-constructed DSS solutions (again fitting the CSS parameter $u^*_{\rm CSS}$ independently at each slice). There are two close approaches within each DSS cycle, corresponding to the two sign choices $\phi = \pm\phi_{\rm CSS}$. Note that the close approaches become closer as η decreases.

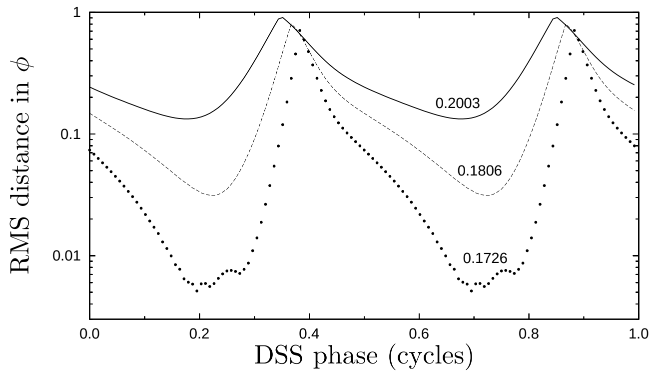

FIG. 4. The distance (same definition as in figure 2) between the CSS and DSS solutions is shown as a function of DSS phase (measured in orbits around the DSS Choptuon), for $\eta = 0.1726$, 0.1806, and 0.2003. The origin of the DSS phase scale is arbitrary at each coupling constant.

Combining our results, we conjecture the following bifurcation scenario in the language of dynamical systems: for large couplings $0.2 \lesssim \eta < 0.5$, the limit cycle representing $\phi_{\rm DSS}$ in phase space and the two fixed points $\pm\phi_{\rm CSS}$ lie far apart (in some suitable norm). As η is decreased, the limit cycle and the CSS points move closer and finally merge at some $\eta_c \approx 0.17$. In this limit the DSS solution becomes a heteroclinic orbit connecting the CSS fixed points and the period of the limit cycle tends to infinity (see below). For $\eta < \eta_c$, DSS ceases to exist.

We conjecture that the DSS solution still plays the role of a critical intermediate attractor even at coupling constants just slightly larger than η_c, where the CSS solutions lie very close to the DSS cycle. An evolution, which is tuned to evolve towards the DSS-CSS region is carried along by the flow of the DSS cycle to periodically come close to the CSS fixed points. We have numerical evidence that the periodical turning away from CSS is dominated by the unstable mode of CSS.

When the DSS evolution is close to one of the CSS fixed points in phase space we can expand the field in terms of linear perturbations around the CSS solution. The departure from CSS must thus happen via the unstable mode of CSS. The amplitude for this mode grows from an initial amplitude A_0 to some fixed amplitude (still in the linear regime) in a time $T = -(1/\lambda)\ln(A_0) + \text{constant}$, where λ denotes the eigenvalue of the unstable mode of the CSS solution. Since this happens twice in a DSS cycle, we can write the total duration of the DSS cycle as $\Delta = 2T + T_{\rm turn}$, where $T_{\rm turn}$ denotes the time spent in the (nonlinear) turnover from one of $\pm\phi_{\rm CSS}$ to the other. From our linear perturbation analysis of the CSS solution we find that $\lambda \approx 5.14$ is only slowly varying for η near η_c [13]. If we assume that $A_0 \sim \eta - \eta_c$ for

η near η_c and that $T_{\rm turn}$ is roughly constant, we have $\Delta = -(2/\lambda)\,\ln(\eta - \eta_c) + \text{const}$. To test this prediction, we have fitted the $\Delta(\eta)$ values shown in Fig. 1 to the 3-parameter functional form $f(\eta) = -a \ln(\eta - \eta_c) + b$ in the range $\eta \in [0.1726, 0.195]$. The fit is very good, with a maximum relative error of 0.3%. We obtain $\eta_c \simeq 0.17$, which is consistent with what we expect from the raise in the number of relevant Fourier coefficients, and $a \sim 2/\lambda$ with a relative error of $\sim 7\%$. Given the fact that we neglected higher order terms and the variation of λ, the fitted value for a is remarkably close to the theoretically predicted one.

Figure 5 gives a schematic overview of all our observations.

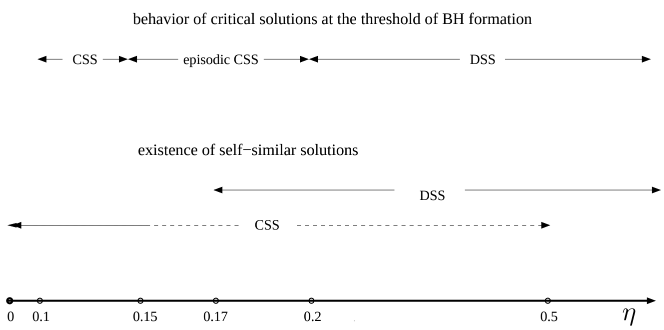

FIG. 5. This figure shows the different phenomenology we observe at various couplings η. The dashed line denotes CSS solutions, that contain marginally trapped surfaces.

We also observe episodic CSS at couplings $0.14 \lesssim \eta < \eta_c$, where we believe the DSS solution ceases to exist. What is tuned out in such a critical search? In the language of dynamical systems the answer could be that below the critical coupling the cycle of DSS is broken i.e. it does not close, but the flow still defines locally an invariant manifold of codimension one. It is this manifold which separates black hole formation from dispersal. The mass scaling from supercritical searches in the intermediate regime $0.14 \lesssim \eta < 0.2$ has not been conclusive so far. Observed deviations from a simple scaling law require further investigation.

Another important question is: Why does the CSS solution cease to be a critical solution for black hole formation for larger couplings? Our stability analysis shows that CSS has a single unstable mode up to $\eta = 0.5$, which in principle could be tuned out in a critical search. We believe that the answer to this is related to the observation by Bizoń and Wasserman [9], that this solution contains a spacelike hypersurface of marginally trapped surfaces outside the backwards light cone of the culmination point for $\eta > 0.152$. We find that numerical evolutions for $\eta = 0.2$ with initial data that are close to the CSS solution inside the backwards lightcone and are asymptotically flat outside, very quickly develop an apparent horizon and thus become a black hole. If this is the generic behavior, then the CSS solution can not lie on the boundary of black hole formation.

Summing up, the SU(2) σ-model shows CSS critical behavior for small and DSS for large values of the coupling constant. In the transition region we observe episodic CSS behavior. We have strong evidence that the CSS/DSS transition is the infinite dimensional analog of a global heteroclinic bifurcation, which is quite different from previously reported bifurcations in self-similar critical collapse, which were found to be characterized by a change of stability (see e.g. [14]). In particular, analogies to finite dimensional dynamical systems pictures have proven essential in interpreting critical collapse (see e.g. Ref. [3] for an overview), and we believe that the bifurcation picture discussed here will stimulate further insights into critical gravitational collapse. It would be interesting to see whether episodic CSS occurs in the critical collapse of different matter models, or also in completely different physical systems. Details of our methods and results, some of which could only be mentioned here will be published in a forthcoming paper.

This work has been supported by the Austrian FWF (project P12754-PHY), the Fundacion Federico, the Alexander von Humboldt Foundation, and G. Rodgers and J. Thorn [J.T.]. We thank Michael Pürrer for his contributions to our numerical evolution code [4], Piotr Bizoń for many useful discussions and for sharing research results in advance of publication, and Carsten Gundlach and José M. Martín-García for stimulating discussions and for drawing our attention to the analysis leading to the logarithmic divergence of Δ.

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