Black hole initial data from a non-conformal decomposition

Nigel T. Bishop,[1, 2] Florian Beyer,[2] and Michael Koppitz[2]

[1]*Department of Mathematics, Applied Mathematics and Astronomy,*
University of South Africa, P.O. Box 392, Pretoria 0003, South Africa
[2]*Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476, Golm, Germany*

We present an alternative approach to setting initial data in general relativity. We do not use a conformal decomposition, but instead express the 3-metric in terms of a given unit vector field and one unknown scalar field. In the case of axisymmetry, we have written a program to solve the resulting nonlinear elliptic equation. We have obtained solutions, both numerically and from a linearized analytic method, for a general perturbation of Schwarzschild.

I. INTRODUCTION

Interferometry antennas (e.g. LIGO [1]) all over the world are going online to measure very fine dynamical variations of spacetime curvature known as gravitational waves. It is hoped that this will serve as a further evidence concerning Einstein's theory of general relativity. Eventually it might lead to a very new kind of observational astronomy because gravitational waves emitted by astronomical sources reach us more or less unattenuated and give us information when other astronomical sources block the electromagnetic spectrum. Our understanding of gravitational waveforms emitted by various events is limited in the strong field case, particularly with processes involving black holes. These are important because they are likely to be powerful and therefore detectable [2]. So for some years there has been a considerable effort to solve the binary black hole collision problem numerically.

In order to solve Einstein's equations numerically one needs initial data, i.e. the description of the state of the system on an initial Cauchy surface. The usual approach is to choose initial data that is conformally flat. The literature on this subject is quite extensive; we do not discuss it in any detail, but instead refer to a recent review [3]. The motivation for using conformally flat initial data as done by e.g. Misner [4] and Bowen and York [5], is mathematical convenience rather than physics. As was pointed out in [6] conformally flat slices of the Kerr metric are not likely to exist. But even in the case of non-spinning black holes it was shown in [7] that astrophysically correct initial data cannot be represented this way.

Conventional wisdom holds that the unphysical part of an initial data set will be radiated away from the system, within the order of the light-crossing time. While this may well be true, the hypothesis has not been well tested. In order to do so, initial data sets constructed by different methods are needed. Of course such methods, including the one presented here, will involve some form of mathematical convenience. Provided the methods are really different, the influence of the precise form of the intial data on the subsequent spacetime evolution, can be investigated.

The Kerr-Schild ansatz which was introduced in [8] and independently in a different form in [9] was a promising ansatz that differs from the conformally flat approach. The case of a particular perturbation has been solved in [8], and subsequent developments have been limited [10]. The motivation for the present work was to find a Kerr-Schild like ansatz for which there is a prospect of being able to find a solution in a variety of circumstances. The ansatz presented here satisfies this condition. It leads to a (nonlinear) elliptic equation in one unknown, and we have written a program to solve the equation. The initial data equation has been solved for a general perturbation of Schwarzschild, using both the numerical program as well as an analytic method that linearizes the problem.

Much use is made of computer algebra in this work, and we have used both Maple and Mathematica so as to provide a check on our calculations.

We first present our ansatz and the constraint equations (Sec. II). Next, in Sec. III, we express initial data for the Schwarzschild geometry in terms of our ansatz. In Sec. IV we use our ansatz to find initial data for a general perturbation of Schwarzschild, solving the problem both numerically and by linearization. Sec. V summarizes our results and also discusses possible further work.

II. MODIFICATION OF THE KERR-SCHILD ANSATZ

We attempted to modify the Kerr-Schild ansatz so that the constraint equations would reduce to a single elliptic equation – because then, under many circumstances, a solution to the problem can be expected to exist. The simplest way to do this is to require that the extrinsic curvature be zero

$$K_{ij} = 0 \qquad (2.1)$$

and the momentum constraints are automatically satisfied. The Hamiltonian constraint reduces to

$$^{(3)}R = 0, \qquad (2.2)$$

and we used computer algebra to investigate the nature of Eq. (2.2) for various forms of the 3-metric that involve

a unit vector field and one scalar function in analogy with [8]. The following ansatz seems to be acceptable

$$\gamma_{ij} = \frac{d_{ij} - 2Vk_ik_j}{1 - 2V} \qquad (2.3)$$

with d_{ij} the Euclidean (flat) 3-metric, V a scalar field, and k_i the normalized gradient of another scalar field

$$k_i = \frac{\Phi_{,i}}{\sqrt{d^{ij}\Phi_{,i}\Phi_{,j}}}. \qquad (2.4)$$

It is useful to note a number of points about the ansatz defined by Eqs. (2.3), (2.1) and (2.4):

- Under the circumstances described later, the ansatz leads to a nonlinear elliptic equation for which a solution can be explicitly constructed. Whether the ansatz leads to solutions in more general circumstances is, of course, unknown.

- The assumptions (2.1) and (2.4), were made to simplify the problem. To what extent these assumptions can be relaxed, needs to be investigated. In order for the method to be astrophysically useful, some relaxation will be necessary so as to be able to allow for situations where there is momentum and spin.

We have evaluated the Hamiltonian constraint in the case of axisymmetry, i.e. $x^i = (r, \theta, \varphi)$, $d_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$, and with V and Φ depending only on r and θ). The reason for the restriction to axisymmetry was problem simplification, while at the same time including non-trivial situations such as two black holes and a perturbed Schwarzschild black hole. We used computer algebra to find

$$c_{20}V_{,rr} + c_{11}V_{,r\theta} + c_{02}V_{,\theta\theta} + c_5\left(V_{,r}\right)^2 + c_4V_{,r}V_{,\theta}$$
$$+ c_3\left(V_{,\theta}\right)^2 + c_2V_{,\theta} + c_1V_{,r} + c_0V = 0. \qquad (2.5)$$

The coefficients are functions of r, θ, V, Φ and higher derivatives of Φ. Some of the coefficients are very long expressions, and so here we give only the coefficients of the principal part

$$c_{20} = -2\frac{(1 - 2V)\,\Phi_{,\theta}^2 + 2\,r^2\,\Phi_{,r}^2}{(1 - 2V)\left(\Phi_{,\theta}^2 + r^2\,\Phi_{,r}^2\right)}$$

$$c_{11} = -4\frac{(1 + 2V)\,\Phi_{,\theta}\,\Phi_{,r}}{(1 - 2V)\left(\Phi_{,\theta}^2 + r^2\,\Phi_{,r}^2\right)}$$

$$c_{02} = -2\frac{2\,\Phi_{,\theta}^2 + r^2\,(1 - 2V)\,\Phi_{,r}^2}{r^2\,(1 - 2V)\left(\Phi_{,\theta}^2 + r^2\,\Phi_{,r}^2\right)}$$

The determinant of the principal part is

$$\Delta = c_{20}c_{02} - \frac{1}{4}c_{11}^2 = \frac{8}{r^2\,(1 - 2V(r, \theta))},$$

showing that the equation is elliptic provided $V < 1/2$.

III. SCHWARZSCHILD GEOMETRY

We now find an explicit representation of the Schwarzschild geometry that satisfies the ansatz (2.3). This will be used for a number of purposes, including the setting of boundary data in more general situations, and being the zeroth order solution about which a perturbed Schwarzschild solution will be constructed.

Consider the Schwarzschild 3-metric in isotropic coordinates $(\bar{r}, \theta, \varphi)$

$$ds^2 = \left(1 + \frac{m}{2\bar{r}}\right)^4\left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\varphi^2\right) \qquad (3.1)$$

with the extrinsic curvature of the spacelike initial slice $K_{ij} = 0$. We can obtain the form (2.3) by requiring

$$\left(1 + \frac{m}{2\bar{r}}\right)^2 d\bar{r} = dr \qquad (3.2)$$

which is easily integrated to give the coordinate transformation

$$r = \bar{r} + m\ln\frac{2\bar{r}}{m} - \frac{m^2}{4\bar{r}} + \frac{m}{2}. \qquad (3.3)$$

where the integration constant has been chosen so that the event horizon is at $r = \bar{r} = m/2$. Then in (r, θ, φ) coordinates the metric satisfies ansatz (2.3) with

$$\Phi = \frac{1}{r}, \quad V(r) = V_S(r) \equiv \frac{1}{2}\left(1 - \left(1 + \frac{m}{2\bar{r}}\right)^{-4}\frac{r^2}{\bar{r}^2}\right) \qquad (3.4)$$

On the event horizon at $r = m/2$ we have

$$V = \frac{15}{32}, \qquad (3.5)$$

and also we can make an asymptotic expansion to find

$$\lim_{r\to\infty} V(r) = \frac{m}{2r}\left(1 - 2\ln\frac{2r}{m}\right). \qquad (3.6)$$

As an additional check on our calculations, we substituted Eq. (3.4) into Eq. (2.5), obtaining $0 = 0$ as expected.

IV. PERTURBATION OF THE SCHWARZSCHILD METRIC

We now explore the case of a perturbed single Schwarzschild black hole with mass $m = 1$. We set $\Phi = 1/r$ and use the simple inner boundary condition

$$V\left(r = \frac{1}{2}, \theta\right) = \frac{15}{32} + \epsilon\,P_n(\cos\theta) \qquad (4.1)$$

to perturb the black hole where P_n is the n-th Legendre polynomial. In principle, the size of ϵ is only limited by

the fact that $V < 1/2$ to preserve ellipticity of Eq. (2.5), so

$$|\epsilon| < \frac{1}{32}.$$

At the analytic level, the outer boundary condition is $V(r, \theta) \to V_S(r)$ as $r \to \infty$, but computationally we set $V(r, \theta)$ to $V_S(r)$ at $r = r_{OB}$ which makes physical sense when r_{OB} is large.

We solve the problem in two ways. First, we linearize, that is we ignore all terms in Eq. (2.5) of order ϵ^2, and we are able to write the solution as an infinite sum of eigenfunctions. Second, we develop a numerical method to solve directly the elliptic problem Eq. (2.5), and use the linearized solution as an analytic solution against which the numerical method can be validated. The reason for doing this is that we expect, in future work, to apply the numerical method to problems for which an approximate analytic solution cannot be found.

A. Linearized Analysis

If the perturbation introduced in Eq. (4.1) is small enough, i.e. $\epsilon \ll 1$, it is a good approximation to neglect terms of order ϵ^2 and higher. In Sec. IV B we consider how small ϵ should be.

It turns out that the ansatz

$$\Phi = 1/r, \quad V(r, \theta) = V_S(r) + \epsilon w_n(r) P_n(\cos \theta) \quad (4.2)$$

separates the linearized Hamiltonian constraint and we are left with an ordinary differential equation for $w_n(r)$

$$d_1 w_n''(r) + d_2 w_n'(r) + d_{3(n)} w_n(r) = 0 \quad (4.3)$$

The coefficients were found using computer algebra

$$
\begin{aligned}
d_1 =& \, r^2 \left(-1 + 2 V_S(r)\right)^2 \\
d_2 =& -r \left(-1 + 2 V_S(r)\right) \left(3 - 6 V_S(r) + 7 r V_S'(r)\right) \\
d_{3(n)} =& \, 1 - \frac{n(n+1)}{2} + (3n(n+1) - 8) V_S(r) \\
& - 2 \left(3n(n+1) - 10\right) V_S(r)^2 \\
& + 4 \left(n(n+1) - 4\right) V_S(r)^3 + 7 r^2 V_S'(r)^2.
\end{aligned}
$$

The fact, that the Schwarzschild solution $V_S(r)$ (Eq. (3.4)) is itself a solution of the Hamiltonian constraint, has been used. We will define the functions $w_n(r)$ to be those solutions of (4.3) which satisfy the boundary conditions

$$w_n\left(r = \frac{1}{2}\right) = 1, \quad w_n(r \to \infty) = 0, \quad (4.4)$$

(although, in numerical calculations, it will be convenient to impose the outer boundary condition at finite r).

Of course, a general solution to the perturbed Schwarzschild problem can be constructed by summing the eigenfunctions, i.e.

$$V(r, \theta) = V_S(r) + \epsilon \sum_{n=1}^{\infty} a_n w_n(r) P_n(\cos \theta) \quad (4.5)$$

where the a_n are arbitrary constants.

1. The York tensor

In order to show that the 3-metric defined by Eq (4.2) is not conformally flat, we need to prove that the York tensor [11, 12]

$$Y_{ijk} = R_{ij;k} - R_{ik;j} + \frac{1}{4}(R_{,j} g_{ik} - R_{,k} g_{ij}), \quad (4.6)$$

does not vanish identically. We have used computer algebra to find Y_{ijk}: if only the first eigenfunction is present (i.e., if $n = 1$) then the York tensor is zero to first order in ϵ; but, for example,

$$Y_{112} = \epsilon \frac{2 - n(n+1)}{4 r^2} w_n(r) P_n'(\cos \theta).$$

So it is proven that the eigenfunctions with $n > 1$ do not represent a conformally flat geometry.

B. Numerical Computation

1. Implementation

We want to solve Eq. (2.5), subject to the boundary condition defined by Eq. (4.1) and

$$V(r, \theta)|_{r=r_{OB}} = V_S(r)$$

with $V_S(r)$ given by Eq. (3.4), with a standard numerical elliptic solver. The computations were done using the Cactus-Computational-Toolkit [13] and the TAT-Jacobi elliptic solver [14] using the Jacobi method [15] implementing Eq. (2.5) and the boundary conditions by means of second order finite differencing.

Due to the symmetry of the problem we impose the additional boundary conditions

$$\left. \frac{\partial V}{\partial n} \right|_{\theta = 0, \pi} = 0$$

where n is normal to the $\theta = 0, \pi$-surfaces. For simplicity we implemented these conditions at $\theta = 0 + \eta, \pi - \eta$ with $\eta \ll 1$ and of the order of magnitude of the accuracy up to which Eq. (2.5) is to be solved. This is a common procedure to avoid numerical problems at the singular points of the equation ($\theta = 0, \pi$) – for example, the same situation arises in the case of the Laplace equation in spherical coordinates.
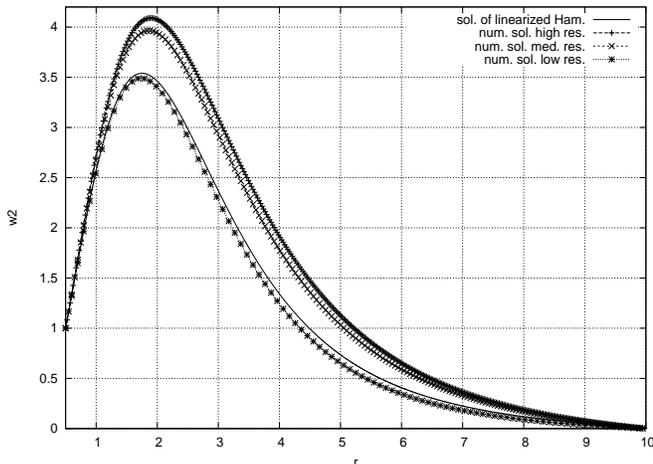
FIG. 1: Linearized and full numerical solutions for $n = 2$, $\theta = 0.59$

FIG. 2: Confirmation of second order convergence in the case $n = 2$, $\theta = 0.59$

To solve non-linear elliptic PDEs using the Jacobi method the choice of the initial guess for $V(r, \theta)$ is crucial. It turns out to be sufficient to use Eq. (3.4) plus a Legendre perturbation of a given order n with linearly decreasing amplitude from the inner to the outer boundary consistent with the boundary conditions above. To compute $V_S(r)$ for the initial guess numerically, Eq. (3.3) was solved for $\bar{r}$ by a numerical integration of Eq. (3.2) which was then substituted into Eq. (3.4) – the problem being that Eq. (3.3) is explicit in the wrong direction.

The convergence of TAT-Jacobi is very slow mainly due to non-linear terms in Eq. (2.5) which are dominant close to the horizon. So we first run the elliptic solver with the residual multiplied by $(1 - 2V)$, which is small near the horizon. This gives the solver the opportunity to get an accurate solution everywhere else before, in a second run, we solve the original equation. By means of this technique the convergence speed was significantly increased.

2. Results

We investigate the case $n = 2$ with the outer boundary at $r_{OB} = 10$ and $\epsilon = 0.005$. We show that three different resolutions exhibit second order convergence, and compare the numerical results to the linearized ones of Sec. IV A. The next table describes the different resolutions used. With *final residual* we mean the residual of Eq. (2.5) after the elliptic solver has finished.

	N_r	N_θ	Δ_r	Δ_θ	final resid. of Eq. (2.5)
low	97	33	0.098	0.095	$4.0 \cdot 10^{-7}$
medium	193	65	0.049	0.048	$1.0 \cdot 10^{-7}$
high	385	129	0.025	0.024	$2.5 \cdot 10^{-8}$

Fig. 1 shows the full numerical results together with the linearized one at $\theta = 0.59$. In the numerical case,
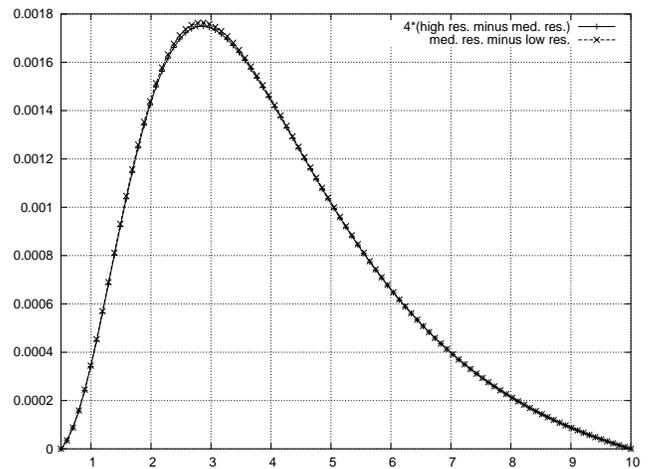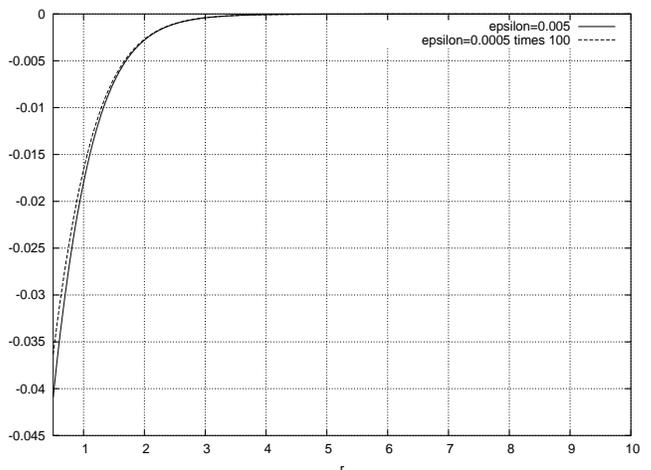

FIG. 3: Residuum of Eq. (2.5) for the linearized solution

we plot the difference $V(r, \theta) - V_S(r)$ and normalized to unity at $r = \frac{1}{2}$. The graphs suggest second order convergence which is confirmed by Fig. 2. It is obvious that the linearized solution is not the limit of infinite resolution.

We have thus shown that our numerics show the right convergence, but it must still be understood why the linearized solution deviates so much from the numerical ones. The answer is that higher order terms get large close to the horizon. Using computer algebra, we substituted the linearized solution which fulfills the linearized Hamiltonian constraint Eq. (4.3) up to an error of 10^{-6} into Eq. (2.5). The residual is shown in Fig. 3, and we also show $100 \times$ the residual when ϵ is smaller by a factor of 10, i.e. $\epsilon = 0.0005$. Although the linearized solution is not accurate close to the boundaries for $\epsilon = 0.005$, the residual shows the right quadratic scaling for decreasing ϵ due to second order terms. This gives us a measure of how small ϵ should be in order to obtain a desired accuracy.
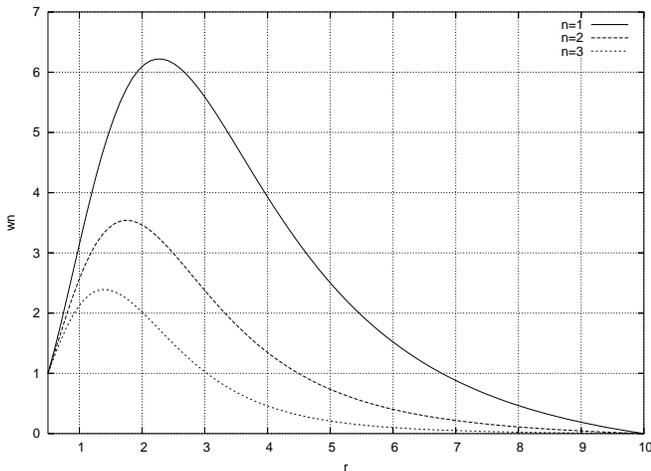
FIG. 4: Linearized solutions for different n

The computations were repeated for different positions of the boundaries and different n, recovering convergence as before. Fig. 4 shows the linearized solutions – i.e. full solutions provided ϵ is small enough – for different n.

V. CONCLUSIONS

The paper has presented a new method of finding initial data for the Einstein equations. The method is not based upon a conformal decomposition, but instead a unit vector field is chosen and then the Hamiltonian constraint reduces to a partial differential equation in one unknown. At least in the case considered here – zero extrinsic curvature, axisymmetry, and with the vector field being a normalized gradient of a scalar field – the Hamiltonian constraint is elliptic, and we have developed a numerical programme for solving this equation. We have tested our numerical routine with boundary data corresponding to both the unperturbed and perturbed Schwarzschild solutions, and the observed second order convergence is a validation of the numerical method.

At the analytic level, we have constructed a coordi-

nate transformation from isotropic Schwarzschild coordinates into coordinates in which the metric satisfies our ansatz. Further, we have investigated small perturbations of Schwarzschild, and have shown that in this case the Hamiltonian constraint is separable so that its solution may be written as an infinite sum of eigenfunctions. The York tensor of the perturbed solution has been found: except for the case of the first eigenfunction, the York tensor is non-zero and thus the 3-metric is not conformally flat.

Two immediate items of further work are envisaged

- In the perturbed Schwarzschild, case it will be interesting to evolve the initial data obtained here and compare it to an evolution from initial data obtained from the standard conformal decomposition.

- The two black hole problem needs to be tackled. This will be difficult technically because at some point between the two black holes the unit vector field must be singular, which means that some of the coefficients in Eq. (2.5) will be singular. At the analytic level this simply means that $V = 0$ at the singular point, but the existence of a singularity will complicate the numerics.

The conformal method for finding initial data has been known and refined over many years. Our method requires further development – and in particular a solution to the two black hole problem – before some form of systematic comparison between the two methods would be worthwhile. Even so, the existence of an alternative to the conformal method may be useful in investigating how different initial data sets, representing the same physics, affect spacetime evolution.

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