Gravitational Self-Force on a Particle Orbiting a Kerr Black Hole

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(Received 26 December 2002; published 20 March 2003)

We present a practical method for calculating the gravitational self-force, as well as the electromagnetic and scalar self-forces, for a particle in a generic orbit around a Kerr black hole. In particular, we provide the values of all the regularization parameters needed for implementing the (previously introduced) *mode-sum regularization* method. We also address the gauge-regularization problem, as well as a few other issues involved in the calculation of gravitational radiation reaction in Kerr spacetime.

The gravitational two-body problem is a long-standing open problem in general relativity. In Newtonian theory the evolution of a binary system is easily analyzed, yielding periodic orbits with conserved energy and angular momentum. In general relativity, in contrast, the orbits in a binary system are never periodic, and the orbital energy and angular momentum decrease due to emission of gravitational radiation. Typically, the two masses will gradually inspiral towards each other, until they merge. An interesting branch of the problem refers to the case where one of the components is a compact object much lighter than the other. In this situation, the lighter object may be treated as an infinitesimally small "particle" moving on the fixed gravitational-field background of the heavier object. The particle then experiences a *self-force* (SF) due to interaction with its own gravitational field (which is distorted by the field of the massive object). The issue, then, becomes that of determining the gravitational SF acting on the particle. The electromagnetic (EM), flat-space analog of this problem was explored many decades ago by Lorentz [1] and Dirac [2], who analyzed the force acting on an accelerating point charge due to interaction with its own EM field. To analyze the evolution of a strong-gravity, large mass-ratio binary system, the classic works by Lorentz and Dirac need to be generalized in two important respects: from the EM to the gravitational SF and from flat to curved spacetime.

DeWitt and Brehme [3] were first to extend the SF concept to curved spacetime, in the EM case. More recently, a theoretical framework has been established [4] for the *gravitational* SF in curved spacetime (cf. [5]). An essential feature of both the EM and gravitational SFs in curved spacetime (unlike the analogous flat-space case) is that they depend, at any given point along the particle's world line, upon the entire past history of the particle. This may be attributed to the scattering of the (EM or gravitational) waves off the spacetime curvature. So far, this nonlocality rendered actual calculations of the SF in curved spacetime most challenging.

The quest for a practical SF calculation method has considerably intensified in recent years, with the deployment of a new generation of gravitational-wave detectors. One of the main sources for the planned space-based gravitational-wave detector LISA [6] would be the inspiral of compact objects [white dwarfs, neutron stars, and stellar black holes (BHs)] into supermassive BHs, with a typical mass ratio of 10^3–10^7. Such supermassive BHs appear to reside in the centers of many galaxies, including our own [7]. In a typical scenario, the compact object spends the last year of inspiral deep inside the highly relativistic region near the BH's event horizon, emitting some 10^5 cycles of gravitational waves [8], which carry detailed information of the massive BH's strong-field geometry [9]. Knowledge of the self-force—which, in turn, determines the orbital evolution—is necessary for predicting the emitted waveform. Such a prediction is also important for the design of templates which should significantly improve the chance of detection in the situation of a small signal-to-noise ratio, typical to the first generation of detectors.

In several, relatively simple situations the orbit's evolution may be inferred using standard energy-momentum balance methods [10]. Namely, one calculates the flux of energy and angular momentum carried away by the gravitational waves, and equates this to (minus) the rate of change in the corresponding entities associated with the orbiting particle itself. This simple method yields the orbital evolution for all orbits around a nonspinning (Schwarzschild-type) BH, and also for strictly equatorial or circular orbits around a rotating (Kerr-type) BH. However, in the astrophysically relevant situation one needs to analyze the evolution of *generic* orbits around a Kerr BH. In this generic case, energy-momentum balance methods are insufficient, because the energy and (azimuthal) angular momentum no longer uniquely determine the orbit. One therefore must calculate the local SF acting on the particle. The goal of this paper is to provide a practical method for calculating this SF, for generic orbits in Kerr spacetime.

Based on the basic formulation of Refs. [4], we previously devised the *mode-sum method,* as a practical technique for SF calculations [11,12]. In this method, one first calculates the (finite) contribution to the SF due to individual multipole modes of the particle's field (this is done by integrating numerically the mode-decomposed linearized field equations). Then, a certain regularization procedure is applied to the mode sum [see Eq. (4) below], which involves certain "regularization parameters" (RP). The latter depend upon the background spacetime and the specific orbit under consideration. The RP values were previously derived for equatorial orbits around a Schwarzschild BH [13–16]. Here we extend the analysis and give the RP values for an arbitrary geodesic orbit around a Kerr BH.

Throughout this Letter we use units where $c = G = 1$, metric signature $-+++$, and Boyer-Lindquist coordinates t, r, θ, φ. For completeness, we shall address here not only the gravitational SF, but also the analogous EM and scalar phenomena. We shall thus consider the following setup: A particle of mass μ, electric charge e, and/or scalar charge q moves along an arbitrary free-fall orbit (i.e., no external forces) $z^\mu(\tau)$ around a Kerr BH with mass $M \gg \mu, |q|, |e|$ and arbitrary spin aM. We shall consider the SF acting on the particle (to the leading order $\propto \mu^2, e^2, q^2$) at an arbitrary point $z_0^\mu = (t_0, r_0, \theta_0, \varphi_0)$ along its trajectory. The particle's equation of motion, including the SF effect, reads

$$\mu u_{\lambda;\beta} u^\beta = F_\lambda^{\text{self}}, \qquad (1)$$

where $u^\beta \equiv dz^\beta/d\tau$ is the four-velocity, a semicolon denotes covariant differentiation with respect to the background (Kerr) geometry, and F_λ^{self} is the SF.

Let x denote a spacetime point in the neighborhood of z_0. We denote the particle's actual perturbation field (the "full field") by $h_{\alpha\beta}(x)$, $\mathcal{A}_\alpha(x)$, or $\Phi(x)$, representing, correspondingly, the metric perturbation ($\propto \mu$), EM vector potential ($\propto e$), and scalar field ($\propto q$). These fields are, in principle, solutions to the linearized Einstein–Maxwell–Klein-Gordon equations, with the particle serving as a source term. To these fields there correspond "full-force" fields, given by [14,15]

$$F_\mu^{\text{full}}(x) = \begin{cases} q\Phi_{,\mu}, & \text{(scalar)}, \\ e\kappa_\mu{}^{\beta\gamma}\mathcal{A}_{\beta;\gamma}, & \text{(EM)}, \\ \mu\kappa_\mu{}^{\beta\gamma\delta}h_{\beta\gamma;\delta}, & \text{(grav.)}, \end{cases} \qquad (2)$$

where the tensors κ are

$$\kappa^{\alpha\beta\gamma}(x) \equiv g^{\alpha\gamma}\hat{u}^\beta - g^{\alpha\beta}\hat{u}^\gamma,$$
$$\kappa^{\alpha\beta\gamma\delta}(x) \equiv -g^{\alpha\beta}\hat{u}^\gamma\hat{u}^\delta + \tfrac{1}{2}g^{\alpha\delta}\hat{u}^\beta\hat{u}^\gamma - \tfrac{1}{2}\hat{u}^\alpha\hat{u}^\beta\hat{u}^\gamma\hat{u}^\delta. \quad (3)$$

Here $g^{\alpha\beta}$ is the background metric at x and $\hat{u}^\alpha$ is the four-velocity parallel propagated from the world line to x along a short geodesic normal to the world line.

In the mode-sum method, the SF is then calculated through [11]

$$F_\mu^{\text{self}} = \sum_{l=0}^\infty [(F_{\mu l}^{\text{full}})^\pm - A_\mu^\pm L - B_\mu - C_\mu/L] - D_\mu. \quad (4)$$

Here $L \equiv l + 1/2$, and $(F_{\mu l}^{\text{full}})^\pm$ is the l multipole mode of $F_\mu^{\text{full}}(x)$, summed over all azimuthal numbers m (for a given l) and evaluated at $(t_0, r \to r_0^\pm, \theta_0, \varphi_0)$. The full modes $F_{\mu l}^{\text{full}}$ are generally discontinued across $r = r_0$, hence the label $\pm$ [11]. The RP, i.e., the quantities $A_\mu^\pm$, B_μ, C_μ, D_μ, are to be derived analytically (by analyzing the local singular behavior of the perturbation field near the particle). The first three of these parameters capture the asymptotic form of the full-force modes at large l, making the sum in Eq. (4) convergent. The parameter D_μ is a certain residual quantity that arises in the summation over l—for a more precise definition of the RP see [11,13].

In deriving the RP in Kerr spacetime, we basically followed the method used in the Schwarzschild case [13], as described in much detail in Refs. [14,15]. Namely, we considered the direct part of the full-force field (i.e., the part associated with waves directly propagating along the light cone), from which all RP are obtained by applying the multipole decomposition and appropriately taking the limit $x \to z_0$. The main technical challenge in extending our analysis to the Kerr case lies in the fact that one can no longer exploit spherical symmetry by choosing a coordinate system in which the orbit is confined to the equatorial plain. Rather, one must carry out the calculation at an arbitrary value of θ_0. This turns out to render the entire analysis significantly more complicated. For lack of space, we skip the derivation of these RP values—the detailed derivation will be given elsewhere [17]. In what follows we merely present the results, i.e., the values of all RP for an arbitrary geodesic in Kerr spacetime.

To write down the RP in a convenient unified form, we introduce a "generalized" charge q_s, where $q_s = q$, e, or μ, for the scalar ($s = 0$), EM ($s = 1$), and gravitational ($s = 2$) cases, respectively. All RP are then independent of s, apart from an overall factor $(-1)^s q_s^2$. For A_μ, C_μ, and D_μ we find

$$C_\mu = D_\mu = 0, \qquad (5)$$

and

$$A_r^\pm = \mp(-1)^s \frac{q_s^2}{V}\left(\frac{\sin^2\theta_0\, g_{rr}}{g_{\theta\theta}g_{\varphi\varphi}}\right)^{1/2}(V + u_r^2/g_{rr})^{1/2},$$
$$A_t^\pm = -(u^r/u^t)A_r^\pm, \qquad A_\theta^\pm = A_\varphi^\pm = 0, \qquad (6)$$

where hereafter $g_{\alpha\beta} \equiv g_{\alpha\beta}(z_0)$ and $V \equiv 1 + u_\theta^2/g_{\theta\theta} + u_\varphi^2/g_{\varphi\varphi}$. The expression for B_μ takes a slightly more complicated form,

$$B_\mu = (-1)^s q_s^2 (2\pi)^{-1} P_{\mu abcd} I^{abcd}. \qquad (7)$$

Hereafter, roman indices $(a, b, c, \dots)$ run over the two Boyer-Lindquist angular coordinates θ, φ. The coefficients $P_{\mu abcd}$ are given by

$$P_{\mu abcd} = (3P_{\mu a}P_{be} - P_{\mu e}P_{ab})C^e_{cd}$$
$$+ \tfrac{1}{2}[3P_{\mu d}P_{abc} - (2P_{\mu ab} + P_{ab\mu})P_{cd}], \quad (8)$$

where

$$P_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \qquad P_{\alpha\beta\gamma} \equiv (u_\lambda u_\gamma \Gamma^\lambda_{\alpha\beta} + g_{\alpha\beta,\gamma}/2) \quad (9)$$

($\Gamma^\lambda_{\alpha\beta}$ denoting the connection coefficients at z_0), and

$$C^\theta_{\varphi\varphi} = \tfrac{1}{2}\sin\theta_0\cos\theta_0, \qquad C^\varphi_{\theta\varphi} = C^\varphi_{\varphi\theta} = -\tfrac{1}{2}\cot\theta_0, \quad (10)$$

with all other coefficients C^e_{cd} vanishing. Finally, the quantities I^{abcd} are

$$I^{abcd} = (\sin\theta_0)^{-N} \int_0^{2\pi} G(\gamma)^{-5/2}(\sin\gamma)^N(\cos\gamma)^{4-N} d\gamma,$$

where $N \equiv N(abcd)$ is the number of times the index φ occurs in the combination $abcd$ (namely, $N = \delta^a_\varphi + \delta^b_\varphi + \delta^c_\varphi + \delta^d_\varphi$) and

$$G(\gamma) = P_{\tilde\varphi\tilde\varphi}\sin^2\gamma + 2P_{\theta\tilde\varphi}\sin\gamma\cos\gamma + P_{\theta\theta}\cos^2\gamma, \quad (11)$$

where

$$P_{\tilde\varphi\tilde\varphi} \equiv P_{\varphi\varphi}/\sin^2\theta_0, \qquad P_{\theta\tilde\varphi} \equiv P_{\theta\varphi}/\sin\theta_0. \quad (12)$$

The integrals I^{abcd} can be expressed in terms of complete elliptic integrals. These relations are given explicitly in Ref. [18].

In the special case of an equatorial orbit in Schwarzschild spacetime, the above RP values reduce to those obtained previously using various methods [11,13].

In the rest of this paper we discuss several important issues that arise when implementing the mode-sum method in Kerr spacetime.

First, to avoid confusion, we emphasize that the quantities $F^{\text{full}}_{\mu l}$ appearing in Eq. (4) refer (in all three cases $s = 0, 1, 2$) to the decomposition of the full-force field components F^{full}_μ in the standard *scalar spherical harmonics*. Practically, in the Kerr case one separates the field equations in the frequency domain, using (spin weighted) *spheroidal harmonics*. Then the full-force field, too, is obtained in terms of the spheroidal harmonics. Thus, in order to implement the mode-sum scheme (at least in its present formulation), one needs to decompose the contributions to the full-force from the various spheroidal-harmonic modes, into scalar spherical harmonics: For a given l (as well as given azimuthal number m and temporal frequency ω) one collects the contributions to the spherical-harmonic modes l, m coming from the full-force fields associated with the spheroidal-harmonic modes l', m, ω. Then one sums over l' (as well as m, ω)

to obtain $F^{\text{full}}_{\mu l}$. The decomposition of the spheroidal harmonics in spherical harmonics is described in Ref. [19].

Our second remark concerns the construction of the full-force field in the EM and gravitational cases. This field involves an extension of u^α off the world line—cf. Equation (3). Here we adopted the most natural extension $\hat{u}^\alpha$, namely, the one obtained by parallel propagating u^α from the world line to x. As discussed in [15], it is possible to use any other (sufficiently regular) extension of u^α. Note, however, that changing the extension (which obviously affects the quantities $F^{\text{full}}_{\mu l}$) will generally affect the value of the parameter B_μ (though it turns out that the other RP are unaffected). Nevertheless, if two extensions share the same values of angular derivatives $u^\alpha_{,b}$ at the world line, they admit the same B_μ [15].

The parallel-propagated extension is an elegant one, and, furthermore, it allows the simplest derivation of B_μ. However, this extension is rather inconvenient for calculating the full-force quantities $F^{\text{full}}_{\mu l}$. (The Legendre decomposition requires the extension of u^α far away from the world line, on the entire two-sphere $r, t = $ const. It is hard to calculate $\hat{u}^\alpha$ explicitly off the world line, and it is even harder to Legendre decompose it. Furthermore, there is no guarantee that $\hat{u}^\alpha$ would be globally well defined.) To overcome this difficulty, we propose the following strategy: Compose any global extension $\tilde{u}^\alpha$ off the world line, with the only demand that $\tilde{u}^\alpha_{,b} = \hat{u}^\alpha_{,b}$ (and, obviously, $\tilde{u}^\alpha = \hat{u}^\alpha$) at the world line itself. It is easy to construct such simple global extensions. Then simply use this $\tilde{u}^\alpha$ instead of $\hat{u}^\alpha$ in Eq. (3). The above values of all RP are unaffected by this change of extension.

There is another technical issue that arises in the EM and gravitational cases: The only known scheme for separating the EM and (linearized) gravitational-field equations in Kerr spacetime is the Teukolsky formalism [20]. By solving the Teukolsky equation, one obtains certain components of the Maxwell or Weyl tensors, respectively. However, as is obvious from Eq. (2), the calculation of F^{full}_μ (and hence of $F^{\text{full}}_{\mu l}$ and the SF) requires the knowledge of the basic perturbation fields, $h_{\alpha\beta}(x)$ or $\mathcal{A}_\alpha(x)$. A method for reconstructing $h_{\alpha\beta}$ and $\mathcal{A}_\alpha$ for a point particle, out of the Teukolsky variables, based on the Chrzanowski-Wald formalism [21,22], was recently provided by Ori [23] (see also [24] for the Schwarzschild case). More recently, this construction was extended to the nonradiative modes $l = 0, 1$, as well as to all higher-l modes with $\omega = 0$ [25].

Finally, we address the crucial issue of the *gauge dependence* of the gravitational SF, and the related problem of gauge regularization in Kerr spacetime. The gravitational SF (unlike the scalar and EM forces) turns out to be gauge dependent [26]. The basic formalism [4] is given in terms of the harmonic gauge. The transformation of the SF to any other gauge is given in Ref. [26]. This transformation, however, is restricted to *regular gauges*, i.e.,

gauges related to the harmonic gauge via a displacement vector ξ^μ that is continuous on the particle's world line. (If the gauge is irregular, then the SF generally becomes ill defined.) For any regular gauge, the SF may, in principle, be calculated through Eq. (4)—provided that one uses the quantities $F_{\mu l}^{\mathrm{full}}$ associated with this same gauge. The RP are gauge invariant in this context [26].

Although the momentary SF is gauge dependent, the long-term radiation-reaction evolution of the orbit, as expressed, e.g., by the drift of the constants of motion, is gauge invariant. The radiation-reaction evolution can therefore be calculated in any regular gauge.

Chrzanowski's construction yields the metric perturbations in the so-called *radiation gauge* [21]. Unfortunately, this gauge is not regular; hence the SF does not have a well-defined value in the radiation gauge. To overcome this difficulty, we previously proposed [26] to calculate the SF in an *intermediate gauge,* namely, a regular gauge which (unlike the harmonic gauge) can be obtained from the radiation gauge via a simple, explicit gauge transformation. A convenient such intermediate gauge was recently constructed explicitly in Ref. [25].

When implementing the mode-sum scheme (4) to the SF in the intermediate gauge, one should, in principle, substitute the quantities $F_{\mu l}^{\mathrm{full(INT)}}$ associated with this same gauge. Alternatively, one may still use the radiation-gauge quantities $F_{\mu l}^{\mathrm{full(RAD)}}$ in Eq. (4), but in this case one must subtract an extra term, which we denote δD_μ, to compensate for the differences $\delta F_{\mu l}^{\mathrm{full}}$ in the values of $F_{\mu l}^{\mathrm{full}}$ in the two gauges. (The derivation of $\delta F_{\mu l}^{\mathrm{full}}$ from ξ^μ is outlined at the end of Ref. [26]; δD_μ is the sum of $\delta F_{\mu l}^{\mathrm{full}}$ over l.) Thus, when applied to the SF in the intermediate gauge, Eq. (4) may be recast as

$$F_\mu^{\mathrm{self(INT)}} = \sum_{l=0}^\infty [(F_{\mu l}^{\mathrm{full(RAD)}})^\pm - A_\mu^\pm L - B_\mu] - \delta D_\mu$$

(recall $C_\mu = D_\mu = 0$). The RP $A_\mu^\pm$, B_μ are those given in Eqs. (6) and (7) (as mentioned above, the RP values are gauge invariant). The term δD_μ was recently calculated for a generic orbit in Kerr, and its expression (whose structure and complexity resemble those of B_μ above) will be given elsewhere [25]. This provides the solution to the gauge-regularization problem. The quantity $F_\mu^{\mathrm{self(INT)}}$ can now be used to analyze the (gauge-independent) long-term radiation-reaction evolution, for any orbit in Kerr spacetime.

[1] H. A. Lorentz, *Theory of Electrons* (Dover, New York, 1952), 2nd ed.
[2] P. A. M. Dirac, Proc. R. Soc. London A **167**, 148 (1938).
[3] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[4] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997); T. C. Quinn and R. M. Wald, *ibid.* **56**, 3381 (1997). The analogous *scalar* SF problem was recently addressed in T. C. Quinn, *ibid.* **62**, 064029 (2000).
[5] For alternative SF regularization schemes see, e.g., C. O. Lousto, Phys. Rev. Lett. **84**, 5251 (2000); H. Nakano and M. Sasaki, Prog. Theor. Phys. **105**, 197 (2001); S. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).
[6] LISA (Laser Interferometer Space Antenna) Web page: http://lisa.jpl.nasa.gov/.
[7] R. Schödel *et al.*, Nature (London) **419**, 694 (2002).
[8] L. S. Finn and K. S. Thorne, Phys. Rev. D **62**, 124021 (2000).
[9] F. D. Ryan, Phys. Rev. D **52**, 5707 (1995).
[10] S. A. Hughes, Phys. Rev. D **64**, 064004 (2001); K. Glampedakis, S. A. Hughes, and D. Kennefick, *ibid.* **66**, 064005 (2002), and references therein.
[11] The mode-sum method was first introduced in L. Barack and A. Ori, Phys. Rev. D **61**, 061502(R) (2000), and was formulated therein for the scalar SF; see also L. Barack, *ibid.* **62**, 084027 (2000). Its extension to the gravitational SF is formulated in L. Barack, *ibid.* **64**, 084021 (2001).
[12] For implementation of the mode-sum method in the scalar case, see, e.g., L. M. Burko, Phys. Rev. Lett. **84**, 4529 (2000); L. Barack and L. M. Burko, Phys. Rev. D **62**, 084040 (2000); L. M. Burko and Y.-T. Liu, *ibid.* **64**, 024006 (2001). A first actual calculation of the *gravitational* SF, using the mode-sum method, is in L. Barack and C. O. Lousto, Phys. Rev. D **66**, 061502(R) (2002).
[13] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).
[14] L. Barack and A. Ori, Phys. Rev. D **66**, 084022 (2002).
[15] L. Barack and A. Ori, gr-qc/0209072 (to be published).
[16] Y. Mino, H. Nakano, and M. Sasaki, gr-qc/0111074.
[17] L. Barack and A. Ori (to be published).
[18] L. Barack and A. Ori, gr-qc/0212103v1.
[19] S. A. Hughes, Phys. Rev. D **61**, 084004 (2000).
[20] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
[21] P. L. Chrzanowski, Phys. Rev. D **11**, 2042 (1975).
[22] R. M. Wald, Phys. Rev. Lett. **41**, 203 (1978).
[23] A. Ori, gr-qc/0207045 (to be published).
[24] C. O. Lousto and B. F. Whiting, Phys. Rev. D **66**, 024026 (2002).
[25] A. Ori (to be published).
[26] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).