Accelerated cosmological expansion due to a scalar field whose potential has a positive lower bound

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Abstract

In many cases a nonlinear scalar field with potential V can lead to accelerated expansion in cosmological models. This paper contains mathematical results on this subject for homogeneous spacetimes. It is shown that, under the assumption that V has a strictly positive minimum, Wald's theorem on spacetimes with positive cosmological constant can be generalized to a wide class of potentials. In some cases detailed information on late-time asymptotics is obtained. Results on the behaviour in the past time direction are also presented.

1 Introduction

Cosmological models with accelerated expansion are currently of great astrophysical interest. Two main themes are inflation, which concerns the very early universe, and the accelerated cosmological expansion at the present epoch as evidenced, for instance, by supernova observations. The simplest way of obtaining accelerated expansion within general relativity is a positive cosmological constant. A more sophisticated variant is the presence of a nonlinear scalar field. For the two themes mentioned above this scalar field is known as the inflaton and quintessence, respectively. A cosmological constant alone may not be an alternative for describing the real universe if

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the observational data indicate different amounts of acceleration at different times. Evidence of this kind based on supernova data is presented in [1]. There is at present no good physical understanding of the exact nature of the inflaton or quintessence and it may even be that both are the same field, producing different phenomena at different times.

Given the limited information available on the nature of the scalar field ϕ supposed to cause accelerated expansion, it makes sense to study the dynamical behaviour of spacetimes containing a nonlinear scalar field with a potential taken from a general class satisfying a few basic assumptions. The aim of this paper is to do this in the context of proving rigorous mathematical theorems which include anisotropic models. Although the literature on cosmological models with accelerated expansion is vast, the number of papers which are mathematical in the sense just mentioned, and which go beyond the isotropic case, is small. Some which are known to the author are [24], [10], [22], [23] and [18] on the case of a cosmological constant, [14] on the case of a quadratic potential (massive scalar field, chaotic inflation) and [8] and [9] on the case of an exponential potential (power-law inflation).

One typical assumption on the scalar field potential $V(\phi)$ is that it should be non-negative. This is a kind of positive energy assumption on the field which is very natural. It implies that the dominant energy condition is satisfied. The weak energy condition then follows. Scalar fields with negative potentials have been considered and lead to interesting dynamical behaviour [6]. In the following, however, it will always be assumed that $V(\phi) \geq 0$. Note that the strong energy condition is in general violated even if V is positive and that this is intimately connected with the occurrence of accelerated expansion.

Consider now spacetimes which are homogeneous but not necessarily isotropic. A classical result of Wald [24] on spacetimes with positive cosmological constant concerns Bianchi types I-VIII. The remaining types of spatially homogeneous spacetimes, namely Bianchi type IX and Kantowski-Sachs, are known to display much more complicated dynamical behaviour. For this reason consideration will be mainly restricted to Bianchi types I-VIII in what follows. A key assumption to be made on the potential is that it has a strictly positive lower bound. This is an essential restriction and rules out many of the cases most frequently considered in the literature. It is to be considered as the first step in a more extended investigation and it is chosen as a starting point because it is likely to be one of the easiest general classes of potentials to analyse. Nevertheless it will be seen that it is not trivial to

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treat rigorously.

To see the meaning of the condition that V is bounded below by a constant $V_0 > 0$ it is useful to consider the following mechanical analogy, which is discussed for instance in chapter 11 of [16]. The equation of motion of the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1}$$

where $H = -\mathrm{tr}k/3$, $\mathrm{tr}k$ is the mean curvature of the hypersurfaces of constant time and a dot denotes d/dt. H is positive for an expanding cosmological model. This is the equation of motion of a ball rolling on the graph of the potential V with a variable friction term. The intuitive picture of the motion of the ball is that it rolls down the potential until it reaches a local minimum where it may then oscillate. If the minimum is strictly positive then there are no oscillations, the phase of accelerated expansion never ends, and the asymptotic dynamics becomes particularly simple. It is shown in section 3 that with an additional mild assumption on the behaviour of the potential where it is large a number of the results of [24] can be generalized to the scalar field case. A further assumption allows detailed late-time asymptotic properties to be derived in section 4, assuming global existence of solutions in the future. Section 5 contains results on global existence of solutions and asymptotic behaviour of matter quantities when matter is modelled as a fluid or collisionless gas.

It is also interesting to know what happens in the past time direction and this is investigated in Section 6. Solutions which exist for infinite proper time in the past often play the role of inflationary attractors. In the case of a positive cosmological constant this leads to the de Sitter solution. It is shown that in the case of a scalar field with assumptions as before a solution which exists for infinite proper time in the past and for which the scalar field remains bounded has very special properties. The results for both the past and future directions are consistent with the intuition provided by the picture of 'rolling' and this picture also guided the development of the proofs.

2   Basics

Consider a spacetime with vanishing cosmological constant which contains a nonlinear scalar field and some other matter. The latter is supposed to satisfy the strong energy condition and therefore to produce no accelerated expansion in the absence of the scalar field. This matter should be thought

of physically as including baryonic matter, radiation and non-baryonic dark matter. The scalar field represents dark energy. The energy-momentum tensor of the spacetime being considered is

$$T_{\alpha\beta} = T_{\alpha\beta}^M + \nabla_\alpha\phi\nabla_\beta\phi - \left[\frac{1}{2}\nabla^\gamma\phi\nabla_\gamma\phi + V(\phi)\right]g_{\alpha\beta}. \tag{2}$$

where $T_{\alpha\beta}^M$ is the energy-momentum tensor of the matter other than the scalar field. It is supposed to satisfy the conditions that for any future-pointing timelike vector fields X^α and Y^β the inequalities $T_{\alpha\beta}^M X^\alpha Y^\beta \geq 0$ (dominant energy condition) and $[T_{\alpha\beta}^M - (1/2)\text{tr}T^M g_{\alpha\beta}]X^\alpha X^\beta \geq 0$ (strong energy condition) are satisfied. The potential V is assumed to be non-negative and sufficiently smooth. For all results in this paper C^2 is enough. Since $T_{\alpha\beta}^M$ is assumed divergence-free the Bianchi identity implies the equation of motion

$$\nabla_\alpha\nabla^\alpha\phi = V'(\phi) \tag{3}$$

for the scalar field.

Now consideration will be restricted to the case that the spacetime is spatially homogeneous. When expressed in terms of a Gaussian time coordinate based on a hypersurface of homogeneity and a left-invariant, time-independent frame, the field equations imply the following relations:

$$\frac{dH}{dt} = -H^2 - \frac{8\pi}{3}[\dot\phi^2 - V(\phi)] - \frac{1}{3}\sigma_{ab}\sigma^{ab} - \frac{4\pi}{3}(\rho^M + \text{tr}S^M) \tag{4}$$

$$\frac{d}{dt}(\sigma^a{}_b) = -3H\sigma^a{}_b + \tilde{R}^a{}_b - 8\pi(\tilde{S}^M)^a{}_b \tag{5}$$

$$H^2 = \frac{4\pi}{3}[\dot\phi^2 + 2V(\phi)] + \frac{1}{6}(\sigma_{ab}\sigma^{ab} - R) + \frac{8\pi}{3}\rho^M \tag{6}$$

Here σ_{ab} and $\tilde{R}^a{}_b$ are the tracefree parts of the second fundamental form k_{ab} and Ricci tensor of the spatial metric respectively and R is the scalar curvature of the spatial metric. The energy density corresponding to $T_{\alpha\beta}^M$ is denoted by ρ^M while S_{ab}^M is the spatial projection of $T_{\alpha\beta}^M$. In the field equations S_{ab}^M has been split into its trace and tracefree parts. Combining (4) and (6) gives

$$\frac{dH}{dt} = -4\pi\dot\phi^2 - \frac{1}{2}\sigma_{ab}\sigma^{ab} + \frac{1}{6}R - 4\pi(\rho^M + \frac{1}{3}\text{tr}S^M) \tag{7}$$

In the case where the spacetime is a Bianchi model of type I-VIII the inequality $R \leq 0$ holds [24]. Assuming that H is positive at some time, an

assumption which will always be made from now on, it follows from (6) that it is positive at all times. For if H vanished at some time t_0 then σ_{ab} and R would vanish at t_0. It follows that the induced metric on $t = t_0$ would be flat and the second fundamental form would vanish. At that time ρ^M would be zero which, by the dominant energy condition, implies $T^M_{\alpha\beta} = 0$. In addition $\dot{\phi} = 0$ and $V(\phi) = 0$ for $t = t_0$. Now

$$\frac{d}{dt}[\dot{\phi}^2 + 2V(\phi)] = -6H\dot{\phi}^2 \tag{8}$$

and so in the case under consideration $\dot{\phi} = 0$ and $V(\phi) = 0$ at all future times. From (1) it follows that $V'(\phi) = 0$. The unique solution of the field equations with initial data of this type is flat spacetime with a constant scalar field and it has $H = 0$ at all times, contradicting our basic assumption. From (7) it can be concluded in general that H is non-increasing.

3 Late-time dynamics

In this section Wald's theorem [24] is generalized to the case of a scalar field whose potential satisfies the following three assumptions:

1. $V(\phi) \geq V_0$ for a constant $V_0 > 0$

2. V' is bounded on any interval on which V is bounded

3. V' tends to a limit, finite or infinite as ϕ tends to ∞ or $-\infty$.

While condition 1. is strong, conditions 2. and 3. are satisfied by many potentials considered in the literature. Condition 2. is used in [13].

Theorem 1 Consider a solution of the Einstein equations of Bianchi type I-VIII coupled to a nonlinear scalar field with potential V of class C^2 satisfying conditions 1. - 3. above and other matter satisfying the dominant and strong energy conditions. If the solution is initially expanding ($H > 0$) and exists globally to the future then for $t \to \infty$, the quantites $\sigma_{ab}\sigma^{ab}$, R, ρ^M and $H^2 - (8\pi/3)[\dot{\phi}^2/2 + V(\phi)]$ decay exponentially. $V(\phi)$ converges to some constant V_1, $V'(\phi) \to 0$ and $H \to (8\pi V_1/3)^{1/2}$.

Proof Since H is bounded in the future it follows from (7) that $\int_{t_0}^{\infty} \dot{\phi}^2 dt < \infty$ for any initial time t_0. From (6) it follows that $V(\phi)$ is bounded to the future. Assumption 2. then implies that $V'(\phi)$ is bounded. Putting this information

into (1) and using the boundedness of $\dot{\phi}$ shows that $\ddot{\phi}$ is bounded. Combining this fact with the integrability of $\dot{\phi}^2$ shows that $\dot{\phi} \to 0$ as $t \to \infty$. The quantity $\dot{\phi}^2 + 2V(\phi)$, being monotone and bounded below, tends to a limit $2V_1$ as $t \to \infty$. Assumption 1. implies that $V_1 > 0$. We have $V(\phi) \to V_1$. Consider now the quantity

$$Z = 9H^2 - 24\pi[\dot{\phi}^2/2 + V(\phi)] = \frac{3}{2}(\sigma_{ab}\sigma^{ab} - R) + 24\pi\rho^M \qquad (9)$$

which is inspired by the quantity S used in [14].

$$\frac{dZ}{dt} = -6H\left[\frac{3}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}R + 4\pi(3\rho^M + \mathrm{tr}S^M)\right] \leq -3HZ \qquad (10)$$

Since $H \geq (8\pi V_0/3)^{1/2}$ it follows that Z decays exponentially as $t \to \infty$, implying several of the assertions of the theorem. In particular the statement on the convergence of H is established. Let ϕ_2 and ϕ_3 be the infimum and supremum respectively of those numbers arising as limits of a sequence $\{\phi(t_n)\}$ for some sequence $\{t_n\}$ tending to infinity. If $\phi_2 < \phi_3$ then V' must vanish on the entire interval (ϕ_2, ϕ_3). It follows that in this case $V'(\phi) \to 0$ as $t \to \infty$. Otherwise, $\phi_2 = \phi_3$ and ϕ converges to a limit, finite or infinite. Under the assumptions of the theorem it then follows that $V'(\phi)$ converges as $t \to \infty$. If this limit were non-zero then $\ddot{\phi}$ would converge to a non-zero limit, contradicting the fact that $\dot{\phi} \to 0$. Hence in fact $V'(\phi) \to 0$ and this completes the proof of the theorem.

The statement of the theorem can now be compared with the 'rolling' picture. For this it is useful to suppose that any points where $V'(\phi) = 0$ are isolated. Then the theorem implies that ϕ converges as $t \to \infty$ to a finite limit ϕ_1 with $V'(\phi_1) = 0$ or to plus or minus infinity. Thus the scalar field converges to a critical point of the potential (possibly at infinity) and the behaviour of H can be interpreted as the emergence of an effective cosmological constant V_1 at late times. The critical point ϕ_1 does not have to be a minimum of the potential and some further remarks on the case it is a maximum will be made in Section 6. Note that in some cases it can be predicted which critical point of V the solution converges to. Using the fact that $V(\phi(t))$ can never exceed the value of $\dot{\phi}^2/2 + V(\phi)$ at any time t_1 with $t_1 < t$ it can be shown that the solution gets trapped in a local minimum of the potential.

To end this section some comments will be made on Bianchi type IX models. These are known to include solutions which recollapse. For a restricted

class of initial data the results of Theorem 1 carry over. An important observation in [24] is that although the spatial scalar curvature may be positive for type IX it satisfies an inequality of the form $R \leq \bar{R}(\det g)^{-1/3}$ where $\det g$ is the determinant of the spatial metric and $\bar{R}$ is the scalar curvature of the isotropic metric with unit determinant. Suppose that at some initial time t_0 the inequality $16\pi V_0 - \bar{R}(\det g)^{-1/3} > 0$ holds and that H is initially positive. It follows that the determinant of the spatial metric is increasing and that the initial inequality is preserved. Thus H remains positive. If the solution exists globally in time then R decays exponentially and H tends to a limit as $t \to \infty$. As a consequence Z decays and this gives statements on decay as in Theorem 1. Recall that the closed universe recollapse conjecture which was proved in [12] implies that a Bianchi type IX spacetime with matter satisfying the dominant and strong energy conditions cannot expand for ever. The above discussion, together with suitable global existence statements, shows that a scalar field (which violates the strong energy condition) prevents the analogue of the result of [12] from holding. The arguments leading to global existence are described in Section 5.

4   Detailed asymptotics in the future

In this section the results of Theorem 1 will be refined in order to get more detailed information about the asymptotics of the solutions at late times under additional assumptions on the potential. Suppose that a solution is as in Theorem 1 so that ϕ tends to a finite limit ϕ_1 as $t \to \infty$. Without loss of generality it may be assumed by a redefinition of the potential that $\phi_1 = 0$. It follows from Theorem 1 that $V'(0) = 0$. In this section it will be assumed in addition that $V''(0) = \beta > 0$ so that V has a non-degenerate minimum at the origin. Given $\epsilon > 0$ the estimates

$$(\beta - \epsilon)\phi^2 \leq V(\phi) - V_1 \leq (\beta + \epsilon)\phi^2 \tag{11}$$
$$(\beta - \epsilon)\phi^2 \leq \phi V'(\phi) \leq (\beta + \epsilon)\phi^2 \tag{12}$$

hold for ϕ sufficiently small. A disadvantage of (8) is that only one of the terms differentiated on the left hand side appears on the right hand side. If the other term also appeared we could hope to prove a rate of decay for the energy-like quantity on the left hand side. As it is we can only show that this quantity is not increasing, The situation can be improved by modifying the energy, adding a term $\alpha\phi\dot{\phi}$ for a suitable constant α. This device has

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previously been used in studying solutions of the Einstein equations in [4] (section 10) and [20] (section 4).

$$d/dt[\dot{\phi}^2/2 + (V(\phi) - V_1) + \alpha\phi\dot{\phi}]$$
$$= (-3H + \alpha)\dot{\phi}^2 - 3\alpha H\phi\dot{\phi} - \alpha\phi V'(\phi) \qquad (13)$$

Note that

$$|\phi\dot{\phi}| \leq \frac{1}{2}\dot{\phi}^2 + \frac{1}{2(\beta - \epsilon)}(V(\phi) - V_1) \leq C\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) - V_1\right) \qquad (14)$$

for a suitable constant C. It can be concluded that for α sufficiently small

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) - V_1 \leq C\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) - V_1 + \alpha\phi\dot{\phi}\right) \qquad (15)$$

For α small enough it can also be concluded that

$$(-3H + \alpha)\dot{\phi}^2 - 3\alpha H\phi\dot{\phi} - \alpha\phi V'(\phi) \leq -C(\dot{\phi}^2 + V(\phi) - V_1 + \alpha\phi\dot{\phi}) \qquad (16)$$

for a positive constant C. Putting these facts together shows that $\dot{\phi}$ and $V(\phi) - V_1$ decay exponentially for $t \to \infty$. This in turn implies that the convergence of H to its limit is exponential. With this information in hand it is possible to proceed as in [10] to show that $g_{ab} = e^{2H_1 t}(g_{ab}^0 + O(e^{-\delta H_1 t}))$ for some $\delta > 0$, $H_1 = \sqrt{8\pi V_1/3}$, and a metric g_{ab}^0 not depending on time. The inverse metric has a corresponding expansion. In the course of the proof it is also shown that $\sigma^a{}_b = O(e^{-H_1 t})$. The following has been shown:

Theorem 2 Consider a solution of the Einstein equations satisfying the hypotheses of Theorem 1 for which $\phi \to \phi_1$ as $t \to \infty$. If $V''(\phi_1) > 0$ then as $t \to \infty$ the quantities $\dot{\phi}$, $V(\phi) - V(\phi_1)$ and $H - H_1$ decay exponentially as $t \to \infty$ while $e^{-2H_1 t} g_{ab}$ converges to a limit. Here $H_1 = \sqrt{8\pi V(\phi_1)/3}$.

5 Perfect fluids and collisionless matter

All the results above have been based on assuming a solution which exists globally to the future. To go beyond this it is necessary to introduce a specific matter model and to prove a global existence theorem for it. After that the asymptotics of the matter fields can be studied in more detail. Consider for a moment the case of a cosmological constant instead of a scalar field. In

that case global existence and asymptotic behaviour have been analysed in [10] for collisionless matter described by the Vlasov equation. For an untilted perfect fluid with a linear equation of state and a cosmological constant a dynamical systems analysis of models of Bianchi class A has been carried out in [5]. In the following analyses for a nonlinear scalar field will be carried out for both perfect fluids with linear equation of state and collisionless matter.

Now the assumptions already made will be specialized to the case where $T^M_{\alpha\beta}$ is the energy-momentum of a perfect fluid with linear equation of state or a collisionless gas. For background on the latter case see [17]. Before tackling the scalar field let us look at the case of a cosmological constant and perfect fluid as a matter model. Thus

$$T^M_{\alpha\beta} = (\mu + p)u_\alpha u_\beta + pg_{\alpha\beta} \tag{17}$$

and $p = (\gamma-1)\mu$. Assume that $1 \leq \gamma < 2$. It will be shown that the solutions exist globally to the future and their asymptotics will be compared with the expansions given by [21]. The basic equations for the fluid, which we take from [18], are

$$
\begin{align}
\rho &= \mu(1 + \gamma|u|^2) \tag{18}\\
j^a &= \gamma\mu(1 + |u|^2)^{1/2}u^a \tag{19}\\
S^a{}_b &= \mu[\gamma u^a u_b + (\gamma - 1)\delta^a_b] \tag{20}
\end{align}
$$

and

$$
\begin{align}
\frac{d}{dt}\rho + 3H\rho + H\mathrm{tr}S &= -\nabla_a j^a + \sigma^a{}_b S^b{}_a \tag{21}\\
\frac{d}{dt}(j^a) + 5Hj^a &= -\nabla^b S^a{}_b + 2\sigma^a{}_b j^b \tag{22}
\end{align}
$$

In order to stop the notation becoming too cluttered the superscript M has been omitted from the matter quantities here.

The Einstein equations with positive cosmological constant coupled to a perfect fluid constitute a system of ordinary differential equations (ODE). Standard ODE theory says that a local solution can be extended as long as g_{ab}, k_{ab}, $(\det g)^{-1}$, ρ, ρ^{-1} and u^a are bounded. If it can be shown that these quantities are bounded for any solution on an interval $[t_0, t_1)$ then global existence follows. It will now be shown that they can be bounded. The procedure used in the proof of Wald's theorem shows that $\sigma_{ab}\sigma^{ab}$ and $\mathrm{tr}k$ are

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bounded. Arguing as in [10], Lemma 1 and 2, then shows that g_{ab}, k_{ab} and $(\det g)^{-1}$ are bounded. With this information it can be seen that $|d\rho/dt|$ can be bounded by $C\rho$ for a suitable constant C. Hence ρ and ρ^{-1} are bounded. Next the evolution equation for j^a can be used to show that j^a is bounded. Since u^a can be expressed as a smooth function of ρ and j^a [18] it follows that u^a is bounded. Thus global existence has been proved.

Let $H_1 = \sqrt{\Lambda/3}$. Consider now the untilted case where, by definition, $u^a = 0$. Then $\rho = \mu$ and μ satisfies the equation $\dot{\mu} = -3\gamma H\mu$. This implies that $d/dt(e^{3\gamma H_1 t}\mu)$ is equal to an exponentially decaying factor times $e^{3\gamma H_1 t}\mu$ and it follows that $\mu = \mu_0 e^{-3\gamma H_1 t} + o(e^{-3\gamma H_1 t})$. In the tilted case it is useful to first look at the behaviour of $|j|$.

$$\frac{d}{dt}(|j|^2) = -8H|j|^2 + 2\sigma_{ab}j^a j^b - 2j_a \nabla^b S^a{}_b \tag{23}$$

Note the inequality $|u^a| \leq |u|e^{-H_1 t}$ where the expression on the left denotes the modulus of the components while that on the right denotes the length of the vector with respect to the spatial metric. A similar relation holds for any vector. Now

$$\nabla^b S^a{}_b = \gamma \nabla_b u^a(\mu u^a) + \gamma \nabla_b u^b(\mu u^a) \tag{24}$$

Putting together these facts shows that the last term in (23) can be bounded by $C|j|^2 e^{-H_1 t}$. Since the second term on the right hand side of (23 admits a similar bound it can be concluded that $|j| = \bar{j}e^{-4H_1 t} + o(e^{-4H_1 t})$ for some constant $\bar{j}$. Putting this information back into the equation for j^a shows that $j^a = j_0^a e^{-5H_1 t} + o(e^{-5H_1 t})$. The comparison with the expansions of [21] is as follows. The leading terms in the expansion of the geometry have been validated, as have those of the matter quantities in the untilted case. For a tilted fluid the leading order behaviour of j^a has been determined but the behaviour of μ and u^a remains open.

In a spacetime as in the previous section where the matter content is a perfect fluid with linear equation of state it is possible to proceed in a very similar way to what was done for a cosmological constant. Once again the global existence question involves controlling the solution of a system of ODE. To ensure continuation of a solution it is necessary to control the quantities listed previously together with ϕ and $\dot{\phi}$. The boundedness of $\dot{\phi}$ follows from that of H and the boundedness of ϕ is an easy consequence. From this point on the arguments used in the case of a cosmological constant apply and the same statements about asymptotics are obtained.

When matter is described by the Vlasov equation the asymptotics for a spacetime with positive cosmological constant have been determined by Lee [10]. For a spacetime with scalar field as in the last section global existence holds [11]. Under the hypotheses of Theorem 2 we obtain a lot of information about the asymptotic behaviour of the solution as $t \to \infty$. The methods of [10] allow further information to be obtained, which will now be summarized briefly. The generalized Kasner exponents converge to $1/3$. The components V_i of the velocity of a particle converge exponentially to a limit. The spacetime is future geodesically complete. Decay rates can be obtained for the components of the energy-momentum tensor. For instance $\rho = O(e^{-3H_1 t})$ and $|j| = O(e^{-4H_1 t})$. These should be compared with the results for dust given above. More details of these methods can be found in [10].

6 Dynamics in the past time direction

A spatially homogeneous spacetime with zero cosmological constant and matter satisfying the strong energy condition which is expanding at one time never exists for an infinite proper time towards the past. This statement is related to the Hawking singularity theorem and is easy to prove directly. It follows from the differential inequality $dH/dt \leq -H^2$ that if H is positive at some time it must blow up in finite time towards the past. If we keep the strong energy condition but introduce a positive cosmological constant there is a solution which exists globally towards the past, namely the de Sitter solution with spatially flat slicing. Its metric is

$$- dt^2 + e^{2H_1 t}(dx^2 + dy^2 + dz^2) \tag{25}$$

In fact it is the only spacetime with these properties, as will now be shown. The Hamiltonian constraint implies that $H^2 \geq \Lambda/3$. If equality holds in this relation at some time then, since H is non-increasing it must hold at all earlier times. This equality implies that $\rho^M = 0$, so that the solution is vacuum, and that $\sigma_{ab} = 0$, so that it is isotropic. Furthermore $R = 0$, so that it is spatially flat. Hence the spacetime is de Sitter. The remaining case to be considered is that where $H^2 > \Lambda/3$ at some time. Then this also holds at all earlier times. There exists $\epsilon > 0$ such that $H^2(1 - \epsilon) > \Lambda/3$ and hence $dH/dt \leq -\epsilon H^2$. As a consequence H blows up at a finite time in the past.

Although the de Sitter spacetime in the form given above exists for an infinite amount of proper time towards the past, it is not past geodesically

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complete. These coordinates only cover half of de Sitter space. As $t \to -\infty$ a Cauchy horizon is approached. In the case where the hypersurfaces of constant time can be compactified by factoring out by a discrete group of isometries these results follow from general theorems of Andersson and Galloway [2] which do not require any symmetry assumptions.

In the case of a scalar field the Hamiltonian constraint implies $H^2 \geq (8\pi/3)V(\phi)$. If equality holds at some time t_0 then, as in the case of a cosmological constant, the spacetime is isotropic and spatially flat and the matter other than the scalar field vanishes. In addition, $\dot{\phi} = 0$ for $t = t_0$. The unique solution with these data is the de Sitter solution with a scalar field which is time independent. The evolution equation for ϕ implies that $V'(\phi) = 0$. If V has critical points solutions of this type exist and otherwise there are no such solutions. The geometry is simply the de Sitter geometry, with the constant scalar field replacing the cosmological constant. Let us call these pseudo-de Sitter solutions. Suppose now that equality never holds and that the solution exists globally to the past with the potential remaining bounded. Under these circumstances several of the steps in the analysis of the expanding direction can be repeated.

Let V_2 be an upper bound for $V(\phi)$ in the past time direction. If H^2 ever exceeded $(8\pi/3)V_2$ then the argument used in the case of a cosmological constant would prove the existence of a singularity after a finite time to the past. Thus under the given assumptions on the solution H is globally bounded towards the past. It can be concluded successively that $\dot{\phi}$, $V'(\phi)$ and $\ddot{\phi}$ are bounded and that $\dot{\phi} \to 0$ as $t \to -\infty$. Then $V(\phi)$ converges to a limit V_1. If V_1 were less than $3/8\pi$ times the limit of H^2 then there would have to be a singularity, a contradiction. The inequality in the opposite direction follows from the Hamiltonian constraint. Hence H^2 converges to $(8\pi/3)V_1$ as $t \to -\infty$. The quantities $\sigma_{ab}\sigma^{ab}$, R and ρ^M converge to zero. The evolution equation for ϕ implies that $V'(\phi) \to 0$.

Do any solutions of the type just discussed actually exist? This can be answered in the affirmative using a dynamical systems analysis due to Foster [7]. In the case where the potential has a local maximum at ϕ_1 there are solutions for which $\phi \to \phi_1$ as $t \to -\infty$. This represents an instability of the pseudo-de Sitter solutions and has been called the deflationary universe by Barrow [3].

It follows from (8) that the limit ϕ_1 cannot be a strict local minimum of the potential. If the potential is such that all critical points of V are strict local minima then it can be concluded that for any solution which exists

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globally to the past the scalar field cannot be bounded at early times.

It would be interesting to have further information about the nature of the singularity in the past in the case that H does blow up in finite time. In the isotropic and spatially flat case it is known that there are many solutions where the potential becomes negligible near the singularity and the dynamics resembles that for a massless linear scalar field. Whether all finite-time singularities in spatially homogeneous solutions of the Einstein equations coupled to a nonlinear scalar field are of this type is not known.

7   Conclusions

Let us sum up what has been learned about scalar fields whose potential has a positive lower bound. Wald's theorem about the asymptotics of homogeneous spacetimes with positive cosmological constant which exist globally in the future has been generalized to this class of scalar fields. This is a cosmic no hair theorem for this setting. It confirms the intuitive picture where the scalar field rolls down to a minimum of the potential. Under a non-degeneracy assumption on the minimum these asymptotics can be refined. For some interesting phenomenological matter models (perfect fluid with a linear equation of state, collisionless matter) solutions exist globally in the future and various aspects of the asymptotic behaviour of geometry and matter can be determined. For collisionless matter and untilted perfect fluids the picture established is rather complete. For tilted perfect fluids only partial results were obtained.

As stated in the introduction the present investigation is the first step in obtaining a mathematical understanding of the dynamical effects of nonlinear scalar fields. The next case which it would be natural to look at is that of power-law and intermediate inflation. This would mean considering a potential whose lower bound is zero but for which this lower bound is not attained for any finite value of ϕ. For power-law inflation, where $V(\phi) = e^{-\lambda\phi}$, results are available in [8] and [9]. The proofs are quite intricate and inflationary behaviour only occurs if λ is not too large. Intuitive considerations suggest that there should be generalizations of the results of this paper under the assumption that the potential does not fall off too fast at infinity. It is to be expected that the behaviour of the spacetime at late times is monotone and that the accelerated expansion continues forever.

A more complicated case is that of chaotic inflation. The standard po-

tential is $V(\phi) = m^2\phi^2/2$ but similar effects are expected for any potential which is zero for some finite value of ϕ. In this case the accelerated expansion does not last for ever and at late times the scalar field oscillates. In order to control the dynamics it seems that some kind of averaging techniques would be necessary. A positive cosmological constant simplifies the late-time dynamics of solutions of the Einstein equations and a potential with a positive lower bound does the same. This allows all Bianchi types I-VIII to be handled in a unified way. This favourable situation can be expected to persist for power-law and intermediate inflation. For chaotic inflation, however, the late-time asymptotics is likely to be much more like that found when the cosmological constant vanishes and there is no nonlinear scalar field. Then the different Bianchi types have individual behaviour and much less is known. For an example of what can happen, see [19].

Of course eventually we would like to handle inhomogeneous spacetimes with as little symmetry as possible. In [22] and [23] the asymptotics of plane symmetric solutions of the Einstein-Vlasov system with positive cosmological constant were determined. Comparable results for spacetimes with a scalar field are not available. The asymptotics of spacetimes with positive cosmological constant which are general (i.e. without symmetries) has been discussed in [18]. Presumably a similar discussion in possible for scalar fields with a positive minimum or for power-law inflation. Formal series solutions for the latter case have been written down in [15].

In conclusion, it is clear from the above discussion that cosmological models with accelerated expansion give rise to many interesting challenges for mathematical relativity. The dynamics of these spacetimes should be examined under general hypotheses and the results compared wherever possible with the conclusions drawn from the rapidly growing body of observational data.

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