The Einstein-Vlasov system

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Abstract

Rigorous results on solutions of the Einstein-Vlasov system are surveyed. After an introduction to this system of equations and the reasons for studying it, a general discussion of various classes of solutions is given. The emphasis is on presenting important conceptual ideas, while avoiding entering into technical details. Topics covered include spatially homogenous models, static solutions, spherically symmetric collapse and isotropic singularities.

1 Introduction

The basic equations of general relativity are the Einstein equations coupled to some other partial differential equations describing the matter content of space-time. There are many choices of matter model which are of physical interest and Yvonne Choquet-Bruhat has published fundamental results on the Cauchy problem for the Einstein equations coupled to a wide variety of matter models. One of these is collisionless matter described by the Vlasov equation. It is the subject of these lectures.

The Vlasov equation arises in kinetic theory. It gives a statistical description of a collection of particles. It is distinguished from other equations of kinetic theory by the fact that there is no direct interaction between particles. In particular, no collisions are included in the model. Each particle is acted on only by fields which are generated collectively by all particles together. The fields which are taken into account depend on the physical situation being modelled. In plasma physics, where this equation is very important, the interaction is electromagnetic and the fields are described either by the Maxwell equations or, in a quasi-static approximation, by the Poisson equation [30]. In gravitational physics, which is the subject of the following, the fields are described by the Einstein equations or, in the Newtonian approximation, by the Poisson equation. (There is a sign difference in the Poisson equation in comparison with the electromagnetic case due to the replacement of a repulsive by an attractive force.) The best known applications of the Vlasov equation to self-gravitating systems are to stellar dynamics [3]. It can also be applied to cosmology. In the first case the systems considered are galaxies or parts of galaxies where there is not too

much dust or gas which would require a hydrodynamical treatment. Possible applications are to globular clusters, elliptical galaxies and the central bulge of spiral galaxies. The 'particles' in all these cases are stars. In the cosmological case they might be galaxies or even clusters of galaxies. The fact that they are modelled as particles reflects the fact that their internal structure is believed to be irrelevant for the dynamics of the system as a whole. The Vlasov equation is also used in cosmology to model non-baryonic dark matter ([4], p. 323). In that case the 'particles' are elementary particles.

These lectures are concerned not with the above physical applications but with some basic mathematical aspects of the Einstein-Vlasov system. First the definition and general mathematical properties of this system of partial differential equations are discussed and then the Cauchy problem for the system is formulated. The central theme in what follows is the global Cauchy problem, where 'global' means global in time. The known results on this and related problems are surveyed and important methods used are highlighted. Further information on kinetic theory in general relativity may be found in [7].

Let $(M, g_{\alpha\beta})$ be a spacetime, i.e. M is a four-dimensional manifold and $g_{\alpha\beta}$ is a metric of Lorentz signature $(-, +, +, +)$. Note that $g_{\alpha\beta}$ denotes a geometric object here and not the components of the geometric object in a particular coordinate system. In other words the indices are abstract indices. (See [32], section 2.4 for a discussion of this notation.) It is always assumed that the metric is time-orientable, i.e. that the two halves of the light cone at each point of M can be labelled past and future in a way which varies continuously from point to point. With this global direction of time, it is possible to distinguish between future-pointing and past-pointing timelike vectors. The worldline of a particle of non-zero rest mass m is a timelike curve in spacetime. The unit future-pointing tangent vector to this curve is the 4-velocity v^α of the particle. Its 4-momentum p^α is given by mv^α. There are different variants of the Vlasov equation depending on the assumptions made. Here it is assumed that all particles have the same mass m but it would also be possible to allow a continuous range of masses. When all the masses are equal, units can be chosen so that $m = 1$ and no distinction need be made between 4-velocity and 4-momentum. There is also the possibility of considering massless particles, whose wordlines are null curves. In the case $m = 1$ the possible values of the four-momentum are precisely all future-pointing unit timelike vectors. These form a hypersurface P in the tangent bundle TM called the mass shell. The distribution function f, which represents the density of particles with given spacetime position and four-momentum, is a non-negative real-valued function on P. A basic postulate in general relativity is that a free particle travels along a geodesic. Consider a future-directed timelike geodesic parametrized by proper time. Then its tangent vector at any time is future-pointing unit timelike. Thus this geodesic has a natural lift to a curve on P, by taking its position and tangent vector together. This defines a flow on P. Denote the vector field which generates this flow by X. (This vector field is what is sometimes called the geodesic spray in the mathematics literature.) The condition that f represents the distribution of a collection of particles moving freely in the given spacetime is that it should be

constant along the flow, i.e. that $Xf = 0$. This equation is the Vlasov equation, sometimes also known as the Liouville or collisionless Boltzmann equation.

To get an explicit expression for the Vlasov equation, it is necessary to introduce local coordinates on the mass shell. In the following local coordinates x^α on spacetime are always chosen such that the hypersurfaces x^0=const. are spacelike. (Greek and Latin indices take the values $0, 1, 2, 3$ and $1, 2, 3$ respectively.) Intuitively this means that x^0, which may also be denoted by t, is a time coordinate and that the x^a are spatial coordinates. A timelike vector is future-pointing if and only if its zero component in a coordinate system of this type is positive. It is not assumed that the vector $\partial/\partial x^0$ is timelike. One way of defining local coordinates on P is to take the spacetime coordinates x^α together with the spatial components p^a of the four-momentum in these coordinates. Then the explicit form of the Vlasov equation is:

$$\partial f/\partial t + (p^a/p^0)\partial f/\partial x^a - (\Gamma^a_{\beta\gamma}p^\beta p^\gamma/p^0)\partial f/\partial p^a = 0 \qquad (1)$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols associated to the metric $g_{\alpha\beta}$. Here it is understood that p^0 is to be expressed in terms of p^a and the metric using the relation $g_{\alpha\beta}p^\alpha p^\beta = -1$. An alternative form of the Vlasov equation which is often useful is obtained by coordinatizing the mass shell using the components of the four-momentum in an orthonormal frame instead of the coordinate components.

The Vlasov equation can be coupled to the Einstein equations as follows, giving rise to the Einstein-Vlasov system. The unknowns are a 4-manifold M, a (time orientable) Lorentz metric $g_{\alpha\beta}$ on M and a non-negative real-valued function f on the mass shell defined by $g_{\alpha\beta}$. The field equations consist of the Vlasov equation defined by the metric $g_{\alpha\beta}$ for f and the Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. (Units are chosen here so that the speed of light and the gravitational constant both have the numerical value unity.) To obtain a complete system of equations it remains to define $T_{\alpha\beta}$ in terms of f and $g_{\alpha\beta}$. It is defined as an integral over the part of the mass shell over a given spacetime point with respect to a measure which will now be defined. The metric at a given point of spacetime defines in a tautological way a metric on the tangent space at that point. The part of the mass shell over that point is a submanifold of the tangent space and as such has an induced metric, which is Riemannian. The associated measure is the one which we are seeking. It is evidently invariant under Lorentz transformations of the tangent space, a fact which may be used to simplify computations in concrete situations. In the coordinates (x^α, p^a) on P the explicit form of the energy-momentum tensor is:

$$T_{\alpha\beta} = -\int f p_\alpha p_\beta |g|^{1/2}/p_0 \, dp^1 dp^2 dp^3 \qquad (2)$$

A simple computation in normal coordinates based at a given point shows that $T_{\alpha\beta}$ defined by (2) is divergence-free, independently of the Einstein equations being satisfied. This is of course a necessary compatibility condition in order for the Einstein-Vlasov system to be a reasonable set of equations. Another

important quantity is the particle current density, defined by:

$$N^\alpha = -\int fp^\alpha |g|^{1/2}/p_0 dp^1 dp^2 dp^3 \tag{3}$$

A computation in normal coordinates shows that $\nabla_\alpha N^\alpha = 0$. This equation is an expression of the conservation of the number of particles. There are some inequalities which follow immediately from the definitions (2) and (3). Firstly $N_\alpha V^\alpha \leq 0$ for any future-pointing timelike or null vector V^α, with equality only if $f = 0$ at the given point. Hence unless there are no particles at some point, the vector N^α is future-pointing timelike. Next, if V^α and W^α are any two future-pointing timelike vectors then $T_{\alpha\beta}V^\alpha W^\beta \geq 0$. This is the dominant energy condition ([9], p. 91). Finally, if X^α is a spacelike vector then $T_{\alpha\beta}X^\alpha X^\beta \geq 0$. This is the non-negative pressures condition. This condition, the dominant energy condition and the Einstein equations together imply that the Ricci tensor satisfies the inequality $R_{\alpha\beta}V^\alpha V^\beta \geq 0$ for any timelike vector V^α. The last inequality is called the strong energy condition. These inequalities constitute one of the reasons which mean that the Vlasov equation defines a well-behaved matter model in general relativity. However this is not the only reason. A perfect fluid with a reasonable equation of state or matter described by the Boltzmann equation also have energy-momentum tensors which satisfy these inequalities.

The Vlasov equation in a fixed spacetime is a linear hyperbolic equation for a scalar function and hence solving it is equivalent to solving the equations for its characteristics. In coordinate components these are:

$$dX^a/ds = P^a, \quad dP^a/ds = -\Gamma^a_{\beta\gamma}P^\beta P^\gamma \tag{4}$$

Let $X^a(s, x^\alpha, p^a)$, $P^a(s, x^\alpha, p^a)$ be the unique solution of (4) with initial conditions $X^a(t, x^\alpha, p^a) = x^a$ and $P^a(t, x^\alpha, p^a) = p^a$. Then the solution of the Vlasov equation can be written as:

$$f(x^\alpha, p^a) = f_0(X^a(0, x^\alpha, p^a), P^a(0, x^\alpha, p^a)) \tag{5}$$

where f_0 is the restriction of f to the hypersurface $t = 0$. This function f_0 serves as initial datum for the Vlasov equation. It follows immediately from this that if f_0 is bounded by some constant C, the same is true of f. This obvious but important property of the solutions of the Vlasov equation is used frequently without comment in the study of this equation.

The above calculations involving $T_{\alpha\beta}$ and N^α were only formal. In order that they have a precise meaning it is necessary to impose some fall-off in the momentum variables on f so that the integrals occurring exist. The simplest condition to impose is that f has compact support for each fixed t. This property holds if the initial datum f_0 has compact support and if each hypersurface $t = t_0$ is a Cauchy hypersurface. For by the definition of a Cauchy hypersurface, each timelike curve which starts at $t = 0$ hits the hypersurface $t = t_0$ at a unique point. Hence the geodesic flow defines a continuous mapping from the part of the

mass shell over the initial hypersurface $t = 0$ to the part over the hypersurface $t = t_0$. The support of $f(t_0)$, the restriction of f to the hypersurface $t = t_0$ is the image of the support of f_0 under this continuous mapping and so is compact. Let $P(t)$ be the supremum of the values of $|p^a|$ attained on the support of $f(t)$. It turns out that in many cases controlling the solution of the Vlasov equation coupled to some field equation in the case of compactly supported initial data for the distribution function can be reduced to obtaining a bound for $P(t)$. An example of this is given below.

The data in the Cauchy problem for the Einstein equations coupled to any matter source consist of the induced metric g_{ab} on the initial hypersurface, the second fundamental form k_{ab} of this hypersurface and some matter data. In fact these objects should be thought of as objects on an abstract 3-dimensional manifold S. Thus the data consist of a Riemannian metric g_{ab}, a symmetric tensor k_{ab} and appropriate matter data, all defined intrinsically on S. The nature of the initial data for the matter will now be examined in the case of the Einstein-Vlasov system. It is not quite obvious what to do, since the distribution function f is defined on the mass shell and so the obvious choice of initial data, namely the restriction of f to the initial hypersurface, is not appropriate. For it is defined on the part of the mass shell over the initial hypersurface and this is not intrinsic to S. This difficulty can be overcome as follows. Let ϕ be the mapping which sends a point of the mass shell over the initial hypersurface to its orthogonal projection onto the tangent space to the initial hypersurface. The map ϕ is a diffeomorphism. The abstract initial datum f_0 for f is taken to be a function on the tangent bundle of S. The initial condition imposed is that the restriction of f to the part of the mass shell over the initial hypersurface should be equal to f_0 composed with ϕ. An initial data set for the Einstein equations must satisfy the constraints and in order that the definition of an abstract initial data set for the Einstein equations be adequate it is necessary that the constraints be expressible purely in terms of the abstract initial data. The constraint equations are:

$$R - k_{ab}k^{ab} + (\mathrm{tr}k)^2 \;\; = 16\pi\rho \tag{6}$$
$$\nabla_a k^a_b - \nabla_b(\mathrm{tr}k) \;\; = 8\pi j_b \tag{7}$$

Here R denotes the scalar curvature of the metric g_{ab}. If n^α denotes the future-pointing unit normal vector to the initial hypersurface and $h^{\alpha\beta} = g^{\alpha\beta} + n^\alpha n^\beta$ is the orthogonal projection onto the tangent space to the initial hypersurface then $\rho = T_{\alpha\beta}n^\alpha n^\beta$ and $j^\alpha = -h^{\alpha\beta}T_{\beta\gamma}n^\gamma$. The vector j^α satisfies $j^\alpha n_\alpha = 0$ and so can be naturally identified with a vector intrinsic to the initial hypersurface, denoted here by j^a. What needs to be done is to express ρ and j_a in terms of the intrinsic initial data. They are given by the following expressions:

$$\rho \;\; = \int f_0(p^a)p^a p_a/(1 + p^a p_a)^{1/2}(^{(3)}g)^{1/2}dp^1 dp^2 dp^3 \tag{8}$$
$$j_a \;\;\;\;\;\; = \int f_0(p^a)p_a(^{(3)}g)^{1/2}dp^1 dp^2 dp^3 \tag{9}$$

If a three-dimensional manifold on which an initial data set for the Einstein-Vlasov system is defined is mapped into a spacetime by an embedding ψ then

the embedding is said to induce the given initial data on S if the induced metric and second fundamental form of $\psi(S)$ coincide with the results of transporting g_{ab} and k_{ab} with ψ and the relation $f = f_0 \circ \phi$ holds, as above. A form of the local existence and uniqueness theorem can now be stated. This will only be done for the case of smooth (i.e. infinitely differentiable) initial data although versions of the theorem exist for data of finite differentiability.

Theorem 1.1 Let S be a 3-dimensional manifold, g_{ab} a smooth Riemannian metric on S, k_{ab} a smooth symmetric tensor on S and f_0 a smooth non-negative function of compact support on the tangent bundle TS of S. Suppose further that these objects satisfy the constraint equations. Then there exists a smooth spacetime $(M, g_{\alpha\beta})$, a smooth distribution function f on the mass shell of this spacetime and a smooth embedding ψ of S into M which induces the given initial data on S such that $g_{\alpha\beta}$ and f satisfy the Einstein-Vlasov system and $\psi(S)$ is a Cauchy hypersurface. Moreover, given any other spacetime $(M', g'_{\alpha\beta})$, distribution function f' and embedding ψ' satisfying these conditions, there exists a diffeomorphism χ from an open neighbourhood of $\psi(S)$ in M to an open neighbourhood of $\psi'(S)$ in M' which satisfies $\chi \circ \psi = \psi'$ and carries $g_{\alpha\beta}$ and f to $g'_{\alpha\beta}$ and f' respectively.

The formal statement of this theorem is rather complicated, but its essential meaning is as follows. Given an initial data set (satisfying the constraints) there exists a corresponding solution of the Einstein-Vlasov system and this solution is locally unique up to diffeomorphism. There also exists a global uniqueness statement which uses the notion of the maximal Cauchy development of an initial data set, but this is not required in the following. The first proof of a theorem of the above kind for the Einstein-Vlasov system was given by Yvonne Choquet-Bruhat in [5].

The problem of extending this local theorem to one which is in some sense global is a very difficult one. In fact with presently available mathematical techniques it is too difficult. One way of making some progress in understanding the general problem is to study the simplified cases obtained by imposing various symmetries on the solutions. Note that if a symmetry is imposed on the initial data for the Cauchy problem this is inherited by the corresponding solutions. (See [8], section 5.6 for a discussion of this.) This ensures the consistency of restricting the problem to a particular symmetry class.

In the following different symmetry classes will be considered in turn, proceeding from the strongest to the weakest assumptions. First spatially homogeneous solutions are discussed. These are simple enough that it is possible to make statements about the Einstein equations coupled to a general class of matter models. After this has been done, further results which can be obtained in the particular case where the matter is described by the Vlasov equation are presented. The first inhomogeneous solutions to be discussed are those which are static and spherically symmetric. Apart from their intrinsic interest these are of relevance for the study of spherically symmetric collapse which is discussed next. Brief comments are made on dynamical cosmological solutions

before concentrating on one question where there are results on solutions of the Einstein-Vlasov system without any symmetry assumptions being required. This concerns the construction of solutions with an isotropic singularity.

2 Spatially homogeneous solutions I: general matter models

A solution of the Einstein equations coupled to some matter equations is said to be symmetric under the action of a Lie group G if G acts by isometries of the metric which also leave the matter fields invariant. A solution of the Einstein-matter equations is called spatially homogeneous if it is symmetric under the action of a Lie group whose orbits are spacelike hypersurfaces. If we think of these as hypersurfaces of constant time then the metric only depends on time and the Einstein equations reduce to ordinary differential equations, an enormous simplification. The equations of motion of matter fields which are defined on spacetime also reduce to ODE's. Since the Vlasov equation is defined on the mass shell it in general still contains derivatives with respect to the momenta and thus remains a partial differential equation in the spatially homogeneous case.

The spatially homogeneous spacetimes can be classified into various types according to the Lie group involved. The conventional terminology is that there are nine Bianchi types I-IX and one additional type, the Kantowski-Sachs models. The latter will not be discussed further here. We might like to use spatially homogeneous spacetimes as cosmological models. Since there is in a sense no spatial dependence we do need to worry about spatial boundary conditions. Eventually, however, we would like to consider more realistic cosmological models which are inhomogeneous perturbations of the Bianchi spacetimes, and then boundary conditions become important. One simple condition to pose is that the spacetimes involved should contain a compact Cauchy surface. Then there is no danger of extra information coming in from infinity. Supposing that it is desired to impose this condition of spatial compactness the question arises if all Bianchi types are compatible with it. Unfortunately this is not the case. In fact the only ones which are are types I and IX.

A larger class of spatially compact spacetimes where the Einstein equations reduce to ODE's are the locally spatially homogeneous ones, as defined in [22]. The idea is to require that the spacetime itself be spatially compact while its universal cover is spatially homogeneous. For details see [22]. This allows a much bigger class of Bianchi types to be included.

The idea now is to consider solutions of the Einstein-matter equations which are spatially compact and locally spatially homogeneous only assuming some general conditions on the matter model. These conditions are satisfied in the case of the Vlasov equation but also, for example, in the case of the Euler equation describing a perfect fluid with a physically reasonable equation of state. This generality is a luxury we can only afford due to the assumption of (local)

spatial homogeneity. In general solutions of the Euler equations can be expected to form shocks which leads to a breakdown of the solution of the evolution equations. In the homogeneous case this possibility does not arise. It is the absence of shock formation which makes the Vlasov equation particularly convenient to work with when studying inhomogeneous spacetimes.

The matter models to be considered will be defined in terms of some general properties. As usual $T^{\alpha\beta}$ denotes the energy-momentum tensor. When a specific matter model has been chosen $T^{\alpha\beta}$ will be a functional of some matter variables, denoted collectively by F, and the spacetime metric $g_{\alpha\beta}$. In the following another quantity N^{α} (called the particle current density) will be required. It is also assumed to be a functional of F and $g_{\alpha\beta}$. Now various properties which will be assumed at appropriate points will be listed.

(1) $T^{\alpha\beta}V_{\alpha}W_{\beta} \geq 0$ for all future-pointing timelike vectors V^{α} and W^{α} (dominant energy condition)

(2) $T^{\alpha\beta}(g_{\alpha\beta} + V_{\alpha}V_{\beta}) \geq 0$ for all unit timelike vectors V^{α} (non-negative sum pressures condition)

(3) for any F and $g_{\alpha\beta}$ the conditions $\nabla_{\alpha}N^{\alpha} = 0$ and $\nabla_{\alpha}T^{\alpha\beta} = 0$ are satisfied (conservation conditions)

(4) for any F and $g_{\alpha\beta}$ the vector N^{α} is future-pointing timelike or zero

(5) for any constant $C_1 > 0$ there exists a positive constant C_2 such that for any F and $g_{\alpha\beta}$ with $-N_{\alpha}N^{\alpha} \leq C_1$ and any timelike vector V^{α} the following inequality holds:
$$T^{\alpha\beta}V_{\alpha}V_{\beta} \geq C_2(N^{\alpha}V_{\alpha})^2$$

(6) for any constant $C_1 > 0$ there exists a positive constant $C_2 < 1$ such that for any F and $g_{\alpha\beta}$ with $-N^{\alpha}N_{\alpha} \leq C_1$ and any unit timelike vector V^{α}

$$(g_{\alpha\beta} + V_{\alpha}V_{\beta})T^{\alpha\beta} \leq 3C_2 T^{\alpha\beta}V_{\alpha}V_{\beta}$$

(7) if a solution with the given symmetry of the Einstein equations coupled to the given matter model is such that the time coordinate defined above takes all values in the interval (t_1, t_2) , if it is not possible to extend the spacetime so as to make this interval longer and if t_1 or t_2 is finite then $\mathrm{tr}k(t)$ is unbounded in a neighbourhood of t_1 or t_2 respectively.

(8) for any constant $C_1 > 0$ there exists a constant $C_2 > 0$ such that $T_{\alpha\beta}T^{\alpha\beta} \leq C_1$ implies $-N_{\alpha}N^{\alpha} \leq C_2$

Some comments will now be made concerning the physical motivation of some of these conditions. If a given type of matter can be considered as being made up of particles then a particle current density N^{α} is defined. If the particles have positive rest mass then this vector is future pointing timelike or zero as

required by condition (4). If the particles are massless then this condition is still satisfied except for very special types of matter where N^α might be null. If particles cannot be created or destroyed then N^α is divergenceless as required in condition (3). It is not easy to give an intuitive interpretation of conditions (5) and (6). The meaning of (5) is roughly as follows. If matter is observed from a boosted frame then the particle density is multiplied by a γ-factor arising from the effect of Lorentz contraction on the volume element. The observed energy density is also affected in this way but picks up an additional γ-factor. Hence when a given matter distribution is considered from a boosted frame the multiplicative factor in the observed energy density behaves like the square of that in the observed particle density. As for condition (6), for a perfect fluid it is related to the condition that the speed of sound should be bounded away from the speed of light. The given symmetry referred to in condition (7) will be the Bianchi symmetry being considered.

In a spatially compact spacetime an interesting quantity is the volume of the hypersurfaces of constant time. The Friedman-Robertson-Walker (FRW) models generally used in cosmology have the property that this volume either increases at all times (open models) or increases to maximum after which it decreases again (closed models). The possibility that it is always decreases, a case which is allowed mathematically, is usually ignored since we know that our universe is expanding at the present time. In any case these models can be obtained from those where the volume is always increasing by reversing the direction of time. Thus they present no essentially new mathematical phenomena.

In local spatially homogeneous spatially compact spacetimes with reasonable matter the same pattern is found. It can be proved that they also share some other significant physical properties with the FRW models. These spacetimes can be parametrized by a Gaussian time coordinate based on a (locally) homogeneous hypersurface. Suppose that the solution is maximal in the sense that it cannot be extended to a larger interval of Gaussian time. Then we would like to know two things. Firstly, if the volume is always increasing then the time of existence in the future is infinite and the spacetime is future geodesically complete. Secondly, if the volume is increasing at some time then the time of existence in the past is finite and as the time of breakdown is approached some geometrical invariant of the spacetime geometry diverges. This rules out the possibility of extending the spacetime in some way which is not globally hyperbolic. This is desirable form the point of view of the strong cosmic censorship hypothesis.

In [22] theorems of the desired type were proved. The first says that if conditions (1), (2) and (7) above are satisfied by some matter model then inextendible locally spatially homogeneous spatially compact solutions of the Einstein-matter equations where the volume is always increasing satisfy the first conclusion mentioned above. In particular, they are future geodesically complete. The second says that if (1)-(8) are satisfied and the spacetime is not vacuum then the curvature invariant $G^{\alpha\beta}G_{\alpha\beta}$ diverges as the finite past limit of the domain of existence is approached. The fundamental intuitive reason for this is that a finite amount of matter is being squeezed to zero volume as the singularity is approached.

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These theorems apply to the Einstein-Vlasov system since in that case, as will now be discussed, conditions (1)-(8) are satisfied. In that case $T^{\alpha\beta}$ and N^{α} have been defined above. It has also been stated that (1) and (3) hold. Condition (2) is a consequence of the non-negative pressures condition mentioned above. To check condition (4) it is merely necessary to observe that if V^{α} is a future-pointing timelike vector then $N_{\alpha}V^{\alpha} < 0$ unless $N^{\alpha} = 0$ at a given spacetime point. Condition (7) is an existence theorem which says, roughly speaking, that as long as the geometry does not break down too badly, the solution of the matter equations (in this case the Vlasov equation) cannot break down in finite time. Here the mean curvature $\mathrm{tr}k$ plays the role of a controlling quantity whose boundedness ensures the continued existence of the solution. It remains to check conditions (5), (6) and (8) and to do this we may choose a frame whose timelike member is V^{α} in order to do the calculation. Then the inequalities of (5) and (6) become $\hat{T}^{00} \geq C_2(\hat{N}^0)^2$ and $\delta^{ij}\hat{T}_{ij} \leq 3C_2\hat{T}^{00}$. (The hats here indicate the use of indices associated to an orthonormal frame.)

$$
\begin{aligned}
- N_{\alpha}N^{\alpha} &= - \left(\int f(p^a)p_{\alpha}/p^0 dp^1 dp^2 dp^3\right)\left(\int f(q^a)q^{\alpha}/q^0 dq^1 dq^2 dq^3\right) \\
&= - \int \int f(p^a)f(q^a)p^{\alpha}q_{\alpha}/(p^0 q^0)dp^1 dp^2 dp^3 dq^1 dq^2 dq^3 \\
&\geq \int \int f(p^a)f(q^a)/(p^0 q^0)dp^1 dp^2 dp^3 dq^1 dq^2 dq^3 \\
&= \left(\int f(p^a)/p^0 dp^1 dp^2 dp^3\right)^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{N}^0 &= \int f(p^a)dp^1 dp^2 dp^3 \\
&\leq \left(\int f(p^a)p^0 dp^1 dp^2 dp^3\right)^{1/2}\left(\int f(p^a)/p^0 dp^1 dp^2 dp^3\right)^{1/2} \\
&\leq (\hat{T}^{00})^{1/2}(-N_{\alpha}N^{\alpha})^{1/4}
\end{aligned}
$$

This shows that (5) holds. It follows directly from the definitions that the inequality of (6) holds with $C_2 = 1/3$ even without restricting $N_{\alpha}N^{\alpha}$ to be bounded. Finally

$$
\begin{aligned}
T_{\alpha\beta}T^{\alpha\beta} &= \left(\int f(p^a)p_{\alpha}p_{\beta}/p^0 dp^1 dp^2 dp^3\right)\left(\int f(q^a)q^{\alpha}q^{\beta}/q^0 dq^1 dq^2 dq^3\right) \\
&= \int \int f(p^a)f(q^a)(p^{\alpha}q_{\alpha})^2/(p^0 q^0)dp^1 dp^2 dp^3 dq^1 dq^2 dq^3 \\
&\geq - \int \int f(p^a)f(q^a)(p^{\alpha}q_{\alpha})/(p^0 q^0)dp^1 dp^2 dp^3 dq^1 dq^2 dq^3 \\
&= -N_{\alpha}N^{\alpha}
\end{aligned}
$$

Thus the general theorems apply to give results on the dynamics of locally spatially homogeneous solutions of the Einstein-Vlasov system. It is known that Bianchi type IX solutions cannot expand forever, while models of the other types force the volume to be monotone. Thus we can say the following about inextendible non-vacuum spatially compact solutions of the Einstein-Vlasov system with Bianchi symmetry. If the Bianchi type is IX they have curvature singularities after finite proper time both in the past and in the future. For all other Bianchi types models which are expanding at some time have a curvature singularity at a finite time in the past and are future geodesically complete. The

statement about geodesic completeness also holds in the vacuum case. The statement about curvature singularities, however, does not. In some cases there is a Cauchy horizon. This has been discussed in detail in [6]. See also [28].

3 Spatially homogeneous solutions II: application of dynamical systems

We have now obtained a crude picture of the dynamics of spatially homogeneous cosmological models of the Einstein-matter equations for a variety of matter models. It is reasonable to hope that in the case of the Vlasov equation this can be considerably refined, in order to get a detailed picture of the asymptotics of the models near an initial singularity or in a phase of unlimited expansion. This has not yet been achieved in general but for certain cases results were obtained in [27]. These extended theorems concerning the simpler case of massless particles obtained in [26]. The models considered were of the simplest Bianchi types I, II and III. It was assumed that a further symmetry is present so that the spacetimes have a total of four Killing vectors. These are the so-called LRS models (locally rotationally symmetric). The reason for making this assumption is that then the Vlasov equation can be solved explicitly and only the Einstein equations remain to be handled. The equations can be reduced to a system of ODE's in contrast to the general case, although the coefficients of the system involve one function which is not known explicitly. Fortunately it suffices to know certain qualitative features of this function in order to determine the asymptotic behaviour of the solutions of the ODE's at early and late times. In fact to do this analysis it was also necessary to assume invariance under certain reflections, a piece of information which will be suppressed in the following for simplicity.

To describe the results it is useful to introduce the generalized Kasner exponents p_i. Each of the homogeneous hypersurfaces has an induced metric g_{ab} and a second fundamental form k_{ab}. Let λ_i denote the eigenvalues of the second fundamental form with respect to the metric. By definition this means that they are the solutions of the eigenvalue equation $\det(k_{ab} - \lambda g_{ab}) = 0$. Suppose now that $\mathrm{tr}k$ is non-zero. Then we can define $p_i = \lambda_i/(\sum_j \lambda_j)$. The quantities p_i are functions of t and their sum is equal to one. The Kasner solution of the vacuum Einstein equations is given by

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{10}$$

for the Kasner exponents p_i, which are constants satisfying $\sum_j p_j = 1$ and $\sum_j p_j^2 = 1$. These equations are called the first and second Kasner relations. In this case the generalized Kasner exponents are equal to the quantities p_i in this metric form and this explains their name.

Consider a solution of the Einstein-Vlasov system of Bianchi type I which is LRS. Choose the time of the initial singularity to be $t = 0$. The following statements hold. For each i the quantity $p_i(t)$ converges to $1/3$ as $t \to \infty$. This is the value of the generalized Kasner exponents in a spatially flat FRW model.

This means that the spacetime isotropizes at late times. At early times there are three possibilities. The first is that the p_i are identically equal to 1/3 at all times. This is the FRW case. The second is that they have limits $(1/2, 1/2, 0)$. The third, which is the generic case, is that they have the limits $(2/3, 2/3, -1/3)$. In this last case the second Kasner relation is satisfied asymptotically and so, in a certain sense, the solution with matter is approximated near the singularity by a vacuum solution.

In type III the initial singularity is similar to that in type I but in type II it is oscillatory. As $t \to 0$ the generalized Kasner exponents approach both the values $(2/3, 2/3, -1/3)$ and $(0, 0, 1)$ as closely as desired in any neighbourhood of $t = 0$. In the expanding direction the type II solution approaches an explicitly known self-similar dust spacetime whose generalized Kasner exponents have the values $(3/8, 3/8, 1/4)$. In type III the generalized Kasner exponents approach $(0, 0, 1)$ in the expanding direction. It is possible to get more detailed information about the asymptotics of these models in the limits $t \to 0$ and $t \to \infty$. The proofs of all these statements make use of the fact that for ODE's the highly developed theory of dynamical systems is available. Getting out the finer features of the expanding phase of Bianchi III models is more delicate that the other cases just discussed and was carried out using centre manifold theory in [24].

A key aspect of the work on the asymptotics of types I, II and III was writing the dynamical system in cleverly chosen variables. As we go to a singularity or to infinity in a phase of unlimited expansion the most obvious variables go to zero or infinity. If dimensionless variables can be found which, at least for some of the solutions, converge to finite non-zero limits in these regimes then this is a great help in analysing the asymptotics. Often the original dynamical system can be extended to a smooth dynamical system on a compact region. This avoids the problem, found with many choices of variables, that the solutions expressed in the given variables run off to infinity in a way which is hard to control. The strategy of dimensionless variables and compactification has been carried much further in the case of spatially homogeneous solutions of the Einstein-Euler system. For an account of this see [31].

4   Static solutions

This section is concerned with static spherically symmetric solutions of the Einstein-Vlasov system. These may play a role in describing the long-time behaviour of solutions of the full dynamical equations and so they have a natural place in these lectures. Before coming to the Einstein-Vlasov system it is worth spending a little time thinking about the corresponding non-relativistic problem, where a lot more is known. There are results on static solutions of the Vlasov-Poisson system which are not spherically symmetric and stationary solutions which are not static [15]. Nothing comparable has yet been done in the case of the Einstein-Vlasov system. This is a gap which should be filled. From now on we restrict consideration to the static spherically symmetric case.

There are two methods which have been used to construct static spherically

symmetric solutions of the Vlasov-Poisson system. The first may be called the ODE method. In a spherically symmetric static spacetime there are two constants of motion of the particles, namely the energy E and the modulus of the angular momentum L, which are useful in constructing solutions of the Vlasov equation. In fact E and L, like any quantity conserved along geodesics, are solutions of the Vlasov equation. The same is true of any function $\Phi(E, L)$ of these quantities. Jeans' theorem says that in a spherically symmetric static solution of the Vlasov-Poisson system the distribution function is a function of E and L. Thus a natural procedure is to make an ansatz for the distribution function by choosing a particular function Φ. The Einstein equations then reduce to a system of integrodifferential equations for the metric coefficients as functions of a radial coordinate. What remains to be done is to analyse the global properties of solutions of this system.

The second method is a variational one. The Vlasov-Poisson system can be expressed as an infinite-dimensional Hamiltonian system. (Cf. [10].) It is degenerate in the sense that instead of a symplectic structure there is only a Poisson structure. This leads to a large class of conserved quantities known as Casimir invariants. In a Hamiltonian system a minimum of the Hamiltonian is a time-independent solution of the equations of motion. This suggests a variational route to finding static solutions. When Casimir invariants are present there are more general possibilities. If C is a Casimir invariant then a minimum of $H + C$ is a time-independent solution of the equations of motion. An advantage of this method is that apart from giving results on the existence of static solutions it can also provide information on the stability of the solutions obtained. This method has been applied extensively to the Vlasov-Poisson system by Guo and Rein (see [16] and references therein).

Returning to the Einstein-Vlasov system, there is a paper [33] where the energy-Casimir method has been applied but it seems to be much harder than the Vlasov-Poisson case and the results are much more limited. More straightforward is the ODE method. One cautionary note is in order. The direct analogue of Jeans' theorem is not true in the case of the Einstein-Vlasov system. Counterexamples were constructed by Schaeffer [29]. Nevertheless we can still assume a distribution function of the form $\Phi(E, L)$ and procede from there. A theorem on global existence in the radial coordinate for a rather general choice of Φ was obtained in [13]. There is a difficulty concerning the physical relevance of these solutions. If we would like to use them to model globular clusters, for instance, then we would like to obtain configurations of finite total mass. The easiest way to prove this is if the spatial density has compact support. The general existence theorem does not give any information on this. In fact whether it is true or not depends on the choice of the function Φ in quite a delicate way. A criterion for the finiteness of the mass in a large class of functions Φ was given in [18].

All known static solutions of the Einstein-Euler system with a physically reasonable equation of state are spherically symmetric and the density is a monotone decreasing function of the radius. In the case of the Einstein-Vlasov system another kind of configuration is possible where the support of the density

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is a thick shell, i.e. the region between two concentric spheres. In order to achieve this the function ϕ must depend on the angular momentum. If it only depends on the energy then the system is equivalent to a solution of the Einstein-Euler system with an equation of state which is in general not explicitly known. The existence of shell solutions was proved in [14].

5 Spherically symmetric collapse

An interesting situation to consider is that of an isolated system consisting of matter undergoing gravitational collapse. The traditional model for this, following Oppenheimer and Snyder, is the collapse of a homogeneous spherical cloud of dust. Unfortunately when inhomogeneities are introduced into the Oppenheimer-Snyder model they often lead to pathologies such as shell-crossing singularities. The advantage of using collisionless matter is that it avoids some of (and perhaps all) the problems associated with dust.

A natural first step towards understanding spherical collapse is to fully understand the case where there is no collapse. If we have only a small amount of matter then it is to be expected that its self-gravitation will not suffice to keep it together and that it will spread out and disperse to infinity. For collisionless matter this has been proved, as described in more detail later. For dust it is not true since even small amounts of matter can develop shell-crossing singularities. Even dust without gravitation can do so and so this effect has nothing to do with gravity at all.

Consider initial data for the Einstein-Vlasov system which are spherically symmetric and asymptotically flat. In a suitable coordinate system only data for the distribution function need be given since the metric can then be determined by solving equations on the hypersurfaces of constant time. This is a reflection of the familiar statement that there is no gravitational radiation in spherical symmetry. Now let us make the initial data small in the following sense. In the presence of fixed bounds on the extent of the support of the initial data for f in position and velocity space we require the maximum of f to be small. For small data it can be shown that the solution exists globally in a suitable time coordinate and, more importantly, that it is geodesically complete. Moreover, various quantities such as the energy density of the matter decay to zero as $t \to \infty$ [17]. Thus it can be seen that for small data the solution disperses and the situation is completely under control.

What happens for large data? It is known that for large data a trapped surface (and presumably a black hole) can form [21]. We might nevertheless get global existence in a singularity-avoiding time slicing like maximal or polar slicing. The latter is also sometimes called a Schwarzschild time coordinate. There is a theorem [19] which says that if a singularity forms in a solution of the spherically symmetric Einstein-Vlasov system then the first singularity (as measured in Schwarzschild time) occurs at the centre of symmetry. Note that the shell-crossing singularities of dust occur away from the centre. A corresponding result for maximal slicing has been proved in [23].

A general mathematical result on the behaviour of spherically symmetric asymptotically flat solutions of the Einstein-Vlasov system has not yet been obtained. In the absence of further analytical progress, attempts have been made to study the problem numerically. One theme which plays an important role is that of critical collapse. Suppose that we have a family of data depending on a parameter λ for the spherically symmetric Einstein-Vlasov system which interpolates between weak and strong data, with $\lambda = 0$ corresponding to data for flat space. For λ sufficiently small the theorem already mentioned tells us that the matter disperses. For λ sufficiently large we might expect collapse to a black hole and this is indeed seen numerically. More precisely, it is seen that for λ smaller than a certain value λ_* the matter disperses while for $\lambda > \lambda_*$ it collapses to a black hole. In general some of the matter falls into the black hole while some escapes. Let $M(\lambda)$ be the mass of the black hole formed when the initial data corresponds to the parameter value λ. If no black hole is formed $M(\lambda)$ is defined to be zero. One of the questions which comes up in the study of critical collapse is whether the function $M(\lambda)$ is continuous at λ_* or not.

In [20] numerical evidence was presented that $M(\lambda)$ is not continuous. In other words, the limit of $M(\lambda)$ as $\lambda \to \lambda_*$ from above is strictly positive. This is different from what is found for some other matter models, such as the massless scalar field. Olabarrieta and Choptuik [12] confirmed this finding and were able to present a more detailed picture of what happens. It is convenient for the numerical calculations to take initial data where there are no particles at the centre and no particles on purely radial orbits. In that case as long as the solution remains regular no particle can reach the centre due to conservation of angular momentum. Thus we have a dynamical configuration of matter with a hole in the middle. This allows difficulties with the singularity of polar coordinates at $r = 0$ to be avoided. It is found in [12] that the solution evolves towards an unstable static shell solution before turning away again and dispersing or collapsing. The mass of the shell solution sets the mass gap in the graph of $M(\lambda)$. The connection between the shell solutions observed numerically in collapse calculations and those whose existence has been shown rigorously is not clear.

6 Isotropic singularities and Fuchsian methods

In the last section results for certain asymptotically flat solutions of the Einstein-Vlasov system were described. These are spherically symmetric and hence have three Killing vectors. There seem to be no other symmetry assumptions on asymptotically flat spacetimes which can be usefully studied at the present time. The obvious symmetry class which comes to mind is axial symmetry. In that case, however, there is only one Killing vector, which is very little, and even that has fixed points, which leads to singularities of the equations obtained when the symmetry is factored out. At the moment it seems to offer no advantage over the general case. In the case of spacetimes evolving from data on a compact Cauchy surface (cosmological spacetimes) there is a variety

of interesting symmetry types with two or three Killing vectors and a number of papers on solutions of the Einstein-Vlasov system with these symmetries. They will not be reviewed here since some choices had to be made in order to limit the volume of the lectures. A good point of entry into the literature is [1].

There is one mathematical result on the Einstein-Vlasov system which does not require any symmetry assumptions and it will be the subject of the remainder of this section. It concerns solutions of the Einstein-Vlasov system with massless particles and it would be interesting to know if analogous results hold for massive particles. The idea is to construct large classes of solutions of the equations whose singularities have a particular structure, the isotropic singularities.

Given a spacetime with a foliation by spacelike hypersurfaces we can define the generalized Kasner exponents as in section 3 in terms of the eigenvalues of the second fundamental form. In the inhomogeneous case these are functions p_i on spacetime which in general depend on both the time and space coordinates. The condition for an isotropic singularity (at least intuitively) is that all generalized Kasner exponents should tend to $1/3$ as the singularity is approached. Thus the solution looks like an isotropic FRW model near the singularity. The actual definition used in the theorem is a different one. A spatially flat FRW model is conformally flat. In the definition of an isotropic singularity it is assumed that the given metric is conformal to a metric which is smooth at the singularity.

The requirement of an isotropic singularity is a restriction on the spacetimes considered but it has been proved by Anguige [2] that there is a very large class of solutions of the Einstein-Vlasov system with isotropic singularities. In particular, he does not have to make any symmetry assumptions. The solutions can be parametrized by certain data on the singularity which can be given freely. The method of proving this is to use Fuchsian methods (although Anguige does not use this terminology). This technique is of wider importance in studying singularities of solutions of the Einstein-matter equations and will now be discussed in a more general context.

Let a system of partial differential equations with smooth coefficients be given and suppose we would like to investigate the existence of solutions which become singular on a certain hypersurface. For simplicity assume that this is the coordinate hypersurface $t = 0$. Suppose that in some way it was possible to guess the asymptotic behaviour of the solutions in the approach to the singularity. This might be done by studying explicit solutions or by using trial and error to get a formally consistent asymptotic expansion. Then express the solution u to be constructed in terms of an explicit function u_0 having the expected asymptotics near the singularity and a remainder v which is expected to be regular and vanish at $t = 0$. Now rewrite the original equation for the unknown u as an equation for v whose coefficients depend on u_0. Since u_0 is singular it is to be expected that the equation for v is singular at $t = 0$. Thus the problem of finding a singular solution of a regular equation has been replaced by that of finding a regular solution of a singular equation.

In favourable cases the singular equation obtained by this method is a Fuch-

sian equation of the form

$$t\partial_t v + N(x)v = tf(t, x, v, v_x) \tag{11}$$

where the matrix-valued function N has some positivity property. There are theorems which guarantee that an equation of this kind has a unique solution v which is regular and vanishes at $t = 0$. One theorem of this kind was proved in [11], where it was applied to study singularities in Gowdy spacetimes. Since then there have been a number of other applications. (See [25], section 6.2, for more information on this.) Unfortunately the Einstein-Vlasov system does not fit into the framework of this theorem due to the fact that it is an integrodifferential equation rather than a differential equation. For this reason Anguige had to prove his theorem by doing a direct iteration.

7   Outlook

In these lectures a selection of work on the Einstein-Vlasov system has been surveyed. Although the results were only discussed on a very general level, without getting into details, it was still necessary to leave out a lot of interesting topics. Some of them were mentioned briefly, some not at all. It should be clear that this is an area of research where there are many open problems and many promising directions to be explored. The references given here should provide a good starting point for those wanting to follow this road.

References

[1] Andréasson, H., Rein, G. and Rendall, A. D. 2001 On the Einstein-Vlasov system with hyperbolic symmetry. Preprint gr-qc/0110089.

[2] Anguige, K. 2000 Isotropic cosmological singularities 3: The Cauchy problem for the inhomogeneous conformal Einstein-Vlasov equations. Ann. Phys. (NY) 282, 395-419.

[3] Binney, J., Tremaine, S. 1987 Galactic dynamics. Princeton University Press, Princeton.

[4] Börner, G. 1993 The early universe. Facts and fiction. Springer, Berlin.

[5] Choquet-Bruhat, Y. 1971 Problème de Cauchy pour le système intégro différentiel d'Einstein-Liouville. Ann. Inst. Fourier 21, 181-201.

[6] Chruściel, P. T. and Rendall, A. D. 1995 Strong cosmic censorship in vacuum space-times with compact locally homogeneous Cauchy surfaces. Ann. Phys. (NY) 242, 349-385.

[7] Ehlers, J. 1973 Survey of general relativity theory. In: Israel, W. (ed.) Relativity, Astrophysics and Cosmology. Reidel, Dordrecht.

[8] Friedrich, H., Rendall, A. D. 2000 The Cauchy problem for the Einstein equations. In B. G. Schmidt (ed) Einstein's Field Equations and Their Physical Implications. Lecture Notes in Physics 540. Springer, Berlin.

[9] Hawking, S. W., Ellis, G. F. R. 1973 The large-scale structure of space-time. Cambridge University Press, Cambridge.

[10] Holm, D. D., Marsden, J. E., Ratiu, T. and Weinstein, A. 1985 Nonlinear stability of fluid and plasma equilibria. Phys. Rep. 123, 1-116.

[11] Kichenassamy, S., Rendall, A. D. 1998 Analytic description of singularities in Gowdy spacetimes. Class. Quantum Grav. 15, 1339-1355.

[12] Olabarrieta, I., Choptuik, M. W. 2002 Critical phenomena at the threshold of black hole formation for collisionless matter in spherical symmetry. Phys.Rev. D65, 024007.

[13] Rein, G. 1994 Static solutions of the spherically symmetric Vlasov-Einstein system. Math. Proc. Camb. Phil. Soc. 115, 559-570.

[14] Rein, G. 1999 Static shells for the Vlasov-Poisson and Vlasov-Einstein systems. Indiana University Math. J. 48, 335-346.

[15] Rein, G. 2000 Stationary and static stellar dynamic models with axial symmetry. Nonlinear Analysis; Theory, Methods and Applications 41, 313-344.

[16] Rein, G. 2002 Stability of spherically symmetric steady states in galactic dynamics against general perturbations. Arch. Rat. Mech. Anal. 161, 27-42.

[17] Rein, G., Rendall, A. D. 1992 Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. Commun. Math. Phys. 150:561-583

[18] Rein, G., Rendall, A. D. 2000 Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. Math. Proc. Camb. Phil. Soc. 128, 363-380.

[19] Rein, G., Rendall, A. D. and Schaeffer, J. 1995 A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system. Commun. Math. Phys. 168, 467-478.

[20] Rein, G., Rendall, A. D. and Schaeffer, J. 1998 Critical collapse of collisionless matter - a numerical investigation. Phys. Rev. D58, 044007

[21] Rendall, A. D. 1992 Cosmic censorship and the Vlasov equation. Class. Quantum Grav. 9, L99-L104.

[22] Rendall, A. D. 1995 Global properties of locally spatially homogeneous cosmological models with matter. Math. Proc. Camb. Phil. Soc. 118, 511-526.

[23] Rendall, A. D. 1997 An introduction to the Einstein-Vlasov system. Banach Centre Publications 41, 35-68

[24] Rendall, A. D. Cosmological models and centre manifold theory. Preprint gr-qc/0112040.

[25] Rendall, A. D. 2002 Theorems on existence and global dynamics for the Einstein equations. Preprint gr-qc/0203012.

[26] Rendall, A. D., Tod, K. P. 1999 Dynamics of spatially homogeneous solutions of the Einstein-Vlasov equations which are locally rotationally symmetric. Class. Quantum Grav. 16, 1705-1726.

[27] Rendall, A. D., Uggla, C. 2000 Dynamics of spatially homogeneous locally rotationally symmetric solutions of the Einstein-Vlasov equations. Class. Quantum Grav. 17, 4697-4714.

[28] Ringström, H. 2000 Curvature blow up in Bianchi VIII and IX vacuum spacetimes. Class. Quantum Grav. 17, 713-731.

[29] Schaeffer, J. 1999 A class of counterexamples to Jeans' theorem for the Vlasov-Einstein system. Commun. Math. Phys. 204, 313-327.

[30] Swanson, D. G., 1989 Plasma Waves. Academic Press, Boston.

[31] Wainwright, J., Ellis, G. F. R. 1997 Dynamical systems in cosmology. Cambridge University Press, Cambridge.

[32] Wald, R. M. 1984 General Relativity. Chicago University Press, Chicago.

[33] Wolansky, G. 2001 Static solutions of the Vlasov-Einstein system. Arch. Rat. Mech. Anal. 156, 205-230.