Higher-order M-theory corrections and the Kac–Moody algebra E_{10}

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Abstract

It has been conjectured that the classical dynamics of M-theory is equivalent to a null geodesic motion in the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of the hyperbolic Kac–Moody group E_{10}. We here provide further evidence for this conjecture by showing that the leading higher-order corrections, quartic in the curvature and related 3-form-dependent terms, correspond to *negative imaginary roots* of E_{10}. The conjecture entails certain predictions for which higher-order corrections are allowed: in particular corrections of type $R^M(DF)^N$ are compatible with E_{10} only for $M+N = 3k+1$. Furthermore, the leading parts of the R^4, R^7, ... terms are predicted to be associated with *singlets* under the $\mathfrak{sl}_{10}$ decomposition of E_{10}. Although singlets are extremely rare among the 4400 752 653 representations of $\mathfrak{sl}_{10}$ appearing in E_{10} up to level $\ell \leqslant 28$, there are indeed singlets at levels $\ell = 10$ and $\ell = 20$ which do match with the R^4 and the expected R^7 corrections. Our analysis indicates a far more complicated behaviour of the theory near the cosmological singularity than suggested by the standard homogeneous *ansätze*.

PACS number: 11.25.Yb

1. Introduction

The analysis, à la Belinskii, Khalatnikov, Lifshitz (BKL) [1], of generic cosmological solutions of $D = 11$ supergravity [2] in the vicinity of a spacelike singularity has revealed a connection with billiard motion in the fundamental Weyl chamber of E_{10} (implying chaotic oscillations of the metric near the singularity) [3–5]. We recall that E_{10} is a rank-10 infinite-dimensional hyperbolic Kac–Moody algebra [6][3] whose root lattice is the canonical hyperbolic extension of the root lattice of the largest exceptional finite-dimensional Lie algebra E_8, and the unique

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[3] For simplicity of notation, E_{10} denotes both the *group*, and its associated Lie *algebra*.

even self-dual Lorentzian lattice $II_{1,9}$ [7]. The cosmological billiard describing the asymptotic behaviour near a spacelike singularity is based on the identification of the ten diagonal metric degrees of freedom of $D = 11$ supergravity with the ten non-compact directions of a Cartan subalgebra (CSA) of E_{10}. This intriguing link between $D = 11$ supergravity and E_{10} was deepened in [8, 9] where it was shown that the bosonic equations of motion of $D = 11$ supergravity at some given spatial point, when restricted to zeroth- and first-order spatial gradients in the metric and the 3-form, can be matched with the equations of motion of a one-dimensional $E_{10}/K(E_{10})\,\sigma$-model restricted to levels $\ell \leqslant 3$. In terms of the heights of the roots of E_{10} involved in these correspondences, the Weyl-chamber billiard retains only roots of height 1 (i.e. simple roots), while the work of [8, 9] has established the correspondence with (null) geodesic motion on the coset space $E_{10}/K(E_{10})$ up to height 29 included. These results underline the potential importance of E_{10}, whose appearance in the reduction of $D = 11$ supergravity to one dimension had been conjectured already long ago in [10, 11], as a candidate symmetry underlying M-theory. A similar, but conceptually different, proposal was made in [12], where E_{11} (or some even larger symmetry containing E_{11} [13]) has been suggested as a fundamental symmetry of M-theory. Let us also note that links between the dynamics of gravitational theories and geodesic motions on certain Lorentzian signature spaces were noticed already in [14, 15]. For recent work in this vein in the context of homogeneous cosmologies (but not related to E_{10}), see [16, 17].

The substantial increase in the height of the roots entering the correspondence between the two dynamics led to the key conjecture of [8], according to which the one-dimensional bosonic $E_{10}/K(E_{10})\,\sigma$-model is equivalent (or 'dual') to the bosonic sector of $D = 11$ supergravity, possibly augmented by further M-theoretic degrees of freedom. To establish this conjecture, one faces two challenges, namely to understand

- how the higher-order spatial gradients are realized in the σ-model, and how the full (untruncated) bosonic sector of $D = 11$ supergravity is thereby embedded into the $E_{10}/K(E_{10})$ coset space dynamics; and

- the meaning and significance of the imaginary (lightlike and timelike) roots of E_{10} and their relation to M-theory degrees of freedom and M-theory corrections of the 'low energy' $D = 11$ supergravity action.

Concerning the first challenge, it was shown in [8] that E_{10} contains three infinite towers of Lie algebra elements, which possess the correct structure for representing the higher spatial gradients of the metric, and of the 'electric' and 'magnetic' components of the 4-form field strength. However, it is still unknown how this identification could work in detail. The progress reported in the present paper concerns the second challenge. Namely, we shall show that certain (partially) known higher-order M-theory corrections to the two-derivative, 'effective' $D = 11$ supergravity Lagrangian can be related, in an approximation which we shall explain, to special imaginary roots of E_{10}. Our 'botanical' approach in searching for new connections between M-theory and the hyperbolic Kac–Moody algebra E_{10} is based on the working hypothesis that *the $E_{10}/K(E_{10})\,\sigma$-model of* [8] *not only describes M-theory degrees of freedom beyond those of $D = 11$ supergravity, but in addition contains hidden information about M-theory corrections to the low energy effective theory (i.e. $D = 11$ supergravity) at arbitrarily high orders*. More specifically, we shall consider the leading (eighth order in derivatives) corrections to the usual supergravity Lagrangian and exhibit their connection with certain imaginary roots of E_{10}. Different routes for interpreting the imaginary roots of E_{10} in terms of brane dynamics have been suggested in [18], and, in the framework of the E_{11} proposal of [12], in [19, 20].

In [5] it was shown that the gravitational Hamiltonian, or more precisely the Hamiltonian constraint at a given spatial point, can be written in the form

$$\mathcal{H}(\beta^a, \pi_a, Q, P) = \tilde{N}\left[\frac{1}{2}G^{ab}\pi_a\pi_b + \sum_A c_A(Q, P, \partial\beta, \partial Q)\exp(-2w_A(\beta))\right] \tag{1}$$

with the rescaled lapse $\tilde{N} \equiv N/\sqrt{g}$, where g is the determinant of the *spatial* metric. Here β^a, π_a are the canonical variables corresponding to the diagonal (spatial) metric degrees of freedom, and G^{ab} the (Lorentzian) 'superspace' metric induced by the Einstein–Hilbert action. (Q, P) denote the remaining canonical degrees of freedom associated with off-diagonal metric and various matter degrees of freedom, and $\partial\beta$, ∂Q denote their *spatial* gradients. The sum runs over various exponential potential 'walls', which are indexed by A. The above form of the Hamiltonian is in particular valid for the bosonic sector of $D = 11$ supergravity, in which case $a, b = 1, \ldots, 10$.

On the other hand, for the one-dimensional 'geodesic' σ-model over the infinite-dimensional coset space $E_{10}/K(E_{10})$, an analogous expression for the Hamiltonian constraint was also derived in [5], namely

$$H(\beta^a, \pi_a, \nu, p) = n\left[\frac{1}{2}G^{ab}\pi_a\pi_b + \sum_{\alpha\in\Delta_+}\sum_{s=1}^{\text{mult}(\alpha)}(\Pi_{\alpha,s}(\nu, p))^2\exp(-2\alpha(\beta))\right] \tag{2}$$

where (β^a, π_a) now denote the ten diagonal CSA degrees of freedom of E_{10}, and (ν, p) stand for infinitely many off-diagonal (Iwasawa-type) canonical variables, on which the quantities $\Pi_{\alpha,s}$ depend. The metric G^{ab} now denotes the restriction of the unique invariant bilinear form on E_{10} to the CSA [6]. This metric happens to be *identical with the metric appearing in* (1) for the bosonic sector of $D = 11$ supergravity; its explicit form is given in (26). The sum on the rhs of (2) ranges over all positive roots α of E_{10} with their multiplicities [= mult(α)], and the number $\alpha(\beta)$ is the result of applying the linear form (= root) α to the CSA element β. Formula (2) is actually valid for any $G/K(G)$ σ-model over an indefinite Kac–Moody group G with 'maximal compact subgroup' $K(G)$. Varying (2) w.r.t. the (new) lapse n implies that the resulting solution is a *null geodesic* on this infinite-dimensional coset manifold.

The similarity of (1) and (2) is obvious, but the coincidence of the two expressions and of the associated bosonic equations of motion has so far only been established for a limited number of terms. The first check consists in matching those terms in (1) and (2) which dominate the dynamics near a cosmological singularity. In this *leading BKL approximation*, one can prove [5] that both the 'off-diagonal' degrees of freedom (Q, P) in (1) and (ν, p) in (2) 'freeze' as one approaches the singular initial hypersurface $T = 0$. (Here T denotes the proper time. We shall reserve the letter t to denote the coordinate time in the gauge $\tilde{N} = 1$, i.e. $N = \sqrt{g}$. This coordinate time goes to $+\infty$ as $T \to 0$.) The precise form of the coefficient functions $c_A(Q, P, \partial\beta, \partial Q)$, which is very complicated, does not matter in this BKL limit. What matters is that the coefficients of the *leading contributions* are all *non-negative*. That is, we have $c_{A'} \geqslant 0$, where the primed index A' labels the leading terms. This implies that, for large β, the Hamiltonian (1) takes the limiting form

$$\mathcal{H}_\infty(\beta^a, \pi_a) = \frac{1}{2}G^{ab}\pi_a\pi_b + \sum_{A'} c_{A'}\exp(-2w_{A'}(\beta)) \tag{3}$$

where the sum is only over the 'dominant walls'. It was also shown that in this limit the exponential (Toda) walls appearing in (3) can be replaced by infinitely high ('sharp') walls. Technically, this means that one can replace the exponential functions in (3) by an infinite step function Θ [5]. The billiard then takes place in the chamber defined by $\Theta(-2w_{A'}(\beta)) = 0$,

or equivalently, $w_{A'}(\beta) \geqslant 0$. An analogous argument applies to the coset dynamics (2). The freezing of all the off-diagonal degrees of freedom ν, p implies that the combinations $\Pi_{\alpha,s}(\nu, p)$ also freeze near the singularity, so that the Hamiltonian (2) takes the limiting form (in the gauge $n = 1$)

$$H_{\infty}(\beta^a, \pi_a) = \frac{1}{2} G^{ab} \pi_a \pi_b + \sum_{\alpha_i} \Pi_{\alpha_i}^2 \exp(-2\alpha_i(\beta)) \qquad (4)$$

where the sum over the positive roots of E_{10} can be asymptotically restricted to the subset of leading positive roots α_i. It is easily seen that these 'leading' positive roots are nothing but the simple roots of E_{10}.

The equivalence between the two limiting dynamics (3) and (4) is now a consequence of the fact that the set of dominant supergravity walls $w_{A'}(\beta)$ coincides with the set of simple roots $\alpha_i(\beta)$ of E_{10} [3]. The validity of this equivalence was extended in [8, 9] from the height 1 simple roots to height 29 in E_{10} roots[4]. At this order, one keeps more exponential terms in both (1) and (2). Now, the precise form of the supergravity coefficient functions $c_A(Q, P, \partial\beta, \partial Q)$ *does* matter, and was shown in [8, 9] to correspond precisely to the form of the coset functions $\Pi_{\alpha,s}(\nu, p)$, with a 'dictionary' relating the variables of both dynamics. This matching between the two Hamiltonians was checked at the level of the equations of motion. It is still unclear whether (and how) one can extend the dictionary relating the dynamical variables in both models so as to prove the equivalence between the two Hamiltonians (1) and (2) beyond height 29. It is quite possible that this dictionary becomes spatially non-local in the gravity variables, and that the simple identification assumed above between the diagonal gravitational degrees of freedom β^a, π_a and the E_{10} CSA variables entering (2) must be modified beyond height 29. Such non-trivial changes of variables may be needed, for instance, in order to reconcile the fact that the only negative contribution in (2) comes from the CSA (the first term on the rhs), whereas in the expansion of the gravity Hamiltonian (1) there may arise negative contributions also from (subleading) terms with $c_A < 0$—as follows from considering, for instance, homogeneous spatial geometries with curvature of arbitrary sign.

Our aim here is to investigate what types of wall forms can be formally associated with higher-order terms (linked to quantum corrections in M-theory), notably the leading ones of the type R^4, $R^2(DF)^2$, $R(DF)^3$ and $(DF)^4$, as well as the topological Chern–Simons term $A_3 \wedge R \wedge R \wedge R \wedge R$. Remarkably, we shall find that these terms correspond, in leading approximation, to *time-like* imaginary roots of E_{10}. Recall that the 'length' of a root is calculated by means of a quadratic form G^{ab}, which is the same as the one appearing in (2) (and (1)), and which, for indefinite Kac–Moody algebras, is *Lorentzian*. Spacelike roots, for which $G^{ab}\alpha_a\alpha_b \equiv \alpha^2 > 0$, are called 'real roots' in the mathematical literature, while light-like or time-like ones, for which $\alpha^2 \leqslant 0$, are called 'imaginary roots' [6]. All the roots up to height 29 that have been explicitly checked to match between the two Hamiltonians (1) and (2) turned out to be *real* and *positive*. By contrast, we will find here that the leading roots associated with eighth-order derivative corrections are, at once, *imaginary* ($\alpha^2 \leqslant 0$) and *negative* (i.e. of the form $\alpha = -\sum n_i \alpha_i$ with $n_i \geqslant 0$). For the leading R^4 corrections, they lie deep inside the root-space lightcone, with $\alpha^2 = -10$, at height $\mathrm{ht}(\alpha) = -115$, and at level $\ell = -10$.

The appearance of exponential walls $\exp(-2w(\beta))$ associated with *negative roots* in a version of the gravitational Hamiltonian (1) with higher-order corrections included might seem to be in plain contradiction with the structure (2) of the coset Hamiltonian, which contains a sum restricted to *positive roots* $\alpha \in \Delta_+$. However, one must remember that the explicit form

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[4] The *height* of a positive root $\alpha = \sum n_i \alpha_i$ ($n_i \geqslant 0$) is defined as $\mathrm{ht}(\alpha) = \sum n_i$ [21, 6]. Correspondingly, the height of a negative root $\alpha = -\sum n_i \alpha_i$ ($n_i \geqslant 0$) is defined as $-\sum n_i$.

of the coset Hamiltonian strongly depends on the parametrization used for a general coset element $\mathcal{V} \in E_{10}/K(E_{10})$. The form (2) results from an Iwasawa parametrization $\mathcal{V} = \mathcal{D}\mathcal{B}$, where $\mathcal{D}$ is 'diagonal' (i.e. in the Cartan torus of E_{10}) and $\mathcal{B}$ of Borel, or 'upper triangular' type, i.e. obtained by exponentiating a sum of positive-root generators. If instead we were to employ a more complicated parametrization of a generic coset element $\mathcal{V} \in E_{10}/K(E_{10})$, namely involving, besides the exponentiation of infinitely many positive-root generators, the exponentiation of some[5] negative-root generators, one would end up with an Hamiltonian of a more general form than (2). The latter would involve some mixture between positive-root walls $\exp(-2\alpha_A(\beta))$ (with $\alpha_A \in \Delta_+$), and negative-root ones $\exp(+2\alpha_B(\beta))$ (with $\alpha_B \in \Delta_+$). Actually, the mixtures of positive-root walls and negative-root ones arising in such non-Borel parametrizations are rather complicated, and do not reduce to a decoupled sum of normal exponentials $\exp(-2\alpha_A(\beta))$, and inverse ones $\exp(+2\alpha_B(\beta))$. A non-Borel parametrization for the root α_B tends to generate superpositions of effects proportional to both $\exp(+2\alpha_B(\beta))$ and $\exp(-2\alpha_B(\beta))$. Therefore, the wrong-sign walls $\sim \exp(+2\alpha_B(\beta))$ can be expected to dominate over the usual-sign ones only in some special corner of the dynamical space. Such an approximate dominance is enough for our purposes here, because we shall see below that the negative-root walls associated with R^4 terms can only be meaningfully considered in some intermediate domain of the dynamical evolution.

No doubt there is still a long way towards the ultimate goal of proving the equivalence between the (essentially unique) E_{10} coset dynamics (2) and the full effective (bosonic) supergravity action (involving the quantum effect of further M-theoretic degrees of freedom), but we hope that the results reported here will allow one to make new and stringent tests of our basic conjecture, and to make new predictions concerning both the structure of higher-order corrections in supergravity and M-theory, which are completely inaccessible by conventional methods, and the existence of special $\mathfrak{sl}_{10}$-representations in the level decomposition of E_{10}.

2. Asymptotics of frames and scale factors

When working with $D = 11$ supergravity fields, we use Lorentz (tangent space) tensors throughout, with the 'mostly plus' metric $\eta^{AB} = (- + \cdots +)$, and the following index conventions:

$$
\begin{aligned}
\text{Flat spacetime indices}: \quad & A, B, C, \ldots \in \{0, 1, \ldots, 10\} \\
\text{Flat spatial indices}: \quad & a, b, c, \ldots \in \{1, \ldots, 10\}.
\end{aligned}
\tag{5}
$$

When needed, the coordinate spacetime indices will be denoted as M, N, P, Q and the coordinate space ones as m, n, p, q.

The Lorentz covariant derivative is defined in the usual manner on any Lorentz vector V_A via

$$
D_A V_B := \partial_A V_B + \omega_{AB}{}^C V_C
\tag{6}
$$

where ∂_A is a (non-commuting) frame derivative linked to the usual (commuting) spacetime derivative ∂_M through $\partial_A \equiv E_A{}^M \partial_M$, where $E_A{}^M$ is the elfbein. The spin connection is given by the standard formula

$$
\omega_{ABC} = \tfrac{1}{2}(\Omega_{ABC} - \Omega_{BCA} + \Omega_{CAB}) = -\omega_{ACB}
\tag{7}
$$

[5] We note that such 'mixed' non-triangular parametrizations of infinite-dimensional Kac–Moody coset spaces have so far not been considered in the literature. It seems likely that, for the exponentiation to be well defined for hyperbolic Kac–Moody groups, one can only have a sparse number of negative-root generators mixed with an infinite number of positive-root ones.

in terms of the coefficients of anholonomicity

$$\Omega_{AB}{}^C := E_A{}^M E_B{}^N \left(\partial_M E_N{}^C - \partial_N E_M{}^C \right) = -\Omega_{BA}{}^C \tag{8}$$

where $E_M{}^A$ is the co-frame, i.e. the inverse of the elfbein $E_A{}^M$. Expressed in flat indices, the Riemann tensor is

$$R_{ABCD} = \partial_A \omega_{BCD} - \partial_B \omega_{ACD} + \Omega_{AB}{}^E \omega_{ECD} + \omega_{AC}{}^E \omega_{BED} - \omega_{BC}{}^E \omega_{AED}. \tag{9}$$

Likewise, for the 4-form field strength we will also use flat indices such that $F_{ABCD} = E_A{}^M E_B{}^N E_C{}^P E_D{}^Q F_{MNPQ}$, etc.

For the further analysis, we will rely on the following (zero-shift) 1+10 split of the elfbein

$$E_M{}^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix} \tag{10}$$

with the spatial (inverse) zehnbein $e_m{}^a$, and adopt the gauge

$$N = \sqrt{g} \equiv \det e_m{}^a. \tag{11}$$

As explained in [5], and as is also obvious from the definition of $\tilde{N}$ in (1), one consequence of this choice is that the leading kinetic term in (1) simplifies to $\propto G^{ab} \pi_a \pi_b$. Splitting indices into $A = 0$ and $A = a$, where 0 denotes a 'flat' (proper) time index and a a flat spatial index, the coefficients of anholonomicity become

$$\begin{aligned} \Omega_{abc} &= 2 e_{[a}{}^m e_{b]}{}^n \partial_m e_{nc}, \\ \Omega_{0bc} &= N^{-1} e_b{}^n \partial_t e_{nc}, \\ \Omega_{a00} &= \omega_{00a} = -e_a{}^m N^{-1} \partial_m N = -e_a{}^m e^{-1} \partial_m e \end{aligned} \tag{12}$$

with all other coefficients of anholonomicity vanishing. From the above formulae we obtain in particular

$$\omega_{0bc} = N^{-1} e_{[b}{}^n \partial_t e_{nc]}, \qquad \omega_{ab0} = N^{-1} e_{(a}{}^n \partial_t e_{nb)} \tag{13}$$

where $(\cdots)$ and $[\cdots]$ denote symmmetrization and antisymmetrization with strength 1, respectively.

In what follows, we will analyse the leading asymptotic behaviour of various fields near $T = 0$ as functions of the logarithmic scale factors β. To this aim we introduce, as an intermediate object, the Iwasawa co-frame $\theta_m{}^a \equiv \mathcal{N}_m^a$, where $\mathcal{N}_m^a$ is the upper triangular matrix entering the Iwasawa decomposition of the metric g_{mn} that was introduced in [5]. (This triangular matrix has ones on the diagonal, so $\det \theta = 1$.) In terms of this intermediate frame, the ten-dimensional spatial Cartan moving frame (zehnbein) is given by (no summation on a)

$$e_m{}^a = e^{-\beta^a} \theta_m{}^a \tag{14}$$

in each spatial section $t = \text{const.}$ (recall that we use a zero-shift gauge). The line element then reads

$$ds^2 = -(N \, dt)^2 + \sum_a (e^{-\beta^a} \theta^a)^2. \tag{15}$$

The decomposition (14) of the zehnbein is most useful in analysing the small T behaviour of the gravity model. More specifically, the dynamical variables β^a and $\theta_m{}^a$ behave very differently as $T \to 0$. While the frame $\theta_m{}^a$ was shown to 'freeze' (i.e. to have a well-defined limit) as $T \to 0$, the logarithmic scale factors β^a (whose time derivatives are proportional to the Kasner exponents) generically diverge towards $+\infty$ near the singular initial ('big bang') spatial hypersurface $T = 0$, while exhibiting chaotic oscillations of BKL type. (Note, in passing, that we never assume the βs to be only functions of time t. All our supergravity

dynamical variables are functions of all 11 coordinates (t, x^m)). With this notation, the gauge condition (11) implies

$$N = \exp\left(-\sum_{a=1}^{10} \beta^a\right) \det\theta = e^{-\sigma}. \tag{16}$$

while the $D = 11$ volume density behaves as

$$E = N\sqrt{g} = e^{-2\sigma}. \tag{17}$$

Here σ denotes the sum of all logarithmic scale factors,

$$\sigma := \sum_{a=1}^{10} \beta^a. \tag{18}$$

This combination will play an important role in the remainder. As $T \to 0$, the volume factor $\sqrt{g} = e^{-\sigma}$ tends towards zero, and σ tends to plus infinity. Similarly, one can exhibit the behaviour of various other fields near the singular initial hypersurface by factoring them into a part that 'freezes' as $T \to 0$ and is independent of the scale factors, and another, dependent on the βs, which diverges and may oscillate near $T = 0$.

Making use of (14), we can compute the asymptotic behaviour of the coefficients of anholonomicity and the spin connection. Suppressing inessential prefactors, we find for the connection components with one (proper) time index

$$\omega_{0bc} \sim 0$$
$$\omega_{ab0} = -N^{-1}\delta_{ab}\partial_t\beta^a + \cdots = -\delta_{ab}\,e^{+\sigma}\,\partial_t\beta^a + \cdots \tag{19}$$

where the symbol $\sim$ indicates that we keep only the leading contributions (from (12) it is easily seen that that the components with *two* time indices are subdominant). For the purely spatial components, we get for the anholonomicity coefficients

$$\Omega_{abc} = e^{\beta^a+\beta^b-\beta^c}\bar\Omega_{abc} + \delta_{ac}\,e^{\beta^b}\,\bar\partial_b\beta^c - \delta_{bc}\,e^{\beta^a}\,\bar\partial_a\beta^c \tag{20}$$

where $\bar\partial_a \equiv \theta_a{}^m\partial_m$, and

$$\bar\Omega_{abc} := 2\theta_{[a}{}^m\theta_{b]}{}^n\partial_m\theta_{nc}. \tag{21}$$

Hence the spatial components of the spin connection are given by a sum of terms:

$$\omega_{abc} = \tfrac{1}{2}\left(e^{\beta^a+\beta^b-\beta^c}\bar\Omega_{abc} - e^{\beta^b+\beta^c-\beta^a}\bar\Omega_{bca} + e^{\beta^c+\beta^a-\beta^b}\bar\Omega_{cab}\right) + \delta_{ac}\,e^{\beta^b}\,\bar\partial_b\beta^c - \delta_{ab}\,e^{\beta^a}\,\bar\partial_c\beta^c. \tag{22}$$

The spatial indices in these and other formulae below always refer to the basic orthonormal frame $e_m{}^a$, whereas the barred quantities associated with the intermediate (non-orthonormal) frame $\theta_m{}^a$ are only used as auxiliary objects.

It is important that the barred quantities $\bar\Omega_{abc}$ and the frame derivatives $\bar\partial_a$ all possess finite limits as $T \to 0$; they correspond to the 'frozen' components in the language of [8]. Hence, for $T \sim 0$

$$\omega_{abc} \propto e^{\beta^a+\beta^b-\beta^c} + \text{permutations of the indices } a, b, c \tag{23}$$

where we may have $a = b$ or $a = c$ (but $b \neq c$ as $\omega_{abc} = -\omega_{acb}$). We will make use of these formulae in section 5, when determining the asymptotic behaviour of the curvature and other 'composite' quantities of interest.

3. Dominant walls and E_{10} roots

In this section, we recall some basic facts and useful formulae concerning the Lie algebra E_{10} and its roots. A crucial fact here is the identification of the logarithmic scale factors appearing in (14) with the CSA degrees of freedom of E_{10}. This will enable us to re-interpret the exponential walls $\exp[-2w(\beta)]$ in the gravitational Hamiltonian in terms of the exponential (Toda) walls $\exp[-2\alpha(\beta)]$ entering the E_{10} σ-model (2). As mentioned already, the scalar product on the space of scale factors induced by the Einstein–Hilbert action for $D = 11$ coincides with the restriction of the invariant bilinear form on E_{10} to its CSA [5]. This scalar product is given by

$$(\beta|\beta) \equiv G_{ab}\beta^a\beta^b = \sum_{a=1}^{10}(\beta^a)^2 - \left(\sum_{a=1}^{10}\beta^a\right)^2. \tag{24}$$

It is indefinite (Lorentzian) due to the lower unboundedness of the Einstein–Hilbert action under variations of the conformal factor. As generally shown in [5], in the limit $T \to 0$ the dynamics of the scale factors is well approximated by a billiard. The region defining this billiard is contained in the forward lightcone in β-space, and bounded by 'sharp walls'; the latter are hyperplanes defined as the zeros of certain linear forms ('wall forms')

$$w(\beta) = \sum_{a=1}^{10} p_a\beta^a. \tag{25}$$

The metric on the dual space of wall forms is the contravariant form of (24)

$$(p|p) \equiv G^{ab}p_ap_b = \sum_{a=1}^{10}p_a^2 - \frac{1}{9}\left(\sum_{a=1}^{10}p_a\right)^2 \tag{26}$$

and this is also the metric appearing in (1) and (2). Below, we will also use the notation

$$w \equiv (p_1, \ldots, p_{10}) \tag{27}$$

to designate a particular wall form (25). In [5], we have explained how to determine these wall forms from the matter coupled Einstein–Hilbert action in a canonical formulation (see also the following section). A main result was that for $D = 11$ supergravity, each wall form w can be associated with a particular real root of E_{10}, in accordance with the fact that the root space is dual to the CSA, which itself is identified with the space of scale factors β. In particular, the *dominant* walls are associated with the *simple roots*. In the above basis (27), the simple roots of E_{10} are

$$\begin{aligned}
\alpha_0 &= (1, 1, 1, 0, 0, 0, 0, 0, 0, 0)\\
\alpha_1 &= (-1, 1, 0, 0, 0, 0, 0, 0, 0, 0)\\
\alpha_2 &= (0, -1, 1, 0, 0, 0, 0, 0, 0, 0)\\
\alpha_3 &= (0, 0, -1, 1, 0, 0, 0, 0, 0, 0)\\
\alpha_4 &= (0, 0, 0, -1, 1, 0, 0, 0, 0, 0)\\
\alpha_5 &= (0, 0, 0, 0, -1, 1, 0, 0, 0, 0)\\
\alpha_6 &= (0, 0, 0, 0, 0, -1, 1, 0, 0, 0)\\
\alpha_7 &= (0, 0, 0, 0, 0, 0, -1, 1, 0, 0)\\
\alpha_8 &= (0, 0, 0, 0, 0, 0, 0, -1, 1, 0)\\
\alpha_9 &= (0, 0, 0, 0, 0, 0, 0, 0, -1, 1)
\end{aligned} \tag{28}$$

and one easily checks that they generate the E_{10} Dynkin diagram via the scalar product (26). In the billiard picture, the dominant wall forms are therefore

$$w_{\alpha_0}(\beta) \equiv \alpha_0(\beta) = \beta^1 + \beta^2 + \beta^3$$
$$w_{\alpha_1}(\beta) \equiv \alpha_1(\beta) = \beta^2 - \beta^1$$
$$\vdots$$
$$\vdots$$
$$w_{\alpha_9}(\beta) \equiv \alpha_9(\beta) = \beta^{10} - \beta^9. \qquad (29)$$

The first simple root corresponds to a leading 'electric wall', while the remaining nine simple roots correspond to the leading 'symmetry walls'. All other walls (electric and symmetry, as well as the magnetic and gravitational walls) are subdominant and can be written as positive linear combinations of these leading walls. Consequently, the billiard takes place in the fundamental Weyl chamber, namely the 'wedge' defined by the conditions $\alpha_i(\beta) \geqslant 0$, or

$$\beta^1 + \beta^2 + \beta^3 \geqslant 0, \qquad \beta^1 \leqslant \beta^2 \leqslant \cdots \leqslant \beta^{10}. \qquad (30)$$

Every positive root of E_{10} can be expressed as a linear combination of simple roots:

$$\alpha = \ell\alpha_0 + \sum_{j=1}^{9} m^j \alpha_j \equiv [l; m^1, m^2, m^3, m^4, m^5, m^6, m^7, m^8, m^9] \qquad (31)$$

where, of course, $\ell, m^j \geqslant 0$, and the height of α is $\mathrm{ht}(\alpha) = \ell + \sum_j m^j$. The brackets ($[\cdots]$) denote the components of a root in the basis of simple roots, and are to be distinguished from the use of round parentheses used to denote the components of a root in the 'coordinate basis' of the βs. The integer ℓ is referred to as the 'A_9 level', or simply the 'level' of the root α. (Different 'levels' can be defined w.r.t. other subalgebras of E_{10} such as D_9 [22] or E_9 [23]). This level provides a grading of the Kac–Moody algebra E_{10}, allowing a decomposition of the E_{10} Lie algebra into an infinite tower of $\mathfrak{sl}_{10}$ representations, which has been worked out up to level $\ell \leqslant 28$ in [24]. The two bases (27) and (31) are related by

$$p_1 = \ell - m^1$$
$$p_2 = \ell + m^1 - m^2$$
$$p_3 = \ell + m^2 - m^3 \qquad (32)$$
$$p_4 = m^3 - m^4, \ldots, p_9 = m^8 - m^9, \qquad p_{10} = m^9.$$

Roots corresponding to wall forms associated with particular higher-order terms will appear at a fixed level ℓ. For the associated roots we will usually give the one corresponding to the root of lowest height within a given $\mathfrak{sl}_{10}$ multiplet, for which the integers p_a are ordered according to

$$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{10}. \qquad (33)$$

Because this is also the lowest weight of the corresponding $\mathfrak{sl}_{10}$ representation, we can read off the associated Dynkin labels directly from the tables of [24], as well as the outer multiplicity μ, i.e. the number of times this representation occurs in the decomposition of E_{10}. Starting from the lowest weight, other weights in the representation can be reached by successive addition of the 'symmetry roots' $\alpha_1, \ldots, \alpha_9$. From (28) it is evident that this operation simply amounts to a permutation of the p_is, and in terms of the scale factors β^a, to a replacement of the spatial indices a by other spatial indices. The highest weight, which corresponds to a root of maximal height, is reached when the above order has been inverted to $p_1 \leqslant \cdots \leqslant p_{10}$.

As an example let us briefly recall how the gravitational billiard walls arise by expanding out the Hamiltonian constraint following from the Einstein–Hilbert action, as described in detail in [5], and how they can be identified with particular E_{10} roots. Using (20), they are obtained by considering the following contribution (no summation on repeated indices):

$$E\Omega_{abc}\Omega_{abc} = \exp(-2\sigma + 2\beta^a + 2\beta^b - 2\beta^c)\bar{\Omega}_{abc}\bar{\Omega}_{abc} + \text{subleading terms.} \tag{34}$$

Because the walls always appear via exponentials $\mathrm{e}^{-2w(\beta)}$, the relevant linear form is

$$w_{abc}^{\mathrm{grav}} = \sigma - \beta^a - \beta^b + \beta^c, \qquad \text{with} \quad a \neq b. \tag{35}$$

Choosing the indices $(a, b, c) = (9, 10, 1)$ so as to obtain the root of lowest height in this multiplet (with Dynkin label (100000010)) we get

$$w_{9101}^{\mathrm{grav}} = (2, 1, 1, 1, 1, 1, 1, 1, 0, 0) \quad \Longleftrightarrow \quad \alpha = [3; 1, 3, 5, 4, 3, 2, 1, 0, 0]. \tag{36}$$

In particular, the associated root is *real* ($\alpha^2 = +2$) and *positive* in our conventions. Consequently, for $D = 11$ supergravity, the above gravitational walls arise at level $\ell = 3$ and are thus 'behind' the level $\ell = 1$ electric walls defined by the simple root α_0. This contrasts with pure gravity, where the leading gravitational wall always corresponds to a simple root of the algebra AE_n [4].

4. Higher-order corrections: what is known

Still as a preparation, we here summarize briefly what is known about the eighth-order derivative corrections to the 11-dimensional supergravity action, which are of the type R^4, $R^2(DF)^2$, $R(DF)^3$, $(DF)^4$, ..., F^8, where R stands for the curvature (Riemann) tensor, and $F = \mathrm{d}A$ is the 4-form field strength (there are no R^3DF terms because any possible contraction of the indices will involve an ε-tensor, and vanish by the Bianchi identity for either R or DF). In addition, there are separate eighth-order derivative terms of Chern–Simons type proportional to A_3R^4 required for the cancellation of anomalies [25]. The supersymmetric completion of these eighth-order terms was studied in [26] and, in a superspace formulation, in [27, 28] (see also the review [29]). The presence of such corrections can be inferred (i) from (string) one-loop corrections to $D = 10$ type IIA string amplitudes [30, 31], which are similar in form to $(\alpha')^3$ corrections at string tree level [32–35]; (ii) from considering M-theory corrections at one loop [36–39], (iii) from the computation of 4-graviton amplitudes in supermembrane theory by use of supermembrane vertices [40–42] or (iv) from the structure of divergences in $D = 11$ supergravity [43]. However, none of these approaches gives the complete 'reduced' (i.e. maximally simplified) kinematical structure of all the (bosonic) eighth-order derivative corrections. The paper [43] does contain explicit expressions for all terms $\propto R^M(DF)^N$ ($M + N = 4$), but in an 'unreduced' form involving the Bel–Robinson tensor, which for our present purposes is rather cumbersome[6]. Moreover, none of these papers gives expressions for the terms involving the undifferentiated 4-form F, such as F^8.

We note that the $D = 11$ results should be compatible with those of the IIA theory, whenever the $D = 10$ results can be 'lifted' to $D = 11$. In this case one must worry about the possible existence of terms that algebraically vanish in $D = 10$, but not in $D = 11$. As the number of independent invariants made out of the Weyl tensor is $= 7$ when $D \geqslant 8$ [44], this is not a concern for the curvature terms. However, the situation is more subtle for invariants built from the covariant derivative DF of the 4-form field strength, whose number does depend on the dimension. For instance, the number of $(DF)^4$ invariants is found to be $= 23$ for $D = 9$, and $= 24$ for $D = 11$, as can be conveniently checked by use of the computer algebra package LiE [45]. For $D = 10$, we would have 29 invariants with a 4-form field strength; however, there are only 15 invariants that can be built from the NSNS 3-form field strength in IIA supergravity (which means that we cannot simply lift the results of [35] to 11 dimensions).

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[6] Beware of a missing factor 12 in the third term, $\propto P^2$, on the rhs of equation (5*a*) of the first reference in [43] (J Plefka, private communication).

Yet another subtlety one must keep in mind is that the lightcone results do not necessarily agree with results of a 'covariant' calculation. A case in point is the 'Euler–Lovelock' term which we will encounter below: this term does not show up in the lightcone calculation of the 4-graviton amplitude, hence is not determined, whereas it does appear in the covariant approach of [39, 26]. However, this term should start to contribute to on-shell amplitudes from 5-point amplitudes onwards, whence it should become visible in a lightcone computation of the 5-graviton amplitude.

We are here only interested in *on-shell counterterms*. That is, we shall ignore all terms proportional to the equations of motion because they can be absorbed into redefinitions of the basic fields. For this reason, the Riemann tensor R_{ABCD} can be replaced by the Weyl tensor C_{ABCD}, since all terms containing the Ricci tensor or Ricci scalar can be absorbed (modulo F-dependent terms) into a redefinition of the metric by use of the equations of motion. For the DF type terms, we can similarly discard the trace and the fully antisymmetric combinations because

$$D^A F_{ABCD} \sim 0, \qquad D_{[A} F_{BCDE]} = 0 \tag{37}$$

by the equations of motion and the Bianchi identity (we neglect terms $\propto F^2$ on the rhs of the equation of motion; we shall see later that they are indeed subdominant). Thus, we must only deal with the irreducible representations of the rotation group represented by the following Young tableaux:

$$C_{ABCD} \sim \quad\boxed{}\quad, \qquad D_A F_{BCDE} \sim \quad\boxed{}\quad . \tag{38}$$

We first focus on the terms quartic in R (or C), because, as we shall see, they are the dominant ones within our analysis. From [39, 26], these terms are given by the expression (modulo a *positive* coefficient)

$$\mathcal{L}^{(4)\text{tot}} = E(J_0 - 2\mathcal{I}_2) \tag{39}$$

where $E = \det E_M{}^A = N\sqrt{g}$,

$$J_0 := t_8 t_8 R^4 + \tfrac{1}{4} E_8 \equiv X - \tfrac{1}{8} Z \tag{40}$$

and

$$\mathcal{I}_2 = \tfrac{1}{4} E_8 + 2\varepsilon_{11} A_3 \big[\operatorname{tr} R^4 - \tfrac{1}{4}(\operatorname{tr} R^2)^2\big] \equiv -\tfrac{1}{8} Z + 2\varepsilon_{11} A_3 \big[\operatorname{tr} R^4 - \tfrac{1}{4}(\operatorname{tr} R^2)^2\big]. \tag{41}$$

Here, we use the same condensed, index-free notation as in [39], while the notation $X \equiv t_8 t_8 R^4$ and $Z \equiv -2 E_8$ is used in [26].[7]

In particular, t_8 denotes the 8-index tensor that enters the kinematic factor of both tree-level and one-loop 4-particle string amplitudes [30, 46][8]. The normalization of t_8 is such that the contraction of t_8 with four times the same antisymmetric matrix M yields

$$t_8 M M M M := 24 \operatorname{tr} M^4 - 6(\operatorname{tr} M^2)^2 \tag{42}$$

$E_8 \equiv \tfrac{1}{3!}\varepsilon_{11}\varepsilon_{11} RRRR$ is shorthand for the so-called Euler–Lovelock density; more explicitly,

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[7] Note a misprint in the definition of Z given in appendix B of [26]: in equation (B.14) $\varepsilon_{10}\varepsilon_{10}$ should be replaced by $-\varepsilon_{10}\varepsilon_{10}$. The sign which appears in the `hep-th` version of that paper is correct. We are grateful to P Vanhove for a discussion on this point.

[8] As usual we mean by t_8 the $D = 11$, 8-index tensor obtained from equation (4.A.21) of this physics report by dropping the first 8-index ε tensor contribution on the right-hand side, and by extending the values of the lightcone indices on the remaining δ^4 tensor contributions to $D = 11$.

writing all indices for once, we have

$$E_8 = \frac{1}{3!} \varepsilon^{ABCD_1 \cdots D_8} \varepsilon_{ABCE_1 \cdots E_8} R_{D_1 D_2}{}^{E_1 E_2} \cdots R_{D_7 D_8}{}^{E_7 E_8}$$

$$= -\delta^{D_1 \cdots D_8}_{E_1 \cdots E_8} R_{D_1 D_2}{}^{E_1 E_2} \cdots R_{D_7 D_8}{}^{E_7 E_8} \tag{43}$$

where $\delta^8_8 = +1, -1, 0$ is (when non-zero) the signature of the permutation between $D_1 \ldots D_8$ and $E_1 \ldots E_8$. Observe that in Minkowskian signature, the product $\varepsilon_{11} \varepsilon_{11}$ contains a *minus* sign with respect to the usual lightcone term entering string-theory amplitudes $\varepsilon_8 \varepsilon_8 = +\delta^8_8$.

There are many subtle issues related to the relative sign between the $t_8 t_8 R^4 = X$ and the $\frac{1}{4} E_8 = -\frac{1}{8} Z$ contributions. In particular the usual lightcone calculation of 4-graviton amplitudes [30] yields the *same* combination $X - Z/8$ for the IIA theory at tree level and at the one-string-loop level. However, according to [39, 26], the correct one-string-loop term (which lifts to the M-theory result) differs from the lightcone calculation by a Euler–Lovelock term (and the Chern–Simons term), and should read $X - Z/8 + 2(Z/8) = X + Z/8$ instead (this is precisely the opposite sign from the one obtained in the lightcone computation of [30]). To recover (or check) this result in the lightcone gauge, one would have to calculate 5-graviton amplitudes in IIA string theory at one loop, but we are not aware of such computations. This issue is important because, as we shall see below, our cosmological billiards are *a priori* sensitive to the Euler–Lovelock term, and hence to the flip of sign between the lightcone result $X - Z/8$, and the complete result $X - Z/8 + 2(Z/8) = X + Z/8$.

In the following, we shall decompose the total contribution (39) into three separate terms: the term $L^{(4)1} = J_0 = t_8 t_8 R^4 + \frac{1}{4} E_8$, the Euler–Lovelock term $L^{(4)2} = -\frac{1}{2} E_8 = +2 \cdot \frac{1}{8} Z$ contained in $-2\mathcal{I}_2$ and the Chern–Simons term L^{CS} contained in $-2\mathcal{I}_2$. The first term (40) (which coincides with the lightcone result $X - Z/8$) can be worked out more explicitly [39, 26, 41], modulo terms that either vanish on-shell, or are equivalent to sub-leading terms (e.g. $\sim R^3 F^2$)

$$\mathcal{L}^{(4)1} = E J_0 = E \left(X - \frac{1}{8} Z \right)$$

$$= 192 E \left(- C^{ABCD} C_{AB}{}^{EF} C_{CE}{}^{GH} C_{DFGH} + 4 C^{ABCD} C_A{}^E{}_C{}^F C_E{}^G{}_B{}^H C_{FGDH} \right). \tag{44}$$

The second term $\mathcal{L}^{(4)2}$ has a much more complicated form when expressed in this way [26, 43], and we will therefore leave it 'in the ε-form' (77). We shall discuss the effects of this term in more detail in section 7, and of the Chern–Simons term in section 8.

The mixed terms containing (DF) factors have so far not been reduced to such a simple form. Let us just note that, in the (11-dimensional) lightcone gauge, the determination of *all* these terms can be reduced to an ε trace over $SO(9)\Gamma$-matrices, and, more specifically, can be obtained by contracting any four factors from

$$C_{\alpha\beta\gamma\delta} := \Gamma^{ij}_{\alpha\beta} \Gamma^{kl}_{\gamma\delta} C_{ijkl}, \qquad (DF)_{\alpha\beta\gamma\delta} := \Gamma^{ij}_{\alpha\beta} \Gamma^{klm}_{\gamma\delta} D_i F_{jklm} \tag{45}$$

with $\varepsilon^{\alpha_1 \cdots \alpha_{16}}$, as follows by inspection of the relevant vertex operators in [38, 40]. Here the indices $i, j, \ldots = 1, \ldots, 9$ label the nine *transverse* dimensions in 11 dimensions, and Γ^{ij} and Γ^{ijk} are the standard $SO(9)\Gamma$-matrices with spinor indices $\alpha, \beta = 1, \ldots, 16$. However, we would expect the *caveats* concerning the difference between lightcone and 'covariant' amplitudes, mentioned above for the C^4 contributions, to also apply to the mixed terms.

5. Scaling behaviour of curvatures and field strengths

Next we exhibit the leading asymptotic behaviour of the various fields and their 'composites' corresponding to the higher-order corrections near $T = 0$, as functions of the logarithmic scale factors β, making use of the decomposition (14) and other formulae derived in section 2.

Exploiting the identification of the scale factors with the CSA degrees of freedom, we shall then try to interpret the exponential walls $\exp[-2w(\beta)]$, as obtained from the higher-order corrections, in terms of new exponential (Toda) walls $\exp[-2\alpha(\beta)]$ in the E_{10} σ-model. To this aim, let us first determine the leading asymptotic behaviour of the curvature, starting with that of the spin connection ω and the anholonomicity coefficients Ω. Because the Iwasawa frame freezes in this limit, the 'leading order' amounts to neglecting the time derivatives of the intermediate frame, i.e. terms proportional to $\partial_t\theta^a$. More precisely, the terms $\partial_t\theta^a$ are proportional to some exponential walls, starting with some simple root walls $\sim \exp(-2\alpha_i(\beta))$. A consequence of this fact is that when discussing the subleading terms in the eighth-order correction (which are smaller than the dominant term by a factor $\sim \exp\left(-2\sum n_i\alpha_i(\beta)\right)$ with $n_i \geqslant 0$), one should, at some stage, take into account their 'mixing' with the terms proportional to $\partial_t\theta^a$ and coming from the leading curvature contribution.

To proceed we now substitute the results derived in section 2 into the expressions for the Riemann tensor in order to see in more detail how its various components behave in the limit $T \to 0$. Retaining only the relevant terms containing time derivatives, we get

$$R_{0a0b} = -\partial_0\omega_{ab0} - \Omega_{0ac}\omega_{cb0} + \cdots$$
$$= \delta_{ab}\,\mathrm{e}^{2\sigma}\left[\partial_t^2\beta^a + \partial_t\sigma\,\partial_t\beta^a - (\partial_t\beta^a)^2\right] + \cdots \tag{46}$$

for the components with two 0s, and

$$R_{0abc} \propto N^{-1}\exp(\beta^a + \beta^b - \beta^c) \equiv \exp(\sigma + \beta^a + \beta^b - \beta^c) + \text{ permutations of } a, b, c. \tag{47}$$

for the component with one 0. For the purely spatial components we have

$$R_{abcd} = R^{(1)}_{abcd} + R^{(2)}_{abcd} \tag{48}$$

where the first term on the rhs contains the contributions with ω_{ab0}

$$R^{(1)}_{abcd} := \omega_{ac0}\omega_{bd0} - \omega_{ad0}\omega_{bc0} = 2\delta_{c[a}\delta_{b]d}\,\mathrm{e}^{2\sigma}\partial_t\beta^a\partial_t\beta^b \tag{49}$$

while the second term $R^{(2)}_{abcd}$ involves all contributions from the purely spatial spin connection (20) and is proportional to

$$R^{(2)}_{abcd} \propto \exp(\beta^a + \beta^b + \beta^c - \beta^d) \tag{50}$$

or

$$\propto \exp(\beta^a + \beta^b + \beta^c + \beta^d - 2\beta^e)$$

or

$$\propto \exp(\beta^a + \beta^b + 2\beta^e - \beta^c - \beta^d)$$

together with all permutations of the indices a, b, c, d; note that some of these indices may coincide. The index e is independent of a, b, c, d; it is the summation index in the terms quadratic in the spin connection (but we here list each of the terms in the sum separately). For later reference, we note that the combinations appearing in the exponentials on the rhs of (50) differ from one another by linear combinations of 'symmetry roots' $(\beta^a - \beta^b)$, corresponding to the so-called symmetry walls [5].

As a check of our computation of the leading curvature components, let us work out the Einstein–Hilbert action to leading order; we get

$$ER \simeq \mathrm{e}^{-2\sigma}\left(\sum_{a,b} R^{(1)}_{abab} - 2\sum_a R_{0a0a}\right)$$
$$= -2\partial_t^2\sigma + \sum_a(\partial_t\beta^a)^2 - (\partial_t\sigma)^2 \equiv -2\partial_t\sigma + \sum_{a,b} G_{ab}\partial_t\beta^a\partial_t\beta^b. \tag{51}$$

Therefore, modulo a total time derivative $-2\partial_t^2\sigma$, we recover the kinetic terms for the diagonal metric components in (1), with the usual (Lorentzian) 'superspace' metric G_{ab} from (24).

It is the inverse ($=G^{ab}$) of this metric which appears in both the Hamiltonians (1) and (2) in the introduction, and was written in (26).

It is easily seen that the leading components of the curvature are R_{a0a0} and $R^{(1)}_{abab}$ (with $a \neq b$, and no summation on repeated indices). These leading components are of order $e^{2\sigma}$ and can be written as (recalling that, in our gauge, β^a is approximately piecewise linear in the coordinate time t)

$$R_{a0a0} = e^{2\sigma} v_a \bar{v}_a$$
$$R^{(1)}_{abab} = e^{2\sigma} v_a v_b \qquad \text{for} \quad a \neq b \tag{52}$$

where, for shortness of notation, we have defined

$$v_a := \partial_t \beta^a, \qquad \bar{v}_a := \sum_{b \neq a} \beta^b \equiv \partial_t \sigma - \partial_t \beta^a. \tag{53}$$

For the Kasner solution, the coordinate time can be taken to be $t = -\ln T$, and the 'velocities' v_a are then just the Kasner exponents.

The remaining components of the curvature tensor are sub-leading in the sense that they are suppressed in comparison with the leading term (of order $\sim e^{2\sigma}$) by exponentially small factors. Before discussing in a unified way how one can define a relative ordering between several contributions, let us collect various results for the *ratios* between subleading curvature (and F-dependent) terms, and the leading curvature $R \sim e^{2\sigma}$. From (47) and (50) we see that

$$R_{0abc}/e^{2\sigma} \sim \exp(-\sigma + \beta^a + \beta^b - \beta^c) + \text{permutations}. \tag{54}$$

Note that the combination appearing in the exponential on the rhs is just (minus) the gravitational wall form (35). Similar estimates hold for $R^{(2)}_{abcd}$; a representative example is

$$R^{(2)}_{abcd}/e^{2\sigma} \sim \exp(-2\sigma + \beta^a + \beta^b + \beta^c - \beta^d) \tag{55}$$

and all other such quotients differing only by symmetry walls.

The asymptotic behaviour of the 4-form and of its derivatives can be analysed in a similar manner. For the electric field strength we have

$$F_{0abc} = \exp(\sigma - \beta^a - \beta^b - \beta^c) \bar{F}_{tabc} \tag{56}$$

where

$$\bar{F}_{tabc} := \theta_{ma} \theta_{nb} \theta_{pc} F_t^{mnp} \tag{57}$$

and the overall factor e^σ comes from the inverse lapse N^{-1}. (We recall that the indices $0, a, b, c$ denote orthonormal-frame indices.) As shown in [5], it is precisely the barred field strength $\bar{F}_{tabc}$ (in the gauge $N = \sqrt{g}$; or, in any gauge, the conjugate momenta $N\sqrt{g} F^{tmnp}$) which possesses a limit as $T \to 0$. Comparing the electric field strength with the square root of the leading curvature $R^{1/2} \sim e^\sigma$, we obtain (no summation on (a, b, c))

$$F_{0abc}/e^\sigma \sim e^{-(\beta^a + \beta^b + \beta^c)} \tag{58}$$

where the combination $(\beta^a + \beta^b + \beta^c)$ appearing in the exponential on the rhs is related to an 'electric wall', see [5]. In particular, for $(a, b, c) = (1, 2, 3)$ we just re-obtain the simple root α_0 in (28), which explains the term 'electric root'. We thus conclude that the electric field strength is 'smaller' than the square root of the dominant curvature by one electric wall.

For the magnetic field strength, we have similarly

$$F_{abcd} = \exp(\beta^a + \beta^b + \beta^c + \beta^d) \bar{F}_{abcd} \tag{59}$$

with

$$\bar{F}_{abcd} := \theta_a{}^m \theta_b{}^n \theta_c{}^p \theta_d{}^q F_{mnpq}. \tag{60}$$

Again the barred field strength is the one that remains finite on the initial hypersurface. Note the different positions of the world indices in these two definitions. (The correct index position for the quantities that freeze is easy to remember as the quantities that freeze are the Qs and the Ps: A_{mnp} and their spatial derivatives, and their conjugate momenta π^{mnp}.) Comparing the magnetic field strength to the square root of the leading curvature, we obtain

$$F_{abcd}/\mathrm{e}^\sigma \sim \mathrm{e}^{-\sum_{e \neq a,b,c,d} \beta^e}. \tag{61}$$

The combination appearing in the exponential on the rhs is now a magnetic wall, which is also equal to the sum of *two* (non-overlapping) electric walls. Consequently, the magnetic field strength is 'smaller' than the electric field strength by one electric wall, and 'smaller' than the square root of the leading curvature term by two electric walls. This shows that all the higher-order terms which are algebraic in R and F are subdominant compared to the leading R^4 terms. For instance, F^8 is 'smaller' than the leading R^4 term by eight electric walls.

Finally, the covariant derivative of the field strength is dominated by the time derivative of an electric component:

$$D_0 F_{0abc} = \exp(2\sigma - \beta^a - \beta^b - \beta^c) D_t \bar{F}_{tabc} + \cdots. \tag{62}$$

Hence,

$$D_0 F_{0abc}/\mathrm{e}^{2\sigma} \sim \mathrm{e}^{-(\beta^a+\beta^b+\beta^c)}. \tag{63}$$

In other words, DF is smaller than R by one electric wall.

We can conveniently summarize the relative ordering of the building blocks entering the eighth-order correction by introducing the symbol $\preceq$ to denote the ordering of exponential factors involving simple wall forms. As will be further discussed at the beginning of section 6, we can treat the negative exponentials of the simple roots as small parameters: $\varepsilon_i \sim \exp(-\alpha_i(\beta)) \preceq 1$. This defines an ordering such that $n_i \geqslant n_i'$ implies $\exp\left(-\sum n_i \alpha_i(\beta)\right) \preceq \exp\left(-\sum n_i' \alpha_i(\beta)\right)$. Now, by looking more closely at the various exponential factors appearing on the rhs's of the comparison equations (54), (55), (58), (61), (63), one sees that they can be rewritten as negative exponentials of sums of simple roots. For instance, the rhs of (54) is $\mathrm{e}^{-(\sigma-\beta^a-\beta^b+\beta^c)}$, where the combination $\sigma - \beta^a - \beta^b + \beta^c$ is the *gravitational* wall form, see (35), which can be written as a sum of simple roots, see equation (36). Actually, it is convenient, for a quick perusal of the relative ordering of various contributions, to neglect the ordering with respect to symmetry roots $\alpha_1, \ldots, \alpha_9$, see (29) in the following section, and to focus on the ordering defined by the number n_0 of leading electric roots $\alpha_0(\beta) = \beta^1 + \beta^2 + \beta^3$. In other words, we shall simply write $\exp\left(-\sum n_i \alpha_i(\beta)\right) \preceq \exp(-n_0 \alpha_0(\beta))$, and use the ordering $\exp(-n_0 \alpha_0(\beta)) \preceq \exp(-n_0' \alpha_0(\beta))$ when $n_0' \geqslant n_0$. For instance, in view of the fact that the gravitational wall form which appears in (54) is at level $\ell = 3$ (see (36)), we can write $\mathrm{e}^{-(\sigma-\beta^a-\beta^b+\beta^c)} \preceq \mathrm{e}^{-3\alpha_0}$. Finally, with this notation, the above results simplify to

$$R_{0abc}/\mathrm{e}^{2\sigma} \preceq \mathrm{e}^{-3\alpha_0} \tag{64}$$

$$R^{(2)}_{abcd}/\mathrm{e}^{2\sigma} \preceq \mathrm{e}^{-6\alpha_0} \tag{65}$$

$$F^2_{0abc}/\mathrm{e}^{2\sigma} \preceq \mathrm{e}^{-2\alpha_0}, \qquad F^2_{abcd}/\mathrm{e}^{2\sigma} \preceq \mathrm{e}^{-4\alpha_0} \tag{66}$$

(in comparing with the curvature which is second order in derivatives, we must square the 4-form field strengths because they have only one derivative)

$$DF/\mathrm{e}^{2\sigma} \preceq \mathrm{e}^{-\alpha_0}. \tag{67}$$

These estimates show that the dominant terms among the various eighth-order corrections $R^4, R^2(DF)^2, R(DF)^3, (DF)^4, \ldots, F^8$ are the curvature terms R^4. Among the R^4 terms, the leading contributions will come either from the time–time components R_{0a0a}, or from the purely spatial ones $\sim R^{(1)}_{abab}$. Then, the next to leading contributions will come either from $R^2(DF)^2$ or from $R^3 F^2$, and will be smaller than the leading one by a factor $\sim \mathrm{e}^{-2\alpha_0}$.

6. Leading behaviour at higher orders: curvature terms

Before proceeding with the analysis of the higher-order derivative contributions in the Lie algebraic setting, let us clarify in which regime our analysis of their effects, and in particular of the eighth-order derivative corrections, will be meaningful. This regime is one of an 'intermediate asymptotics' of the type $T_P \ll T \ll T_0$. Here, T_P is the (11-dimensional) Planck time, and T_0 is some initial (or final) time away from the big crunch (or big bang) singularity located at $T = 0$. We assume that the Cauchy data at T_0 are such that spatial derivatives of the metric and the 3-form are mildly smaller than their time derivatives, say $\partial_a g < 0.2\partial_T g$. We also assume that the time derivatives correspond to curvatures much smaller than the 11-dimensional Planck curvature T_P^{-2}. Then, for a while, the BKL-type analysis becomes better and better for times $T \ll T_0$, in the sense that the potential walls become sharper and sharper, and that the Weyl chamber of E_{10} appears as a dominant billiard description of the dynamics of the diagonal components of the metric at any given spatial point. Such a billiard description of the Einstein-3-form system remains accurate until the higher-order corrections to the action, $\sim R + T_P^6 R^4 + \cdots$ become important. This happens for curvatures $R \sim T_P^{-2}$, i.e. for cosmological times $T \sim T_P$. As there are no small coupling parameters in M-theory, when the leading correction $\sim R^4$ becomes important, all higher-order ones $\sim T_P^{2(N-1)} R^N$ also become important as $T \sim T_P$. However, in the intermediate regime $T_P \ll T \ll T_0$, it is meaningful to consider only the terms $\sim R^4$ and to treat them as a small correction to the usual supergravity billiard defined by $S_0 = R + F^2 + AFF$. In other words, in such a regime we can meaningfully combine two separate expansion schemes: (i) a *height expansion* in which the negative exponentials of the simple roots are treated as small parameters, say $\varepsilon_i \sim \exp(-\alpha_i(\beta))$, and (ii) a *small curvature expansion* in which $\varepsilon_c \sim T_P^2 R \sim (T_P/T)^2$ is treated as a separate small parameter. Note that the basic rule behind the first expansion is to order the exponential terms by their height. Indeed, a generic wall $\exp(-2\alpha(\beta))$, with a positive-root $\alpha = \sum n_i \alpha_i$ is seen as the product of small numbers, $\prod(\varepsilon_i)^{2n_i}$, which gets smaller and smaller as the n_i increase. Another, and more heuristic, way to think about the height expansion would be to replace the ten individual β^as, or better their simple-root combinations $\alpha_i(\beta)$, by some 'average scale factor' $\overline{\beta}$, or some 'average simple root' $\alpha_s \equiv \overline{\alpha}_i$. This is a legitimate approximation because the $\alpha_i(\beta)$ increase towards $+\infty$ and remain positive during their chaotic motion within the Weyl chamber $\alpha_i(\beta) \geqslant 0$. In terms of such averaged quantities, one can estimate the effect of a generic wall $\exp(-2\sum n_i \alpha_i(\beta))$ as being on average equivalent to $\exp(-2(\sum n_i)\alpha_s) = \exp(-2\mathrm{ht}(\alpha)\alpha_s)$.

As we have seen in section 4, the dominant terms, in the sense of the height expansion just explained, among the eighth-order derivative corrections R^4, $R^2(DF)^2$, $R(DF)^3$, $(DF)^4, \ldots, F^8$ are the curvature terms R^4. Therefore, these are the terms that will define the leading corrections to the dynamics in the intermediate asymptotics $T_P \ll T \ll T_0$.

Let us examine more carefully the contribution of the terms R^4 in the Hamiltonian constraint $\mathcal{H}$. In any gauge an additional term J in the (invariant) Lagrangian density contributes the term $-EJ$ in the Hamiltonian constraint. We recall from (17) that the 11-dimensional volume density behaves as $E = N\sqrt{g} = \mathrm{e}^{-2\sigma}$. As we have seen above, the leading frame components of the curvature blow up proportionally to $\mathrm{e}^{2\sigma}$. Hence, a term $J \sim R^4$ contributes $-EJ \sim \mathrm{e}^{-2\sigma}(\mathrm{e}^{2\sigma})^4 \sim \mathrm{e}^{6\sigma}$. (Certain special quartic curvature invariants R^4 might only contribute terms $\ll (R_{0a0a})^4$, but we shall see below that both the $t_8 t_8 R^4$ and E_8 do contain terms $\sim(\mathrm{e}^{2\sigma})^4$.) For more generality, let us consider a correction containing an arbitrary power of the curvature, say $J \sim R^N$ (the precise kinematical structure of R^N does not matter at this point). This contributes to the Hamiltonian constraint a term

$$ER^N \propto \mathrm{e}^{2(N-1)\sigma}. \tag{68}$$

The important point for our analysis is that, in the Hamiltonian constraint (1) such a term has the form of an exponential wall $\sim \mathrm{e}^{-2w(\beta)}$, where $w(\beta)$ is a linear form in the βs. The associated 'wall form' $w_N \equiv w[R^N]$ is easily read off, namely

$$w_N(\beta) = -(N-1)\,\sigma \quad \Longleftrightarrow \quad w_N = -(N-1)\cdot(1,1,1,1,1,1,1,1,1,1). \tag{69}$$

Let us now see for which values of N the wall form (69) happens to be a root of E_{10}. A simple *necessary* criterion for checking whether a wall form w might be a root of E_{10} consists in computing the squared length of w with the scalar product of the CSA of E_{10} (26). Indeed, all forms w on the CSA which belong to the E_{10} root lattice[9], and which are such that $w^2 = -2j$, for some integer $j \geqslant 0$, are (imaginary) roots of E_{10} and vice versa. For the wall form (69), one easily finds

$$w_N^2 = -\tfrac{10}{9}(N-1)^2. \tag{70}$$

This is negative, and is an even integer only when $N = 3k+1$, with some positive integer k. This includes the case of main interest here, namely $N = 4$, i.e. $k = 1$.

It is easily checked that when the necessary condition $N = 3k+1$ is met, the corresponding $w_{3k+1} = -3k\sigma$ does belong to the root lattice, and is therefore an (imaginary) root. For instance, the quartic correction R^4 corresponds to the E_{10} root

$$w_4 = -(3,3,3,3,3,3,3,3,3,3) \Longleftrightarrow$$
$$\alpha = -[10; 7, 14, 21, 18, 15, 12, 9, 6, 3] \tag{71}$$

which is of squared length $\alpha^2 = -10$ and height $\mathrm{ht}(\alpha) = -115$; its level is $\ell = -10$. We see that this root lies rather deep inside the lightcone in root space. More generally, for $N = 3k+1$ (i.e. for corrections $R^4, R^7, R^{10}, \ldots$), one gets wall forms corresponding to E_{10} roots of squared length $-10k^2$, height $\mathrm{ht}(\alpha) = -115k$, and level $\ell = -10k$ for $k = 1, 2, 3, \ldots$.

Looking back at the potential wall (69) associated with the correction R^N we note something special. Barring special cancellations, all terms in the correction R^N lead to a unique dominant potential wall $\mathrm{e}^{2(N-1)\sigma}$. This is quite different from what happens for the usual walls that have appeared in the analysis of [5, 8]. Indeed, the walls associated with the electric energy of the 3-form, or its magnetic energy, or to the 'gravitational energy' all came in multiplets of the group of permutations of spatial indices. For instance, the electric walls were of type $\exp(-2(\beta^a + \beta^b + \beta^c))$, where a, b, c are three different spatial indices. The lowest-height electric root $\alpha_0 = \beta^1 + \beta^2 + \beta^3$ is accompanied by other electric roots at the same level $\ell = 1$, which are obtained by acting on α_0 by the generators of $\mathfrak{sl}_{10}$ (with roots $\beta^2 - \beta^1, \ldots$), which have the effect of permuting the $SL(10)$ indices among $a = 1, 2, \ldots, 10$. This fact was associated with the fact that the basic electric root $\alpha_0 = \beta^1 + \beta^2 + \beta^3$ was simply the lowest weight associated with a generator of E_{10} which was given by an antisymmetric 3-tensor $SL(10)$ representation, say E^{abc}. In other words, the walls which come in permutation multiplets are associated with non-trivial tensor representations of $\mathfrak{sl}_{10}$.

The 'isotropy' of the wall form (69), i.e. the fact that it does not depend on the choice of a special permutation of $SL(10)$ indices, suggests that w_N, equation (69), should be associated with a root of E_{10} whose associated generator is a $\mathfrak{sl}_{10}$ *singlet*. We do find in the tabulated results of [24] that there are indeed the requisite singlets matching the R^4 terms at level $\ell = 10$, and the predicted R^7 terms at level $\ell = 20$. Moreover, these are the *only singlets* among the 4 400 752 653 representations identified up to level $\ell \leqslant 28$ in [24]. In view of the 'contravariant' character of the 'raising generators' of E_{10}, namely, $E^{(3)} \equiv E^{abc}$, $E^{(6)} \equiv E^{a_1 \cdots a_6} \sim [E^{(3)}, E^{(3)}]$, etc it is easy to see that $SL(10)$ singlets can only

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[9] We recall that the *root lattice*, i.e. the set of integer combinations of simple roots, contains, but is strictly larger than, the set of roots.

occur at levels $\ell = 10k$ (so that one can use $3k$ covariant ε tensors of $SL(10)$ to soak up the 3ℓ contravariant indices of $E^{(3\ell)})^{10}$. It is, however, remarkable that: (i) singlets *do occur* at these levels with non-zero outer multiplicities, and (ii) there exists such a nice compatibility between the algebraic structure of E_{10} (built by 'commuting' 3-form level-1 generators E^{abc}), and the wall structure entailed by curvature corrections. See the final section for further discussion of this point.

Moreover, among the highest weights of the representations at levels $\ell = 10$ and $\ell = 20$ the associated root (71) and its integer multiples are the ones of largest root height. *Therefore, for any N, among the roots associated with the $2N$th order derivative corrections, (71) is always the root that lies deepest inside the past lightcone.* For the moment, no data are available beyond $\ell > 28$, but we confidently predict the requisite $\mathfrak{sl}_{10}$ singlets to appear at levels $10k$ with roots $3k\sigma$ and non-vanishing outer multiplicities.

We can summarize our findings in a little lemma.

Lemma. *Corrections proportional to the Nth power of the Riemann (or Weyl) tensor are compatible with E_{10} if and only if $N = 3k + 1$, i.e. for $R^4, R^7, R^{10}, \ldots$. Moreover, such corrections must be associated with $\mathfrak{sl}_{10}$-singlet generators of E_{10} at levels $\ell = -10k$.*

For the special value $k = 0$, the 'leading term' is just the kinetic term (51), and there is no 'leading wall'. For $k = 1, 2$, i.e. $N = 4, 7$, the outer multiplicities of the singlets are known: $\mu = 3$ for $\ell = 10$, and $\mu = 913$ for $\ell = 20$ [24]. We have no physical interpretation for these values in terms of the R^4 and R^7 corrections to the action. It might have been tempting to associate the outer multiplicity $\mu = 3$ with the three different contributions ($t_8 t_8 R^4$, E_8 and the Chern–Simons term) entering (39). However, we shall see that the Chern–Simons one is subdominant. Also note that for E_{11} the root corresponding to (71) is associated with the $SL(11)$ representation (0000000003), which is no longer a singlet.

Below, we will show that this result extends to corrections of mixed type $R^{N-n}(DF)^n$. Remarkably, the very same constraint $N = 3k + 1$ on the allowed powers of curvature, which we found here assuming compatibility with the E_{10} root lattice, was arrived at by a very different argument in [47][11] (to wit, an expansion in integer powers of the inverse membrane tension). Also note that, as far as we know, no detailed information is yet available about the R^N corrections for $N > 4$.

7. Positivity properties of leading curvature terms

Let us comment on two interesting aspects of the E_{10} root (71) associated with the leading R^4 corrections. First, as announced in the introduction, the root (71), besides being imaginary (or more precisely, time-like, $\alpha^2 < 0$), is also negative, $\alpha \in \Delta_-$. Physically, this corresponds to the fact that the R^4 corrections grow faster than the leading Einstein terms near the singularity. As we already explained, this fact limits the validity of our analysis to some intermediate asymptotic $T_P \ll T \ll T_0$. Within such an intermediate asymptotic expansion, we see no *a priori* conflict between the occurrence of negative E_{10} roots in the M-theory action, and the form (2) of the coset action, involving only *positive* roots of E_{10}. Indeed, as already pointed out above, the positive-root form (2) is the result of using a special nilpotent (i.e. purely upper triangular) parametrization of the coset $E_{10}/K(E_{10})$. The use of a more complicated parametrization, involving a mixture of upper and lower triangular parametrization, would entail the presence of negative roots in the coset action.

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[10] We thank Axel Kleinschmidt for pointing out this algebraic fact.
[11] This result has been contested in [48], which claims that only $N = 6k + 4$ is allowed.

However, no matter how one parametrizes the solution, the trajectory that extremizes the coset action is simply a null geodesic on $E_{10}/K(E_{10})$, and therefore geometrically the same solution. In the case of finite-dimensional Lie algebras one can prove that a generic geodesic curve is conjugate to one in the CSA. However, no proof of this result is known for hyperbolic Kac–Moody algebras, nor is it known whether such a statement holds at all, see [49]. If that were true for geodesics on $E_{10}/K(E_{10})$ one would immediately conclude that a generic null geodesic is conjugate to a null geodesic in the CSA, i.e. to the curve $\beta_a(t) = v_a t + \beta_a^{(0)}$ (with $v^2 = 0$) which runs straight from 'past null infinity' towards 'future null infinity' in the CSA. Fortunately, we can qualitatively control the global behaviour of null geodesics without having to assume the unproven equivalence with a CSA geodesic[12]. Indeed, we can remark that, in the Borel parametrization of equation (2), the potential terms $\propto \Pi^2$ are all non-negative, so that they define a non-negative 'squared-mass term', $-\pi^2 = M^2 \geqslant 0$ for the motion projected in β-space. As $\dot{\beta}^a \propto \pi^a$ this implies that the β^a worldline is everywhere timelike or null. Remembering that, in terms of the logarithmic scale factors β_as, 'future infinity' is $\sigma \equiv \sum \beta_a \to +\infty$, and corresponds to a 'big crunch' singularity where the volume $\sqrt{g} = e^{-\sigma} \to 0$, we conclude that a generic null geodesic, which is initially future directed, will never be able to turn back to become past directed. In other words, this proves that a generic E_{10} null geodesic cannot correspond to a 'bouncing' cosmological solution, where the volume of space reaches a minimum before it re-expands. In view of this argument, and if there is any truth to the E_{10} conjecture, we expect that the *sign* of the higher-order corrections to the supergravity action will be such as to allow *no bouncing solutions*.

We have studied the sign and magnitude of the various leading quartic corrections in (39) to see if they might be compatible with our expectations. (Strictly speaking, the sign of the leading correction is not logically related to the bounce/non-bounce behaviour. Indeed, even if the leading correction indicates a bounce behaviour, the next order corrections may prevent the completion of this bounce.) Actually, the analysis of the sign of the quartic-in-R contribution to the gravity action is rather involved, because the precise kinematical structure of the R^4 now does matter. The first important fact is that the sign of the coefficient of the loop correction (39) is *positive*[13]. Then there arises the issue of the actual sign of the two leading contributions to the action, (40) and the E_8 term in (41) (which, as we shall see below, is dominant compared to the Chern–Simons one) when they are evaluated 'on-shell', i.e. along a solution of the leading Einstein-3-form dynamics. If we can prove that the sum of the on-shell actions (40) and the E_8 term in (41) is numerically *positive*, this will prove that the corresponding contribution to the correction of the gravity Hamiltonian is[14] *negative*. In that case, the wall associated with the R^4 corrections (characterized by a negative, time-like root, with negative coefficient) will no longer represent a 'mountain', which would obstruct the motion of β towards infinity and thereby induce a bounce, but instead a 'canyon' or a 'crevice' precipitating the collapse of space towards zero volume (i.e. $\sigma \to +\infty$).

Let us therefore have a closer look at the various R^4 contributions to the action. We consider first the contributions coming from (40), which explicitly read as (44), and will later consider the E_8 one contained in (41). The first observation is that in leading order (in the sense explained above), only terms with R_{a0a0} and $R_{abab}^{(1)}$ need to be considered in (44). Secondly,

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[12] See, in this respect, the appendix where we emphasize that null geodesics on $E_{10}/K(E_{10})$ are 'strongly chaotic' in the sense of being extremely sensitive to small changes in their initial conditions. This result suggests that geodesics which do not lie within the CSA might be somewhat different from the simpler (Kasner) CSA ones.

[13] Due to a remnant Euclidean-type convention, one should reverse the overall sign of the actions in [39].

[14] We use here the usual first-order result $H_1 = -L_1$ for the correction to the Hamiltonian corresponding to a correction to the Lagrangian. This result applies to the case where one introduces no new degrees of freedom, but only modifies the action for given degrees of freedom, possibly by higher derivative terms.

it is straightforward to verify that all terms proportional to either $(C_{0a0a})^4$ or $(C_{abab})^4$ (no summation on indices!) cancel between the two contributions in (44). Substitution of (52) into (44) and some further calculation then reveal for the leading part of the density (40) the explicit expression

$$E J_0 = 768 \, e^{6\sigma} V_2^4 \tag{72}$$

where

$$V_2^4 \equiv \sideset{}{'}\sum_{a,b} v_a^2 v_b^2 (v_a \bar{v}_a + v_b \bar{v}_b - \bar{v}_a \bar{v}_b)^2 + \frac{1}{3} \sideset{}{''}\sum_{a,b,c} v_a^2 v_b^2 v_c^2 (v_a + v_b + v_c)^2 \geqslant 0 \tag{73}$$

and the sum $\sum'$ over a, b is restricted to $a \neq b$, while the sum $\sum''$ ranges over the three indices a, b, c with the restriction that $a \neq b$, $b \neq c$, and $c \neq a$. Note that each ordered pair $a < b$ contributes two equivalent terms to the sum $\sum'$, while each ordered triplet $a < b < c$ contributes six equivalent terms to the sum $\sum''$. The explicit expression (73) as a sum of squares proves that $J_0 \geqslant 0$. If that were the only contribution to the action, this argument would suffice to prove the negativity of the corresponding term in the Hamiltonian. However, we still need to consider the term proportional to the Euler–Lovelock density in (41). Using the explicit expression of the Euler–Lovelock densities in Kasner spacetimes [50],[15] we get the following explicit, leading expression for the relevant term in (41):

$$\mathcal{L}^{(4)2} = -\tfrac{1}{2} E E_8 = +\tfrac{1}{4} E Z = -46\,080 [\partial_t (e^{6\sigma} V_7) + e^{6\sigma} V_8] \tag{74}$$

where

$$V_7 := \sum_{a_1 < \cdots < a_7} v_{a_1} \cdots v_{a_7} \tag{75}$$

and

$$V_8 := \sum_{a_1 < \cdots < a_8} v_{a_1} \cdots v_{a_8}. \tag{76}$$

Note that, contrary to what happened for the sums $\sum'$ and $\sum''$ above, the sums appearing in V_7 and V_8 contain only one contribution per ordered multiplet of indices. In the following, we shall discard the total time derivative appearing as the first term on the right-hand side of (43), and evaluate

$$\mathcal{L}^{(4)2'} = -46\,080 \, e^{6\sigma} V_8. \tag{77}$$

Note, however, in passing that if one were to evaluate $\mathcal{L}^{(4)2}$ *on-shell* before discarding this term, it would yield a non-zero contribution (but a contribution which is equivalent to a reparametrization of the basic degrees of freedom). In order to see whether the term (77) is likely to modify the positivity of (73) we need to compare $(46\,080/768) V_8 = 60 V_8$ to the term V_2^4, i.e. to the expression defined by the restricted sums $\sum'$, $\sum''$ in (73). The latter quantity, V_2^4, was shown above to be a sum of squares, and therefore to be always $\geqslant 0$. We need to study the sign and magnitude of the additional contribution $-60 V_8$.

We can obtain rather simply an approximate knowledge of the magnitude of V_8 in the following way[16]. We consider the tenth degree polynomial $P(x) = \prod_a (x - v_a)$. Let us consider on-shell values of the v_as, i.e. such that $v^2 = -2 \sum_{a<b} v_a v_b = 0$. Let us also normalize the values of the vs by imposing $\sum_a v_a = 1$. (So that we have both $\sum_a v_a = 1 = \sum_a v_a^2$.) In the expansion of the polynomial $P(x) = x^{10} - c_9 x^9 + c_8 x^8 + \cdots + c_2 x^2 - c_1 x + c_0$,

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[15] After discussion with the author, it appears that the overall coefficient 2^{2p-1} in equations (22a) and (22b) of [50] should be replaced by the, *alas*, larger coefficient $(2p)!/(2p - 1)$.

[16] We thank Ofer Gabber for suggesting this method for bracketing the value of V_8.

we then know $c_9 = \sum_a v_a = 1$, $c_8 = \sum_{a<b} v_a v_b = 0$ and we have $c_2 = V_8$, where V_8 is the quantity we wish to bracket. Because *all* the roots of $P(x)$ are real, it is well known that *all* the roots of its successive derivatives $P'(x)$, $P''(x)$, ... will also be real. By considering the expansion in powers of x of the second derivative $P''(x)/90 = x^8 - 0.8x^7 + 0x^6 + \cdots + V_8/45$, and relating it to its real roots, say x_i (for $i = 1, \ldots, 8$), we get the following expression for V_8: $V_8 = 45 \prod_i x_i$ where the x_is are constrained by $\sum_i x_i = 0.8$ and $\sum_i x_i^2 = (0.8)^2$. It is then easy to get both upper and lower limits to V_8 by considering the various possible cases where p values among the x_is are negative. (It is easily seen that p must be $p \geqslant 1$.) Using then the convexity inequality $a_1^2 a_2^2 \cdots a_n^2 \leqslant \left[\left(\sum_i a_i^2 \right)/n \right]^n$, with $n = 8 - p$ and/or $n = p$, and minimizing over the average value of the negative x_is, we get the double-sided inequality

$$-45(0.8)^8/8^{4.5} \leqslant V_8/(\sigma_v)^8 \leqslant +45(0.8)^8/8^5 \tag{78}$$

where we denote $\sigma_v \equiv \sum_a v_a$. Numerically, this yields for the quantity we are interested in:

$$-0.0391 \leqslant 60 V_8/(\sigma_v)^8 \leqslant +0.0138. \tag{79}$$

These rather small allowed values for V_8 are due to the fact that V_8 is a sum of products of eight different vs. By contrast, we expect the quantity $V_2^4/(\sigma_v)^8$ in (73) to typically take significantly larger values because it contains products of only two or three different vs. This suggestive (if not quite conclusive) argument makes it plausible that the difference $V_2^4 - 60 V_8$ will always be positive. Actually, the place where this difference is most likely to become negative is in the vicinity of the points where $(V_2)^4$ vanishes. This happens (normalizing again by $\sum_a v_a = 1$) when one specific v is very close to 1, and the other ones are near zero. (Note that the Kasner solution where $(v_a) = (1, 0, 0, \ldots, 0)$ is flat spacetime in disguise.) However, it is easy to see that, near such a point, $V_2^4 = \mathcal{O}(v^4)$ while $V_8 = \mathcal{O}(v^7)$. Therefore, near the points where V_2^4 is the smallest, the difference $V_2^4 - 60 V_8$ remains positive. We performed some partial numerical investigations of the sign of the difference $V_2^4 - 60 V_8$ and always found it to be positive.

We take all this as evidence that the total eighth-order action (39) is equivalent to a *negative* contribution in the Hamiltonian, i.e. to an 'inverted wall', or a 'crevice'. As we said above, this is compatible with the E_{10} expectation that there should be no bounce.

8. Subleading terms: some examples

We now wish to rephrase the results derived in section 4 in a Lie algebraic setting, by giving more details concerning the subleading eighth-order derivative corrections, and by illustrating their asymptotic behaviour with some examples. More specifically, we will demonstrate explicitly how *all these terms can be associated either with (in general imaginary) roots of E_{10}, or, more generally, with elements of the E_{10} root lattice*. Comparing the relevant highest roots with the corresponding highest weights in the tables of [24] we can then read off to which $\mathfrak{sl}_{10}$ multiplets they belong when they do correspond to actual roots. Because the relevant roots turn out to be *negative*, the order of the integer entries from section 3 has to be inverted for this comparison: the Dynkin label is associated with the ordering $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_{10} \leqslant 0$, while the reverse ordering $0 \geqslant p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{10}$ will give the root that lies 'deepest' within the past lightcone.

In section 4, we estimated the various tensor structures in comparison with the leading curvature terms $R \sim e^{2\sigma}$. We have already remarked that the subleading terms are smaller than the latter terms by exponential factors involving electric and symmetry wall forms, which are linear combinations of the simple wall forms appearing in (29). Above, we discussed a coarse ordering of the effects of these wall forms by focussing on the number of basic electric
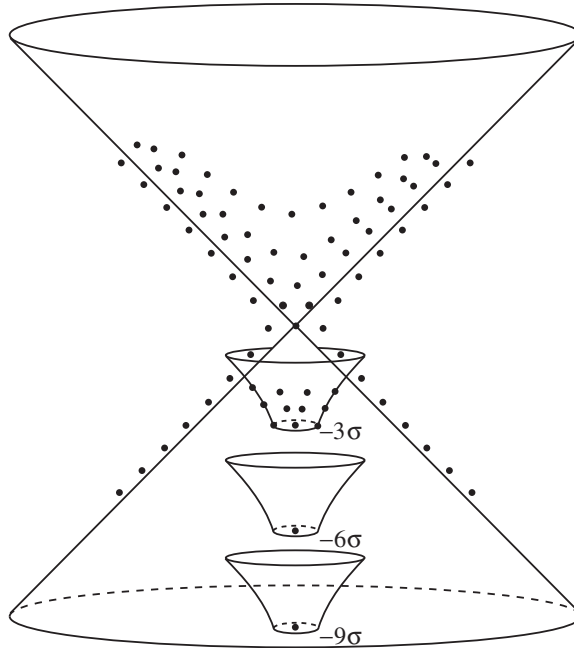
Figure 1. Schematic representation of the roots of E_{10} in the Lorentzian space of the Cartan degrees of freedom β. Positive roots are in the upper part of the figure, while negative ones are in the lower part. The real roots are on the outer (timelike) hyperboloid $\alpha^2 = +2$. The imaginary roots are either null ($\alpha^2 = 0$) or timelike ($\alpha^2 = -2j$). The basic root associated with the leading R^4 correction to the supergravity action is the negative, timelike point -3σ indicated. From it stems the solid half-hyperboloid of roots associated with subleading contributions contained in R^4 as well as in the other eighth-order derivative terms: $R^2(DF)^2$, $R(DF)^3$, $(DF)^4$, ..., F^8, as well as the Chern–Simons contribution AR^4. Similar half-hyperboloids stem from the roots $-3k\sigma$ predicted to exist and to be associated with higher-order contributions $\sim R^7$, R^{10}, ...

roots α_0 entering them. We shall now have a closer look at the various wall forms associated with the different types of higher-order corrections, and their relation to the root lattice of E_{10}.

From the results in section 4, and from the basic fact that in the final expression all indices must be contracted in pairs, and thus appear *an even number of times*, we can see that the subleading terms in the Hamiltonian constraint are all obtained by multiplying the leading Hamiltonian contribution $e^{6\sigma}$ (corresponding to the leading root $\alpha_{\text{leading}} = -3\sigma$) by an exponential factor of the form $\exp\left(-2\sum n_i \alpha_i\right)$, with n_i being some non-negative integers. Because of the presence of a factor 2 in the exponent (linked to the fact that all indices are finally contracted), we see that all the wall forms associated with the corrections can be written as $\exp(-2w_{\text{subleading}})$, with subleading wall forms of the type

$$w_{\text{subleading}} = \alpha_{\text{leading}} + \sum n_i \alpha_i \equiv -3\sigma + \sum n_i \alpha_i. \tag{80}$$

We see from (80) that the set of wall forms associated with higher-order corrections is a subset of the root lattice of E_{10}. To better understand this subset, it is convenient to introduce at this point a geometric picture of this subset. This is done in figure 1.

The above reasoning shows only that the wall forms are of the form (80) for *some* non-negative integers n_i. The tools we used above are too coarse to determine whether *all* values of the integers n_i are realized, or whether there are restrictions of any kind on the values of the n_i. In view of the complicated structure of the higher-order corrections, which as we pointed

out, is not known in an algebraically completely reduced form, especially for the crucial next-to-leading terms $R^2(DF)^2$ and R^3F^2), we have made no attempt at an exhaustive analysis as to whether the integers n_i are indeed subject to restrictions. Instead, we have proceeded 'experimentally' by studying in detail some specific, but typical, terms that we could check to be indeed present among the available expressions for various higher-order corrections. In all the cases we have checked, we did find a rather remarkable pattern: the vector in root space joining the 'leading' R^4 root $\alpha_{\text{leading}} = -3\sigma$ to any subleading higher-order wall form $w_{\text{subleading}}$, i.e. the quantity

$$\zeta := w_{\text{subleading}} - \alpha_{\text{leading}} = \sum n_i \alpha_i \qquad (81)$$

was 'experimentally' found to satisfy the restriction

$$\zeta^2 \equiv \left(\sum n_i \alpha_i \right)^2 \leqslant 2 \qquad (82)$$

in all cases. Below, we will refer to ζ as the 'relative vector'. This leads us to formulate the following:

Conjecture. *The relation (82) holds for all kinematically allowed combinations, and thereby encodes information about the algebraic structure of the eighth-order corrections.*

We shall illustrate below with some examples the interrelation between this conjecture and the properties of the E_{10} root lattice on one hand, and certain 'kinematical cancellations', as they follow from the kinematical structure of the known eighth order correction terms, on the other hand. However, we will leave a systematic investigation of this aspect to future work.

Geometrically, the restriction (82) (together with $n_i \geqslant 0$) means that the relative position vector ζ, equation (81), belongs to the set of positive roots, Δ_+. In other words, if we assume that this restriction is indeed true for all terms, we can geometrically describe the set of wall forms generated by higher-order corrections as a solid half-hyperboloid, which is congruent to Δ_+, and with the leading R^4 root $\alpha_{\text{leading}} = -3\sigma$ as its basis, see figure 1. From this picture there seems to be no evident upper limit on the height $\sum n_i$ of the relative vector (81). We initially thought that the wall forms $w_{\text{subleading}}$ might be constrained to lie inside the root diagram *stricto sensu* (instead of lying anywhere in the root lattice), which would have implied the further restriction $(w_{\text{subleading}})^2 \leqslant 2$. However, as we shall see below, we found wall forms lying high enough in the solid half-hyperboloid (82) to extend beyond the outer hyperboloid $w^2 = 2$ corresponding to real roots. More precisely, by adding simple roots of A_9 one generates $\mathfrak{sl}_{10}$ weight diagrams starting at a certain highest weight inside this solid half-hyperboloid. Many of the $\mathfrak{sl}_{10}$ weight diagrams are such that only a subset of the weights are actually roots of E_{10}, while the weight diagram itself extends beyond the hyperboloid $w^2 = 2$. We emphasize that there is no *a priori* inconsistency or incompatibility in this feature: indeed, starting from a Hamiltonian containing only roots as wall forms, any change of parametrization can generate arbitrary combinations of the roots, and therefore arbitrary elements of the root lattice.

Let us now substantiate our conjecture (82) by giving examples of wall forms associated with various higher-order contributions. We first consider the Chern–Simons term in (41), which is proportional to

$$\varepsilon^{ABCD_1 \cdots D_8} A_{ABC} R_{D_1 D_2}{}^{E_1 E_2} \cdots R_{D_7 D_8}{}^{E_7 E_8} t^{(8)}_{E_1 \cdots E_8} \equiv \varepsilon_{11} A_3 [24 \operatorname{tr} R^4 - 6(\operatorname{tr} R^2)^2]. \qquad (83)$$

Here the tensor t_8 is, by virtue of (42), equivalent to the combination of traces appearing in (41). Though the above contribution is also quartic in the curvature, and contains the spatial components A_{abc} of the 3-form which simply freeze near the singularity, it is actually

subleading wrt the leading curvature contribution $\sim E R^4 \sim e^{6\sigma}$. The reason for this is the interplay of the ε tensor, which constrains the first pairs of indices $[D_1 D_2] \cdots [D_7 D_8]$ on the curvature tensors to be all different, with the effect of the t_8 tensor, which contains Kronecker deltas and thereby obliges the remaining indices on the curvature tensors to coincide pairwise. As a consequence, it is not possible to have only leading curvature components of the type R_{0a0a} or R_{abab}. A typical contribution is[17]

$$E A_{8910} R_{0101} R_{0123} R_{0145} R_{0167} \sim \exp(3\sigma + 3\beta^1 + \beta^2 - \beta^3 + \beta^4 - \beta^5$$
$$+ \beta^6 - \beta^7 - \beta^8 - \beta^9 - \beta^{10}) \tag{84}$$

where we used (47). After reshuffling some indices, we obtain the wall form

$$w = -(3, 2, 2, 2, 1, 1, 1, 1, 1, 1) \equiv -[5; 2, 5, 8, 6, 5, 4, 3, 2, 1]. \tag{85}$$

Here, and below, we give the results both in the 'coordinate basis' (27) and the 'root basis' (31). The second form displays the wall form as an element of the E_{10} root lattice (w yields a root whenever $w^2 \leqslant 2$), with the level ℓ as the first entry as in (31). (85) is a root because $w^2 = 2$, with $\ell = -5$, and Dynkin label (100 100 000). The corresponding relative vector

$$\zeta = w - \alpha_{\text{leading}} \equiv w + 3\sigma = (0, 1, 1, 1, 2, 2, 2, 2, 2, 2) \tag{86}$$

satisfies $(w + 3\sigma)^2 = +2$, in agreement with (82). We thus see that this Chern–Simons wall form is so much higher than the basic R^4 root -3σ that the associated wall form lies even beyond the interior of the past CSA lightcone and belongs to the timelike hyperboloid $\alpha^2 = +2$.

Next we consider the subleading curvature terms. Let us first recall that the kinematic structure of the C^4 terms in (44) eliminates certain combinations; for instance, from (73) it is obvious that there are no contributions of the type C^4_{abab} (no summation on indices a, b). As is evident from (50), the curvature terms coming from $R^{(2)}_{abcd}$ and the above estimates are strongly suppressed. Furthermore, not all of them produce actual roots, as we shall see presently.

Obviously, from (50), there are many such terms, which we can group into certain 'permutation multiplets'. A first set of terms corresponds to terms of type

$$E R^{(2)}_{abcd} R^{(2)}_{abef} R^{(2)}_{cegh} R^{(2)}_{dfgh} \sim e^{-2\sigma} \exp(\beta^a + \beta^b + \beta^c - \beta^d) \cdot \exp(\beta^a + \beta^b + \beta^e + \beta^f - 2\beta^i)$$
$$\cdot \exp(\beta^c + \beta^e + \beta^g + \beta^h - 2\beta^j) \cdot \exp(\beta^d + \beta^f + \beta^g + \beta^h - 2\beta^k). \tag{87}$$

We can write this result schematically as

$$E[R^{(2)}]^4 \sim e^{-2\sigma} \cdot \exp(2(\beta^a + \beta^b + \beta^c + \beta^e + \beta^f + \beta^g + \beta^h - \beta^i - \beta^j - \beta^k)). \tag{88}$$

Taking all the indices different, and choosing $(i, j, k) = (8, 9, 10)$, this gives

$$w = -(0, 0, 0, 0, 0, 0, 0, 2, 2, 2) \equiv -[2; 2, 4, 6, 6, 6, 6, 6, 4, 2] \tag{89}$$

which is clearly *not* a root as $w^2 = +8$. Nevertheless, the relative vector

$$\zeta = w + 3\sigma = (3, 3, 3, 3, 3, 3, 3, 1, 1, 1) \tag{90}$$

obeys $\zeta^2 = 2$ and therefore lies on the boundary of the half-hyperboloid in agreement with our conjecture. A second set of terms derives from

$$E[R^{(2)}]^4 \sim e^{-2\sigma} \cdot \exp(+2(\beta^a + \beta^b + \beta^c + \beta^d + \beta^e + \beta^f - \beta^g - \beta^h)) \tag{91}$$

and differs from the one above by one symmetry wall. Again taking all indices different and choosing them appropriately, we get

$$w = -(0, 0, 0, 0, 0, 0, 1, 1, 2, 2) \equiv -[2; 2, 4, 5, 6, 6, 6, 6, 4, 2] \tag{92}$$

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[17] In the remainder, we will no longer distinguish between the Riemann and the Weyl tensors.

which is again not a root as $w^2 = +6$. The relative vector

$$\zeta = w + 3\sigma = (3, 3, 3, 3, 3, 3, 2, 2, 1, 1) \tag{93}$$

is lightlike, $\zeta^2 = 0$, and lies within the half-hyperboloid. A third set of terms comes from

$$E[R^{(2)}]^4 \sim e^{-2\sigma} \cdot \exp(+2(\beta^a + \beta^b + \beta^c + \beta^e + \beta^g - \beta^h)) \tag{94}$$

and gives rise to, choosing values for the indices conveniently,

$$w = -(0, 0, 0, 0, 0, 1, 1, 1, 1, 2) \equiv -[2; 2, 4, 6, 6, 6, 5, 4, 3, 2] \tag{95}$$

with $w^2 = +4$, and

$$\zeta = (3, 3, 3, 3, 3, 2, 2, 2, 2, 1) \tag{96}$$

with $\zeta^2 = -2$. Finally, there is a fourth set with

$$w = -(0, 0, 0, 0, 1, 1, 1, 1, 1, 1) \equiv -[2; 2, 4, 6, 6, 5, 4, 3, 2, 1] \tag{97}$$

which now *is* a root because $w^2 = 2$, and

$$\zeta = (3, 3, 3, 3, 2, 2, 2, 2, 2, 2) \quad \Rightarrow \quad \zeta^2 = -4 \tag{98}$$

again confirming the conjecture. The maximal height for any of these wall forms w is -44 for (89), well above the height -115 computed above for the leading term, see figure 1. We observe that all the terms displayed above differ from one another either by simple permutations of the wall form components, or by the addition of symmetry roots.

Similar comments apply to the curvature terms with one or more temporal indices. From (44), we find, for instance, that terms $\propto (C_{0abc})^4$ or $(C_{0abc})^2(C_{0ade})^2$, with the indices a, b, c, d, e all different (and no summation on repeated indices!), cancel between the two contributions in (44). Remarkably, these terms are precisely of a form which is disallowed by our above conjecture: for instance, if the term

$$E(R_{01ab})^2(R_{01cd})^2 \sim e^{-2\sigma} \cdot (e^{\sigma + \beta^1 + \beta^a - \beta^b})^2(e^{\sigma + \beta^1 + \beta^c - \beta^d})^2 \tag{99}$$

with a, b, c, d all different, had contributed instead of cancelling, it would have yielded (choosing indices conveniently)

$$w = -(3, 2, 2, 1, 1, 1, 1, 1, 0, 0) \quad \Rightarrow \quad \zeta = (0, 1, 1, 2, 2, 2, 2, 2, 3, 3) \tag{100}$$

whence $\zeta^2 = +4$, in violation of (82).

A set of terms which *does* contribute, by inspection of (44), is

$$E(R_{01ab})^2(R_{0101})^2 \sim e^{4\sigma - 2\beta^1 + 2\beta^a + 2\beta^b}. \tag{101}$$

Choosing $(a, b) = (9, 10)$, we get the maximal height contribution

$$w = -(1, 2, 2, 2, 2, 2, 2, 2, 3, 3) \equiv -[7; 6, 11, 16, 14, 12, 10, 8, 6, 3] \tag{102}$$

at level $\ell = -7$, of height $-93 > -115$. The Dynkin label is obtained from the wall form in reverse ordering $w = -(3, 3, 2, 2, 2, 2, 2, 2, 2, 1) \equiv -[7; 4, 8, 13, 11, 9, 7, 5, 3, 1]$ and is $(010\,000\,001)$. The corresponding relative vector is

$$\zeta = (2, 1, 1, 1, 1, 1, 1, 1, 0, 0) \quad \Rightarrow \quad \zeta^2 = 2 \tag{103}$$

and therefore on the boundary of the half-hyperboloid. From the tables of [24] we deduce that the set of roots corresponding to w belongs to the $\mathfrak{sl}_{10}$ representation (010000001) with outer multiplicity $\mu = 1$.

There are numerous mixed terms involving the curvature and the 4-form field strength, which we again illustrate with some examples (we have checked from [43] that the structures

displayed below actually do appear). For instance, the term $E(D_0 F_{0123})^2 (D_0 F_{0456})^2$ leads to the $\ell = 8$ root

$$w = -(3, 3, 3, 3, 2, 2, 2, 2, 2, 2) \equiv -[8; 5, 10, 15, 12, 10, 8, 6, 4, 2] \quad (104)$$

with $w^2 = -4$, and relative vector

$$\zeta = (0, 0, 0, 0, 1, 1, 1, 1, 1, 1) \quad \Rightarrow \quad \zeta^2 = 2. \quad (105)$$

At level $\ell = -8$, it corresponds to the $\mathfrak{sl}_{10}$ representation (000100000) with $\mu = 2$. The highest root in the multiplet is

$$w = -(2, 2, 2, 2, 2, 2, 3, 3, 3, 3) \equiv -[8; 6, 12, 18, 16, 14, 12, 9, 6, 3] \quad (106)$$

of height $-106 > -115$, well above the height of the leading R^4 singlet.

From the $C(DF)^3$ terms given in [43] we read off the leading term which is $E D_0 F_{3456} D_0 F_{78910} D_a F_{a012} C_{0b0b}$. The associated root is

$$w = -(2, 2, 2, 2, 2, 2, 2, 2, 1, 1) = -[6; 4, 8, 12, 10, 8, 6, 4, 2, 1] \quad (107)$$

with $w^2 = -2$, and

$$\zeta = (1, 1, 1, 1, 1, 1, 1, 1, 2, 2) \quad \Rightarrow \quad \zeta^2 = 0. \quad (108)$$

The relevant $\mathfrak{sl}_{10}$ representation at level $\ell = 6$ is (000000010) with $\mu = 1$. The maximal height in the representation is again above -115.

Our final example is the combination $E(D_0 F_{0123})^2 (C_{0a0a})^2$. It is associated with the $\ell = 9$ root

$$w = -(3, 3, 3, 3, 3, 3, 3, 2, 2, 2) = -[9; 6, 12, 18, 15, 12, 9, 6, 4, 2] \quad (109)$$

with relative vector

$$\zeta = (0, 0, 0, 0, 0, 0, 0, 1, 1, 1) \quad \Rightarrow \quad \zeta^2 = 2. \quad (110)$$

Indeed, in agreement with the result (63), we see that ζ is just an electric root. On the other hand, $w^2 = -6$, and w corresponds to the $\mathfrak{sl}_{10}$ representation (000000100) with $\mu = 4$. The highest root in the multiplet is

$$w = -(2, 2, 2, 3, 3, 3, 3, 3, 3, 3) \equiv -[9; 7, 14, 21, 18, 15, 12, 9, 6, 3] \quad (111)$$

of height $-114 > -115$, barely above the height of the leading singlet C^4 contribution.

9. Discussion

Our analysis of higher-order corrections to M-theory provides further evidence for the validity of the conjecture [8] that the classical (bosonic) dynamics of M-theory is 'dual' to a one-dimensional σ-model on the infinite-dimensional coset space $E_{10}/K(E_{10})$. In particular, we find it remarkable that the leading wall form associated with the known R^4 corrections, namely the *permutation singlet* $w_4 = -3\sigma$, does match with a root of E_{10} whose corresponding generator is a $\mathfrak{sl}_{10}$ *singlet*. This compatibility between a gravity structure and a Kac–Moody algebra one was not *a priori* guaranteed, and can be viewed as a deep confirmation of the hidden role of E_{10} in M-theory. Indeed, as a foil, let us consider the simple case of pure gravity in any spacetime dimension $D \equiv d + 1$. The study of the corresponding cosmological billiard has found that one should associate with pure gravity the Kac–Moody algebra AE_d [4]. Now, for pure (bosonic) gravity, one generally expects that the first higher-order corrections will be $\sim (R_{\mu\nu\rho\sigma})^2$. However, by using equation (69) such terms quadratic in curvature correspond to the wall form $w_2 = -\sigma$ whose squared length is $(w_2)^2 = -d/(d-1)$. As the latter squared

length is never an integer[18], we conclude that w_2 *never corresponds to a root of* AE_d. Let us then consider the problem of determining which values, if any, of the nonlinearity order N, for corrections $\sim R^N$, might be compatible with the algebraic structure of AE_d. For instance, let us consider the case $d = 3$ corresponding to the usual $(3 + 1)$-dimensional Einstein gravity. By using again equation (69), one gets the wall form $w_N = -(N-1)\sigma$ whose squared length in AE_d is $-\frac{d}{d-1}(N-1)^2$; i.e. $-\frac{3}{2}(N-1)^2$ when $d = 3$. One then concludes that one needs $N = 2k+1 = 3, 5, 7, \ldots$ The lowest candidate for compatibility with AE_3 is then $\sim R^3$. The corresponding wall form is $w_3 = -2\sigma = -2(\beta^1 + \beta^2 + \beta^3)$. Its squared length is $(w_3)^2 = -6$, and w_3 is easily seen to correspond to a root α of AE_3 of level $\ell = 3$ wrt the $SL(3)$ subalgebra [5]. However, as for E_{10} above, the wall form $w_3 = -2(\beta^1 + \beta^2 + \beta^3)$ is invariant under permutations of the spatial indices. Therefore, for the conjectured correspondence between $(3+1)$-dimensional Einstein gravity and AE_3 to hold, the wall form w_3 should correspond to a root of AE_3 which parametrizes a *singlet* of $SL(3)$. However, by using the results given in equation (8.30) of [5] one sees that *there is no singlet representation* of AE_3 at level $\ell = 3$. This negative result exemplifies that the compatibility found above between $\sim R^4$ corrections and the presence of 'singlet roots' of E_{10} is rather non-trivial, and could well have failed to hold.

Our work has also provided evidence for the 'no bounce' behaviour of big crunches in M-theory, as naively expected from the dual dynamics, i.e. the global structure of null geodesics on $E_{10}/K(E_{10})$. If we admit the validity of this conjecture, what conclusions can we draw for big crunches in M-theory? The 'dual' description is an infinite affine length null geodesic going towards $\beta \to \infty$. This suggests that the quantization of the $E_{10}/K(E_{10})$ model will exhibit no information loss at the big crunch. On the other hand, the dictionary of [8, 9] between the supergravity description and the coset description is defined only in the quasi-classical regime $T \gg T_P$. By contrast, the infinite future of the null geodesic motion corresponds to the regime $T \ll T_P$. The fact that the dictionary between the two descriptions becomes ill-defined in this limit (and that we find no evidence for a bounce) suggests that the infinite 'affine life' near the singularity can only be described in the coset variables. This situation is somewhat reminiscent of the recent results of [51] based on an AdS/CFT analysis of certain cosmological singularities in anti-deSitter solutions of supergravity—as is the fact that the dual E_{10} coset picture emerges in full only in the 'strongly coupled limit' $T \ll T_P$, while the gravity picture corresponds to the 'weakly coupled limit' $T \gg T_P$. (For a contrasting suggestion based, similarly to the E_{10} coset model, on geodesic motion in auxiliary Lorentz spaces, see [16, 17]).

However, we feel that at this stage of development of the $E_{10}/K(E_{10})$ conjecture, speculating on the ultimate quantum fate of big crunches is premature. Many technically challenging tasks remain before one can seriously consider the eventual physical consequences of the dual E_{10} picture. First, one should extend the dictionary to prove the heretofore unseen roots between height 29 and height 115 do match in the two descriptions. Second, it would be interesting to explore in more detail the validity of the conjecture (82), which was 'experimentally' observed to hold in quite a few specific cases. If our conjecture could be fully verified for the *known* eighth-order corrections, this would open the tantalizing possibility that it still holds for higher corrections $\sim (R + DF + F^2)^7$, $(R + DF + F^2)^{10}, \ldots$. It would then provide a strong constraint on the algebraic structure of these corrections. As these corrections are very difficult to obtain by conventional methods, the hidden E_{10} structure might be of great help in pinning down their structure.

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[18] For $d \geqslant 3$. Actually, we should restrict ourselves to $d \geqslant 4$ as all R^2 terms are known to be on-shell trivial in $d = 3$ because of the topological nature of the R^2 Euler–Gauss–Bonnet density.

Finally, among other pressing issues let us also mention: the role of fermions, the consequences of compactifying 11-dimensional spacetime (which is expected to reduce the continuous symmetry $E_{10}(\mathbf{R})$ to the discrete symmetry $E_{10}(\mathbf{Z})$) and the effect of quantizing the coset model (see, in this respect, also the remarks in the appendix). It would furthermore be interesting to explore the link, if any, between the non-compact T-duality symmetries of string theories in the presence of Killing vectors (which have been shown to survive the addition of higher-order terms [52]) and the conjectured E_{10} symmetry.

Acknowledgments

Appendix

Geodesic deviation and sectional curvature on $E_{10}/K(E_{10})$

As shown in [5, 8], geodesic motion on the coset space $E_{10}/K(E_{10})$ is formally integrable in the sense that one can exhibit as many constants of motion as degrees of freedom (i.e. a doubly infinite tower $\mathcal{J}$) and, moreover, that one can formally write a generic solution of the geodesic equation in terms of the constants of motion $\mathcal{J}$. We wish, however, to emphasize here that, even if formally integrable, geodesic motion on $E_{10}/K(E_{10})$ is at the same time strongly chaotic in the sense that it is extremely sensitive to small deviations in the initial conditions. Let us, indeed, consider the 'geodesic deviation equation' which governs the deviation $\boldsymbol{\xi}$ between two neighbouring geodesics. For simplicity, we focus on the neighbourhood of a base geodesic lying entirely in the CSA (corresponding to a simple Kasner solution), i.e. where only β^a and π_a are excited. This solution is simply described by $\beta^a = v^a t + \beta_0^a$ where t is an affine parameter, β_0^a some constants and where the 'velocity' v^a is null: $G_{ab} v^a v^b = 0$. The other ('off-diagonal') variables, v, p, are all zero. The geodesic deviation equation governing the affine-parameter evolution of the deviation vector $\boldsymbol{\xi}$ connecting this geodesic to a neighbouring one can be obtained in two different ways: either by 'moving' the base geodesic $\beta^a = v^a t + \beta_0^a$ by a generic element in the Lie algebra of E_{10}, or by computing the appropriate components of the sectional curvature of the symmetric space $E_{10}/K(E_{10})$ and by writing the general geodesic-deviation (Jacobi) equation. We have checked that both methods give the same result. If we decompose the deviation vector $\boldsymbol{\xi}$ into its various components:

$$\boldsymbol{\xi} = \xi^a h_a + \sum_{\alpha,s} \xi_{(s)}^\alpha \big(E_\alpha^{(s)} + F_\alpha^{(s)}\big)/\sqrt{2},$$

where $h_a, a = 1, \ldots, 10$, is a basis of the CSA, and where $\xi_{(s)}^\alpha$ denote the components along the non-CSA coset directions. The Jacobi equation yields $\ddot{\xi}^a = 0$, corresponding to a vanishing sectional curvature in the vv' two-planes (where $v = v^a h_a$ is the vector tangent to the base geodesic and $v' = v'^a h_a$ a generic direction in the CSA), and $\ddot{\xi}_{(s)}^\alpha = +(\alpha(v))^2 \xi_{(s)}^\alpha$, corresponding to a sectional curvature

$$\mathcal{R}\big(v, E_\alpha^+, v, E_\alpha^+\big) = -(\alpha(v))^2$$

where $E_\alpha^+ \equiv (E_\alpha + F_\alpha)/\sqrt{2}$ is a unit-norm vector in the 'transverse' coset direction associated with the root α. (For simplicity, we suppressed the degeneracy index s on $E_\alpha^{(s)}$ and $F_\alpha^{(s)}$.) Here $\alpha(v) \equiv \alpha_a v^a$ denotes the value of the linear form α acting on the vector $v = v^a h_a$ tangent to the geodesic. The point we wish to emphasize here is twofold: (i) the sectional curvature $\mathcal{R}(v, E_\alpha^+, v, E_\alpha^+)$[19] is *negative* (as for geodesics on hyperbolic space), and (ii) this sectional curvature *decreases without limit towards* $-\infty$ as the height of the root increases to infinity. This result shows that small deviations from the geodesic exponentially increase with affine length in a manner which becomes faster and faster as we consider components of the deviation in transverse directions corresponding to higher and higher roots. This indicates that geodesics on $E_{10}/K(E_{10})$ are more and more exponentially sensitive to initial conditions as one examines them in directions of increasing height. This property might play an important role for the quantization of the geodesic motion on $E_{10}/K(E_{10})$. Indeed, we recall that quantum motion for a scalar particle on curved space involves an ambiguous coupling to the scalar curvature, which appears to be given by a formally infinite series for $E_{10}/K(E_{10})$. This suggests that it is important to include fermions in the E_{10} coset model before drawing conclusions about the strong sensitivity of classical geodesic motion on initial conditions.

References

[1] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 *Adv. Phys.* **19** 525
 Belinskii V A, Khalatnikov I M and Lifshitz E M 1982 *Adv. Phys.* **31** 639
[2] Cremmer E, Julia B and Scherk J 1978 Supergravity theory in 11 dimensions *Phys. Lett.* B **76** 409–12
[3] Damour T and Henneaux M 2001 E_{10}, BE_{10} and arithmetical chaos in superstring cosmology *Phys. Rev. Lett.* **86** 4749–52 (*Preprint* hep-th/0012172)
[4] Damour T, Henneaux M, Julia B and Nicolai H 2001 Hyperbolic Kac–Moody algebras and chaos in Kaluza–Klein models *Phys. Lett.* B **509** 323–30 (*Preprint* hep-th/0103094)
[5] Damour T, Henneaux M and Nicolai H 2003 Cosmological billiards *Class. Quantum Grav.* **20** R145–200 (*Preprint* hep-th/0212256)
[6] Kac V G 1990 *Infinite Dimensional Lie Algebras* 3rd edn (Cambridge: Cambridge University Press)
[7] Conway J H and Sloane N J A 1991 Sphere packings, lattices and groups *Grundlehren der mathematischen Wissenschaften* vol 290 2nd edn (Berlin: Springer)
[8] Damour T, Henneaux M and Nicolai H 2002 E_{10} and a small tension expansion of M-theory *Phys. Rev. Lett.* **89** 221601 (*Preprint* hep-th/0207267)
[9] Damour T and Nicolai H 2004 Eleven dimensional supergravity and the $E_{10}/K(E_{10})\sigma$-model at low A_9 levels *Preprint* hep-th/0410245
[10] Julia B 1981 *Superspace and Supergravity* ed S W Hawking and M Rocek (Cambridge: Cambridge University Press)
[11] Julia B 1984 Kac–Moody symmetry of gravitation and supergravity theories invited talk given at *AMS-SIAM Summer Seminar on Applications of Group Theory in Physics and Mathematical Physics (Chicago, IL 6–16 July, 1982) Vertex Operators in Mathematics and Physics* (Publications of the Mathematical Sciences Research Institute no 3) (Berlin: Springer)
[12] West P C 2001 E_{11} and M-theory *Class. Quantum Grav.* **18** 4443 (*Preprint* hep-th/0104081)
[13] West P C 2003 $E(11)$, $SL(32)$ and central charges *Phys. Lett.* B **575** 333 (*Preprint* hep-th/0307098)
 West P C 2004 Brane dynamics, central charges and $E(11)$ *Preprint* hep-th/0412336
[14] DeWitt B S 1967 Quantum theory of gravity. 1. The canonical theory *Phys. Rev.* **160** 1113
[15] Gibbons G W and Townsend P K 1987 Cosmological evolution of degenerate vacua *Nucl. Phys.* B **282** 610
[16] Townsend P K and Wohlfarth M N R 2004 Cosmology as geodesic motion *Class. Quantum Grav.* **21** 5375
[17] Russo J G and Townsend P K 2005 Cosmology as relativistic particle mechanics: from big crunch to big bang *Class. Quantum Grav.* **22** 737
[18] Brown J, Ganor O and Helfgott C 2004 M-theory and E_{10}: billiards, branes, and imaginary roots *J. High Energy Phys.* JHEP08(2004)063 (*Preprint* hep-th/0401053)

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[19] As $E_{10}/K(E_{10})$ is a homogeneous space the sectional curvature components computed here should be the same in the neighbourhood of an arbitrary geodesic.

[19] Englert F and Houart L 2004 $\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: exact BPS solutions *J. High Energy Phys.* JHEP01(2004)002 (*Preprint* hep-th/0311255)
Englert F and Houart L 2004 $\mathcal{G}^{+++}$ invariant formulation of gravity and M-theories: exact intersecting brane solutions *J. High Energy Phys.* JHEP05(2004)059 (*Preprint* hep-th/0405082)

[20] Englert F, Henneaux M and Houart L 2004 From very extended to overextended gravity and M theories *Preprint* hep-th/0412184

[21] Humphreys J E 1980 *Introduction to Lie Algebras and Representation Theory (Graduate Texts in Mathematics, vol 9)* (Berlin: Springer)

[22] Kleinschmidt A and Nicolai H 2004 E_{10} and $SO(9, 9)$ invariant supergravity *J. High Energy Phys.* JHEP07(2004)041 (*Preprint* hep-th/0407101)

[23] Kac V, Moody R and Wakimoto M 1988 On E_{10} *Differential Geometrical Methods in Theoretical Physics* ed K Bleuler and M Werner (Dordrecht: Kluwer)

[24] Nicolai H and Fischbacher T 2003 Low level representations of E_{10} and E_{11} *Proc. Ramanujan Int. Symposium on Kac–Moody Lie Algebras and Related Topics (Contemporary Mathematics* vol 343*)* ed N Sthanumoorthy and K C Misra (Providence, RI: American Mathematical Society) (*Preprint* hep-th/0301017)

[25] Duff M J, Liu J T and Minasian R 1995 Eleven-dimensional origin of string/string duality: a one-loop test *Nucl. Phys.* B **452** 261 (*Preprint* hep-th/9506126)

[26] Peeters K, Vanhove P and Westerberg A 2001 Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace *Class. Quantum Grav.* **18** 843 (*Preprint* hep-th/0010167)

[27] Cederwall M, Gran U, Nielsen M and Nilsson B E W 2000 Manifestly supersymmetric M-theory *J. High Energy Phys.* JHEP10(2000)041 (*Preprint* hep-th/0007035)

[28] Howe P and Tsimpis D 2003 On higher order corrections in M-theory *J. High Energy Phys.* JHEP09(2003)038 (*Preprint* hep-th/0305129)

[29] Howe P S 2004 R**4 terms in supergravity and M-theory *Preprint* hep-th/0408177

[30] Green M B and Schwarz J H 1982 Supersymmetric dual string theory (III). Loops and renormalization *Nucl. Phys.* B **198** 441

[31] Sakai N and Tanii Y 1987 One loop amplitudes and effective action in superstring theories *Nucl. Phys.* B **287** 457

[32] Gross D J and Witten E 1986 Superstring modifications of Einstein's equations *Nucl. Phys.* B **277** 1

[33] Grisaru M T and Zanon D 1986 Sigma model superstring corrections to the Einstein–Hilbert action *Phys. Lett.* B **177** 347

[34] Myers R C 1987 Superstring gravity and black holes *Nucl. Phys.* B **289** 701

[35] Gross D J and Sloan J H 1987 The quartic effective action for the heterotic string *Nucl. Phys.* B **291** 41–89

[36] Green M B and Vanhove P 1997 D-instantons, strings and M-theory *Phys. Lett.* B **408** 122 (*Preprint* hep-th/9704145)

[37] Green M B, Gutperle M and Vanhove P 1997 One loop in eleven dimensions *Phys. Lett.* B **409** 177 (*Preprint* hep-th/9706175)

[38] Green M B, Kwon H and Gutperle M 1999 *J. High Energy Phys.* JHEP08(1999)012 (*Preprint* hep-th/9907155)

[39] Tseytlin A A 2000 R**4 terms in 11 dimensions and conformal anomaly of $(2, 0)$ theory *Nucl. Phys.* B **584** 233 (*Preprint* hep-th/0005072)

[40] Dasgupta A, Nicolai H and Plefka J 2000 Vertex operators for the supermembrane *J. High Energy Phys.* JHEP05(2000)007 (*Preprint* hep-th/0003280)

[41] Pioline B, Nicolai H, Plefka J and Waldron A 2001 R**4 couplings, the fundamental membrane and exceptional theta correspondences *J. High Energy Phys.* JHEP03(2001)036 (*Preprint* hep-th/0102123)

[42] Plefka J 2001 Vertex operators for the supermembrane and background field matrix theory *Int. J. Mod. Phys.* A **16** 660 (*Preprint* hep-th/0009193)

[43] Deser S and Seminara D 1999 Counterterms and M-theory corrections to $D = 11$ supergravity *Phys. Rev. Lett.* **82** 2435
Deser S and Seminara D 2000 Tree amplitudes and two-loop counterterms in $D = 11$ supergravity *Preprint* hep-th/0002241

[44] Fulling S, King R, Wybourne B and Cummins C 1992 *Class. Quantum Grav.* **9** 1151

[45] http://wwwmathlabo.univ-poitiers.fr/maavl/LiE/

[46] Schwarz J H 1982 Superstring theory *Phys. Rep.* **89** 223

[47] Russo J G and Tseytlin A A 1997 One-loop four-graviton amplitude in eleven-dimensional supergravity *Nucl. Phys.* B **508** 245 (*Preprint* hep-th/9707134)

[48] Kehagias A and Partouche H 1998 D instanton corrections as (p, q) string effects and non-renormalization theorems *Int. J. Mod. Phys.* A **13** 5075 (*Preprint* hep-th/9712164)

[49] Peterson D and Kac V G 1983 *Proc. Natl Acad. Sci. USA* **80** 1778

[50] Deruelle N 1989 On the approach to the cosmological singularity in quadratic theories of gravity: the Kasner regimes *Nucl. Phys.* B **327** 253

[51] Hertog T and Horowitz G T 2005 Holographic description of AdS cosmologies *Preprint* hep-th/0503071

[52] Meissner K A 1997 Symmetries of higher-order string gravity actions *Phys. Lett.* B **392** 298 (*Preprint* hep-th/9610131)