One-Loop QCD Spin Chain and its Spectrum

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Abstract

We study the renormalization of gauge invariant operators in large N_c QCD. We compute the complete matrix of anomalous dimensions to leading order in the 't Hooft coupling and study its eigenvalues. Thinking of the mixing matrix as the Hamiltonian of a generalized spin chain we find a large integrable sector consisting of purely gluonic operators constructed with self-dual field strengths and an arbitrary number of derivatives. This sector contains the true ground state of the spin chain and all the gapless excitations above it. The ground state is essentially the anti-ferromagnetic ground state of a XXX_1 spin chain and the excitations carry either a chiral spin quantum number with relativistic dispersion relation or an anti-chiral one with non-relativistic dispersion relation.

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1 Introduction and summary

The computation of anomalous dimensions of composite operators is of central importance in many areas of physics, most notably condensed matter and particle physics. In particle physics, their phenomenological interest stems from hadronic processes at high energy. For instance, in the study of deep inelastic scattering [1] one is led to consider the operator product expansion of two hadronic currents. The anomalous dimensions of twist-two operators appearing in the expansion control the logarithmic deviations to the Bjorken scaling. Considering operators with low twist amounts to considering only operators made up of very few fundamental fields $F_{\mu\nu}$ and q (two in most cases) and possibly many derivatives. These are the operators that are important in collider physics.

On the other hand, the study of operators containing many fundamental fields has received new impetus in connection with the attempts to construct a string description of QCD at low energies [2]. There is probably still a long way to go in fulfilling this dream (if indeed the idea is viable), but for other gauge theories the answer has been found. We now have a string theory description of the maximally supersymmetric $\mathcal{N} = 4$ $SU(N_c)$ theory in terms of type IIB strings in an $AdS_5 \times S^5$ background [3,4]. This was rather surprising at first since the $\mathcal{N} = 4$ theory is not confining and there are no color flux tubes in the ordinary sense. The connection with string theory was instead obtained by identifying the string spectrum with the spectrum of anomalous dimensions of the theory [4].

After many checks performed in the supergravity regime, a new important step was taken in [5] where it was shown that by taking an appropriate limit of both sides of the correspondence one could match anomalous dimensions of near BPS operators with the corresponding string states. On the gauge theory side this amounts to considering long operators with large classical dimension and R-charge. These operators are constructed by taking the BPS operator $\operatorname{tr} Z^J$, where Z is one of the three chiral superfields making up the $\mathcal{N} = 4$ multiplet, and adding a few "impurities", i.e. replacing some of the Z with other fields in the theory. The analysis of near BPS states led to an agreement to all loops in the 't Hooft coupling[1] $\alpha_s N_c$ [6].

States with large quantum numbers should correspond to semiclassical string states whether they are near BPS or not. In fact, as noticed in [7] one can get a qualitative agreement between the anomalous dimension of twist-two operators of large spin S computed at small $\alpha_s N_c$ and the energy of a classical rotating string in AdS space, valid for large values of $\alpha_s N_c$. In both cases, the anomalous dimension grows like $\log S$, one of the most famous "trade-marks" of non-abelian gauge theories [1]. Of course, the dependence of the coefficient of $\log S$ on the 't Hooft coupling $\alpha_s N_c$ is polynomial on the gauge theory side, as always in perturbation theory, whereas it grows like $\sqrt{\alpha_s N_c}$ in the semiclassical AdS analysis, a "screening" phenomenon that has already been observed e.g. in the analysis of the Wilson loops [8].

The above qualitative observations can be made quantitative by considering, once again, the $\mathcal{N} = 4$ theory [9–11]. In this case the string can instead be allowed to rotate in the compact S^5 dimensions. Giving the string two large angular momenta J_1 and J_2

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[1]We use the standard definition $\alpha_s = g_{YM}^2/4\pi$.

on the sphere (recall that the isometry group is $SO(6)$ and has rank three) one obtains an expression for the energy that is analytic in $\alpha_s N_c$ and thus can be expanded around $\alpha_s N_c = 0$. In order to match this expansion with the perturbative analysis one needs to compute the anomalous dimension of operators that are far from the BPS condition. This is successfully accomplished by noticing that the matrix of planar anomalous dimensions can be interpreted as the Hamiltonian of an integrable spin chain – not only at one-loop [12, 13], but also at higher loops [14] – and its eigenvalues can thus be obtained by applying the Bethe ansatz [12, 13, 10, 15]. For an introduction to this subject, c.f. [16]. (See also [17] for a computation of correlation functions from the Bethe ansatz.)

Following this line of thought, Kruczenski [18] showed that the agreement is already present at the level of the effective sigma model (see also [19] for related work). Namely, the effective action obtained by taking the long wavelength limit of the one-loop spin chain Hamiltonian and the action of the rotating string in the corresponding space-time background are the same. This leads to the exciting possibility of extracting information about the string dual from the perturbative study of anomalous dimensions and was the main motivation behind the previous work by some of us [20] in the context of large N_c (non-supersymmetric) QCD. In [20] we showed that the closed spin chains constructed with only the selfdual field strength form an integrable subsector. In this paper we continue this investigations in the hope that it will reveal the nature of the excitations required in the description of the QCD string.

Before moving to describing the results of our investigation, we must stress that the subject of integrability in the context of QCD has a long parallel history starting with the introduction and the study of operators on the light cone in [21], (see also [22]) and continuing with the discovery that their renormalization is described by an integrable, non-compact spin chain [23–25]. The early papers on the subject dealt only with "short" operators, i.e. operators containing a small number of elementary fields but with an arbitrary number of derivatives. We refer the reader to the reviews [26, 27] for a survey of the literature on this subject. Very recently, the complete matrix of anomalous dimension for light-cone operators of arbitrary length has been given in [28] for theories with any number of supersymmetries. One of the results in the present paper is the complete one-loop matrix of anomalous dimension for large N_c QCD without the restriction to the light cone. This was obtained by first solving the counting problem for operators and then truncating the known $\mathcal{N} = 4$ answer of [29] but it can also be thought of as a "lifting" of the light cone results of [28] to the full conformal group. For earlier partial results that were useful in the initial stages of this investigation see [30]. Recently, anisotropic spin chains have also made an appearance in this context [31, 28].

The main results of the present work are the following:

We give the complete one-loop matrix of anomalous dimension for large N_c QCD and study its eigenvalues. We use a particularly convenient choice of basis for such operators that avoids the necessity of restricting oneself to light-cone operators. In order to use the intuition from condensed matter we will be thinking of the matrix of anomalous dimensions as the Hamiltonian of a spin chain. We are particularly interested in the "thermodynamic limit" consisting of studying operators with a large number of elementary fields.

We are able to construct the exact (anti-ferromagnetic) ground state, corresponding

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to the operator with the lowest (negative) anomalous dimension for any given length and we show that all gapless excitations above the ground state are described by an integrable system based on the conformal group $SO(4,2)$. There are in fact two equivalent ground states, corresponding to scalar operators constructed by only using the selfdual field strength or by only using the anti-selfdual component. Operators constructed using both, or operators containing quarks, are separated by a "mass gap" from the ground state, i.e. their anomalous dimension remains higher than the ground state by a finite amount in the thermodynamic limit.

With the above "stringy" motivation in mind we then analyze the structure of the operators that contain "gapless" excitations. These are the QCD equivalents of the BMN operators for $\mathcal{N} = 4$.

The analysis of the quantum numbers carried by the gapless excitations reveals some surprises. Focusing on the chiral ground state for definitiveness, we find, as already expected from [20], excitations carrying a (space-time) chiral index α corresponding to the "spinons" of the compact XXX_1 chain embedded in the conformal chain ($(SU(2)_L \in SO(4,2))$. As well known from the condensed matter literature, such excitations along the chain are characterized by a linear ("relativistic") dispersion relation.

But there is another type of excitation that was not accessible by the previous analysis which did not include derivatives. Namely, by "spreading" a covariant derivative along the ground state chain, very much like the impurities in the BMN case, we are able to construct new excitations along the chain that are characterized by a quadratic ("non-relativistic") dispersion relation. One would naively expect the quantum number carried by these excitations to be a space-time vector since a covariant derivative carries a space-time index μ. However, by writing $\mu = (\alpha, \dot\alpha)$ and recalling that α propagates in an anti-ferromagnetic background, we are able to show that the presence of D_μ gives rise to two independent elementary excitations, one being the above mentioned spinon and the other carrying a space-time anti-chiral index $\dot\alpha$. It is well known since the work of Faddeev and Takhtajan [32,33] that scattering off background spins can modify quantum numbers of excitations in anti-ferromagnetic spin chains. Here we found another manifestation of this phenomenon.

The picture that emerges from the one-loop analysis has some intriguing similarities to the twistor string theory of Witten [34] and it would be very interesting to make the connection more explicit. What we can say is that the thermodynamic limit of the one-loop QCD chain is described by chiral/anti-chiral twistorial excitations with linear/quadratic dispersion relation. Whether these are (part of) the correct degrees of freedom required to formulate the string description of QCD, what their space-time dynamics is and the connections (if any) with the work of [34] remain a project for the future.

The paper is organized as follows:

We begin in Section 2 by discussing some basic preliminary facts about the renormalization of composite operators, the simplifications occurring in the large $N_{\rm c}$ limit and the definition of the spin chain Hamiltonian.

In Section 3 we discuss in detail the construction of the complete basis of operators that form the Hilbert space of our spin chain. Having such a complete basis makes the restriction to the light-cone no longer necessary.

Section 4 contains the complete one-loop chain of anomalous dimensions expressed in terms of projectors on irreducible representations of the conformal group. Recall that the use of conformal symmetry to classify the operator to one-loop is possible, even in a non-conformal field theory such as QCD, since the beta function is of order $O((\alpha_s N_c)^2)$ whereas the anomalous dimensions are $O(\alpha_s N_c)$. The relevant mathematical formalism is summarized in Appendix A. The coefficients of the projection operators are obtained by a truncation of the corresponding matrix in the $\mathcal{N} = 4$ theory with the additional modifications required by the different wave-function renormalization. We also give a table of anomalous dimensions of all conformal primary operators up to dimension seven for pure glue and five for those involving quarks. We conclude the section by showing briefly how the same results could also be obtained by "lifting" the light-cone results of [28]. To this purpose, we consider for definitiveness the gluon-quark coupling. Although this is not the way we arrived at the results, we feel it may be useful to include it in order to establish a connection with the previous literature.

Section 5 presents the Bethe ansatz for the chiral sector of the theory. We show that the closed chain composed of selfdual gluon field strengths and derivatives is integrable and we give the Bethe equations corresponding to the conformal group $SO(4, 2)$. We begin our discussion of the possible excitations over the ground state. As a check of our computation we present all the Bethe roots corresponding to the conformal primaries in the integrable sector up to dimension seven. We conclude with some remarks on the open chain, which is also integrable for selfdual gluons in the bulk. We present the Bethe ansatz for the case of open chains without derivatives.

Section 6 contains a more detailed analysis of the excitations over the ground state. First, we show that operators containing a mixture of selfdual and anti-selfdual fields or quarks have an anomalous dimension that remains above the previously found ground state by a finite amount (gap) in the thermodynamic limit. This establishes that we have indeed found the true ground state. We then move on to study the gapless excitations. As mentioned before they are characterized by a dispersion relation that is linear or quadratic for chiral and anti-chiral objects respectively.

In Appendix A we review the oscillator representation of the conformal group and the decompositions required in the paper.

In Appendix B we make the comparison with the length two primaries some of which are well know from the literature on deep inelastic scattering. We do this partly as a check and partly as an illustration of how to use the Hamiltonian obtained in Section 4.

In Appendix C we give a proof of integrability for the chiral sector of pure YM by solving the corresponding Yang–Baxter equations.

2   Preliminaries

In this paper we consider anomalous dimensions of local, gauge invariant operators for massless QCD:

$$L = -\frac{1}{2} \operatorname{tr} F^2 + i\bar{q} \, \slashed{D} q, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{\text{YM}}[A_\mu, A_\nu]$ is the gauge field strength, thought of as an $N_c \times N_c$ hermitian matrix, and q is the quark field, transforming in the fundamental representation of the gauge group $SU(N_c)$ and possibly carrying a finite number of flavors. It will be very convenient in the following to express such field content in the chiral basis by decomposing $F_{\mu\nu}$ into its selfdual and anti-selfdual parts f and $\bar{f}$

$$F_{\mu\nu} = \sigma_{\mu\nu}^{\alpha\beta} f_{\alpha\beta} + \bar{\sigma}_{\mu\nu}^{\dot{\alpha}\dot{\beta}} \bar{f}_{\dot{\alpha}\dot{\beta}} \tag{2.2}$$

and the quark field in its chiral and anti-chiral parts ψ and $\bar{\chi}$:

$$q = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.3}$$

We employ the 't Hooft limit of letting the number of colors N_c go to infinity and the usual QCD coupling g_{YM} to zero while holding the 't Hooft coupling $\alpha_s N_c$ finite. In fact, we will deal with one-loop perturbation theory in $\alpha_s N_c$ which must therefore be assumed to be small.

The latter condition would make the limit rather simple were it not for the fact that we apply it to the study of anomalous dimensions of gauge invariant composite operators made out of a large number of elementary fields where even ordinary perturbation theory becomes quite involved.[2]

Let us briefly recall how anomalous dimensions of composite operators are defined in quantum field theory. We use the conventions of [35] throughout the paper. In particular, let us denote by $\mathcal{O}$ a multiplicatively renormalized operator, i.e. an operator that can be renormalized by multiplying it with a (divergent) renormalization constant. We set

$$\mathcal{O}_{\text{bare}} = Z_{\mathcal{O}} \mathcal{O}_{\text{ren.}}, \tag{2.4}$$

where $Z_{\mathcal{O}}$ is constructed so that $\mathcal{O}_{\text{ren.}}$ has finite correlation functions with the renormalized quark and gluon fields. The anomalous dimension of $\mathcal{O}$ is defined through

$$\gamma_{\mathcal{O}} = \mu \frac{\partial}{\partial \mu} \log Z_{\mathcal{O}}. \tag{2.5}$$

To one-loop in $\alpha_s N_c$ it takes the form

$$\gamma_{\mathcal{O}} = c_{\mathcal{O}} \frac{\alpha_s N_c}{2\pi}, \tag{2.6}$$

where $c_{\mathcal{O}}$ is a numerical constant to be determined.

In general, operators carrying the same quantum numbers and having the same *classical* dimension will mix under renormalization and we write:

$$\mathcal{O}_{\text{bare}}^i = Z_{\mathcal{O}j}^{\phantom{\mathcal{O}}i} \mathcal{O}_{\text{ren.}}^j. \tag{2.7}$$

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[2]Actually, with some more work it would be possible to obtain the mixing matrix at finite N_c but we decided not to include it in this paper as our main motivation is to learn about the stringy description where $1/N_c$ plays the role of the string coupling.

The anomalous dimension also becomes a matrix and we write, in matrix notation:

$$\Gamma = \mu \frac{\partial}{\partial \mu} \log Z_{\mathcal{O}} = Z_{\mathcal{O}}^{-1} \mu \frac{\partial}{\partial \mu} Z_{\mathcal{O}} = \frac{\alpha_{\mathrm{s}} N_{\mathrm{c}}}{2\pi} H. \tag{2.8}$$

Multiplicatively renormalizable operators correspond to eigenvectors of Γ and their eigenvalues correspond to the anomalous dimensions:

$$\Gamma \hat{\mathcal{O}} = \gamma_{\hat{\mathcal{O}}} \hat{\mathcal{O}}. \tag{2.9}$$

The problem of the computation of anomalous dimensions is thus split into two parts: First one must derive the mixing matrix Γ and second one must find its eigenvalues. In the past few years new techniques, inspired by the study of $\mathcal{N} = 4$ SYM have been developed that bypass the computation of the divergences in the loop diagrams and allow a direct determination of Γ. These techniques are based on the analysis on the dilatation operator, thought of as one of the generators of the conformal group. The second part, involving the diagonalization of Γ, also requires analytical work since we are mostly interested in letting the size of the matrix go to infinity by considering "long" operators. Fortunately, even this second problem can be handled by analytic methods such as the Bethe ansatz for the cases of interest.

We shall refer to a generic composite operator as a "chain", where the name originates by the interpretation of such operators as a spin chain, as we will discuss in detail later.

Neglecting for the moment the details of the Lorentz structure of these operators, such chains can be grossly divided into two main groups: (see Fig. 1)
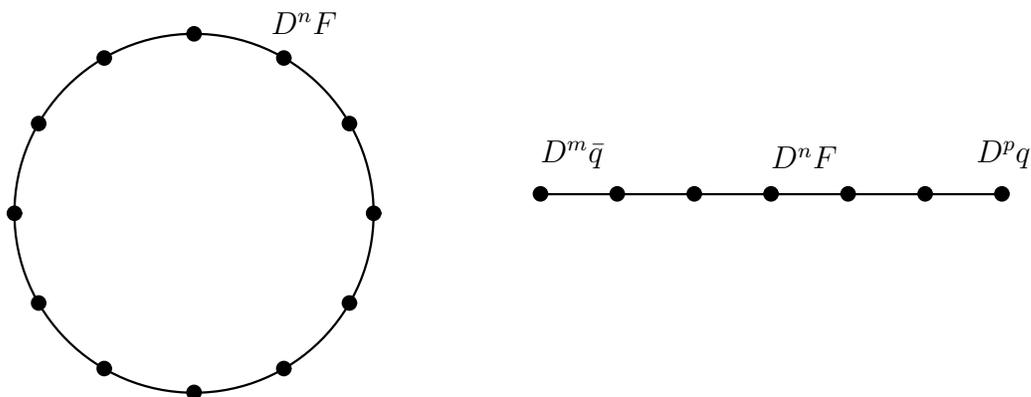

Figure 1: Closed and open "spin chains" corresponding to purely gluonic operators or mesonic operators respectively.

- Closed chains, i.e. operators schematically of the type

$$\mathrm{tr}(D^{n_1} F) \dots (D^{n_L} F), \tag{2.10}$$

- Open chains, i.e. operators schematically of the type

$$(D^{n_1} \bar{q})(D^{n_2} F) \dots (D^{n_{L-1}} F)(D^{n_L} q), \tag{2.11}$$

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Notice that we will distinguish between different conformal descendants, i.e. will not identify operators differing by a total derivative. Therefore it does not suffice to ignore derivatives on the first site $\bar{q}$ as is commonly done in QCD.

In the large N_c limit the following three crucial simplifications occur:

- If we consider two eigenvectors $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$, linear combinations of either type of operator described above, with anomalous dimensions $\gamma_{\hat{\mathcal{O}}_1}$ and $\gamma_{\hat{\mathcal{O}}_2}$ respectively, then $\Gamma(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2) = (\gamma_{\hat{\mathcal{O}}_1} + \gamma_{\hat{\mathcal{O}}_2})(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2)$. Hence, it is enough to consider operators that cannot be split into products of gauge singlets, commonly referred to as "single-trace" operators, although, in the presence of quarks the name "irreducible" would be more appropriate.

- Open and closed chains do not mix. This can be seen by noticing that with the usual normalization, the mixing matrix between open and closed chains becomes upper triangular. Alternatively, one can show that by scaling the quark fields by an extra factor $\sqrt{N_c}$ the mixing becomes block diagonal. We shall refer to the mixing matrix for open and closed operators as Γ_{open} and Γ_{closed} respectively.

- Finally, the relevant Feynman diagrams for the computation of the anomalous dimension are only those connecting neighboring fields (see Fig. 2). Thus we can identify either type of basic operator with a chain of "spins", each spin corresponding to a basic building block $D^n f$, $D^n \psi$, etc. The length L of the chain is conserved to one-loop and the matrix of anomalous dimensions can be regarded as a Hamiltonian of the spin chain with only nearest neighbor interactions.
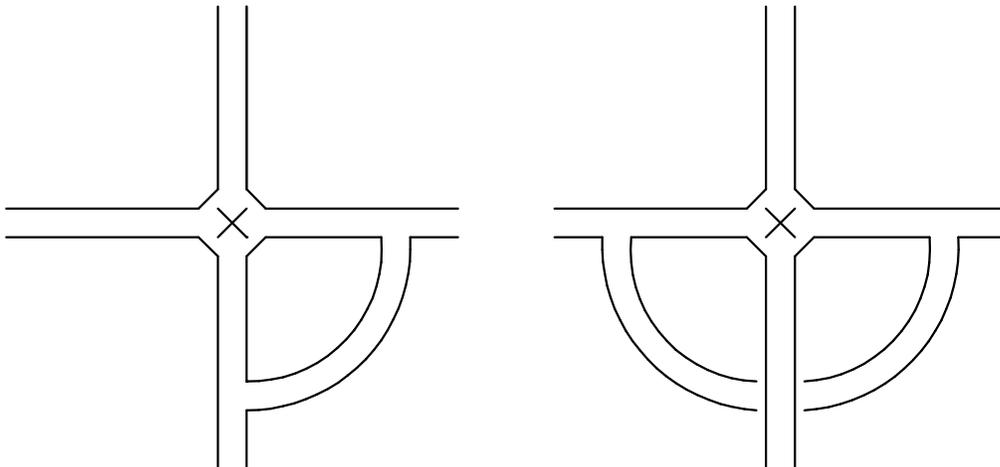

Figure 2: Renormalization of a four gluon composite operator (denoted by a cross) in double line notation. To the left is depicted a diagram connecting two adjacent gluon legs containing a closed color loop. To the right is shown a diagram connecting two non-adjacent gluon legs, not containing any color loop and thus subleading in $1/N_c$.

Thus we write, for the closed chain, with the usual periodic identification $L + 1 \to 1$:

$$\Gamma_{\text{closed}} = \frac{\alpha_{\text{s}} N_{\text{c}}}{2\pi} \sum_{l=1}^{L} \mathbf{H}_{l,l+1}^{\text{FF}} \tag{2.12}$$

and, for the open chain ($L > 2$):

$$\Gamma_{\text{open}} = \frac{\alpha_{\text{s}} N_{\text{c}}}{2\pi} \left(\mathbf{H}_{1,2}^{\text{qF}} + \sum_{l=2}^{L-2} \mathbf{H}_{l,l+1}^{\text{FF}} + \mathbf{H}_{L-1,L}^{\text{Fq}} \right). \tag{2.13}$$

For completeness we should also add the rather trivial Hamiltonian for an open chain of length two

$$\Gamma_{\text{open},L=2} = \frac{\alpha_{\text{s}} N_{\text{c}}}{2\pi} \mathbf{H}_{1,2}^{\text{qq}}. \tag{2.14}$$

Obviously, only (2.12) and (2.13) admit a continuum limit $L \to \infty$.

The matrix of anomalous dimensions is thus completely specified if we give the expression for the various "link Hamiltonians" connecting two neighboring sites.

3   Lorentz structure

Before describing the link Hamiltonian, we must carefully define the space of operators on which it acts. This space must be constructed in such a way as to avoid double counting or missing some allowed operator.

Seemingly different operators can be mapped into each other by repeatedly using either of the following two identities:

$$[D_\mu, D_\nu] = -ig_{\text{YM}} F_{\mu\nu} \quad \text{and} \quad D_\mu F_{\nu\rho} + \text{cyclic} = 0. \tag{3.1}$$

Moreover, we must identify operators that differ by the classical equations of motion:

$$D^\mu F_{\mu\nu}^a = -g_{\text{YM}} \bar{q} \gamma_\nu T^a q \quad \text{and} \quad \gamma^\mu D_\mu q = 0. \tag{3.2}$$

Equations (3.1) and (3.2) will be used to maximize the number of building blocks in a given operator as much as possible in order to come to a unique basis. Before showing that, we should mention the reason why the equations of motions also need to be imposed. In [36] it is shown that if one separates gauge invariant operators into those not vanishing by the classical equations of motion, henceforth referred to as OS (on shell) operators, and those vanishing by the equations of motion (referred to as EOM operators), the full matrix of anomalous dimensions has the form:[3]

$$\Gamma = \begin{pmatrix} \Gamma_{\text{OS}}^{\text{OS}} & \Gamma_{\text{OS}}^{\text{EOM}} \\ 0 & \Gamma_{\text{EOM}}^{\text{EOM}} \end{pmatrix}. \tag{3.3}$$

Thus, although it might be necessary to add counterterms vanishing on shell in the renormalization of ordinary OS operators, we are assured that the former operators do

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[3]To be precise, the EOM part might also contain BRST variations of gauge variant operators.

not alter the anomalous dimension of the latter since their mixing is upper triangular. From now on we shall consider only $\Gamma_{\rm OS}^{\rm OS}$.

We now claim that a generic gauge singlet operator can be constructed, up to the above identifications, out of the following elementary building blocks:

$$D^n f \in \mathbb{V}_{(n/2+1,n/2)}, \quad \text{and} \quad D^n \psi, D^n \chi; \in \mathbb{V}_{(n/2+1/2,n/2)} \tag{3.4}$$

together with their complex conjugates. The fields f, ψ and χ are the chiral components of the gluon field strength and the quark fields discussed in the introduction. The labels (S_1, S_2) in (3.4) refer to the Lorentz quantum numbers and characterize a unique irreducible representation of the Lorentz group. The *module* $\mathbb{V}_{(S_1,S_2)}$ is a vector space on which the representation (S_1, S_2) acts. In fact, these irreducible representations of the Lorentz group can be combined to form infinite-dimensional irreducible representations of the conformal group $SO(2,4)$ by putting together all terms of the type, e.g. $D^n f$, $n = 0 \ldots \infty$, c.f. Appendix A. The use of the conformal group is what will allow us to obtain the full matrix of anomalous dimensions to one-loop.

To be specific, $D^n f$ written in the ordinary Lorentz basis would look like:

$$D^n f = D_{(\mu_1} \ldots D_{\mu_n} F_{\nu)\rho}^+ - \text{traces}, \tag{3.5}$$

where $F_{\nu\rho}^+$ is the selfdual component of the field strength. By "$-$traces" we mean that the tensor has been reduced to be totally traceless and the brackets around the indices correspond to total symmetrization. In the chiral basis, setting $D_\mu = \sigma_\mu^{\alpha\dot\alpha} D_{\alpha\dot\alpha}$ and $F_{\mu\nu}^+ = \sigma_{\mu\nu}^{\alpha\beta} f_{\alpha\beta}$, one would write

$$D^n f = D_{\alpha_1\dot\alpha_1} \ldots D_{\alpha_n\dot\alpha_n} f_{\beta\gamma} + \text{symmetrized}, \tag{3.6}$$

where "+symmetrized" means that the tensor has been totally symmetrized in the un-dotted and dotted indices, respectively. One usually writes simply $D^n f_{\alpha_1\ldots\alpha_{n+2},\dot\alpha_1\ldots\dot\alpha_n}$ for the resulting irreducible component in direct analogy with the representation labels $(n/2 + 1, n/2)$, where each symmetrized spinor index contributes spin $1/2$.

To show the validity of the claim, let us look at a few examples.

Consider first an operator containing the element $D_\mu F_{\nu\rho}$. We can decompose such an object into a totally anti-symmetric tensor, $D_{[\mu} F_{\nu\rho]}$ vanishing by the Bianchi identity in (3.1), a vector $\eta_{\mu\nu} D^\lambda F_{\lambda\rho} + \eta_{\mu\rho} D^\lambda F_{\nu\lambda}$ proportional to the gauge current by the equations of motion, and thus quadratic in its field content, and a remaining tensor belonging to $\mathbb{V}_{(3/2,1/2)} \oplus \mathbb{V}_{(1/2,3/2)}$ and thus precisely of the type stated above. Hence:

$$D_\mu F_{\nu\rho} = Df + D\bar{f} + O(\bar{q}q). \tag{3.7}$$

Consider now an operator containing an element with two derivatives: $D_\mu D_\nu F_{\rho\lambda}$. It is clear that the covariant derivatives can always be symmetrized, up to terms containing an extra $F_{\mu\nu}$ due to (3.1). Once the symmetrization has been done, the two indices of $F_{\rho\lambda}$ cannot further be anti-symmetrized with the ones of D_μ or D_ν because of (3.1)[4] and in total three indices will have to be symmetric. Finally, tracing any two indices will, by

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[4]This involves splitting off further $F_{\mu\nu}$'s to bring D_μ close to $F_{\rho\lambda}$.

the equations of motion, yield a derivative of the current and thus a term quadratic in the fields. Perhaps the only slightly non-trivial identity occurs when the trace is between indices both belonging to the covariant derivative. In this case one must use once again the Bianchi identities to recover the equations of motion:

$$D^\mu D_\mu F_{\rho\lambda} = -D^\mu D_\rho F_{\lambda\mu} - D^\mu D_\lambda F_{\mu\rho} = -D_\rho D^\mu F_{\lambda\mu} - D_\lambda D^\mu F_{\mu\rho} + O(F^2)$$
$$= 0 + O(\bar{q}q) + O(F^2). \tag{3.8}$$

Again, the remaining tensors in the decomposition of the original tensor containing two covariant derivatives belong to the irreps $(2,1) \oplus (1,2)$.

Similarly, the element $D_\mu q$ appearing at the end of an open chain will be equivalent to operators in $\mathbb{V}_{(1,1/2)} \oplus \mathbb{V}_{(1/2,1)}$ up to equations of motions.

One can proceed by induction and show that:

$$D_{\mu_1} \dots D_{\mu_n} F_{\nu\rho} = D^n f + D^n \bar{f} + O(\bar{q}q) + O(F^2),$$
$$D_{\mu_1} \dots D_{\mu_n} q = D^n \psi + D^n \bar{\chi} + O(\bar{q}qq) + O(Fq). \tag{3.9}$$

Since (3.9) may raise but not decrease the length of a chain, one can proceed systematically from the chains of lowest length (two) and show that the building blocks (3.4) are sufficient to construct uniquely all the on-shell operators.

Note that we do *not* identify operators that differ by a total derivative. Although in some other applications this may be desirable, in our case we should keep them and we will identify them as descendants in the representation of the conformal group.

4 Complete QCD spin chain

We are now ready to discuss the form of the link Hamiltonian. We can think of it as a map from a generic pair of elementary building blocks previously discussed into two others. We can make use of Lorentz symmetry to simplify the form somewhat. The crucial insight is that the Hamiltonian can only map components from one irreducible module to the same component of a multiplet of the same type (or even the same multiplet). In general this is still a complicated problem because of the large number of Lorentz multiplets appearing in the product. Fortunately, for one-loop anomalous dimensions, conformal symmetry comes to the rescue by vastly reducing the number of occurring irreps as we shall now proceed to discuss.

Although QCD is not a conformal theory, the use of the conformal group to classify composite operators in QCD has a long history [37] (see [26] for a recent review). The conformal symmetry is especially useful at one loop because the β-function, which is responsible for the breaking of conformal invariance, has leading term $O((\alpha_s N_c)^2)$:

$$\beta(\alpha_s N_c) = \mu \frac{\partial}{\partial \mu} \alpha_s N_c = -\frac{11}{3} \frac{(\alpha_s N_c)^2}{2\pi} \tag{4.1}$$

and thus cannot affect the one-loop anomalous dimensions which are of order $\alpha_s N_c$. We shall make use of the conformal symmetry to write the full matrix of one-loop anomalous dimensions for large N_c QCD. (The conformal algebra and its representations are reviewed in Appendix A).

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Each fundamental field, namely f, ψ and χ and their conjugates, is a primary state[5] of an irreducible module (or multiplet; a vector space transforming in some irreducible representation) of the conformal algebra. The remaining elements in the module (descendants) are obtained by acting with derivatives on the primaries. Thus, we have the following structure for the modules:

$$
\begin{aligned}
\mathbb{V}^f &= \left\langle f_{\alpha_1\alpha_2}, Df_{\alpha_1\alpha_2\alpha_3\dot{\alpha}_1}, D^2 f_{\alpha_1\alpha_2\alpha_3\alpha_4\dot{\alpha}_1\dot{\alpha}_2} \cdots \right\rangle, \\
\mathbb{V}^\psi &= \left\langle \psi_{\alpha_1}, D\psi_{\alpha_1\alpha_2\dot{\alpha}_1}, D^2 \psi_{\alpha_1\alpha_2\alpha_3\dot{\alpha}_1\dot{\alpha}_2} \cdots \right\rangle, \\
\mathbb{V}^\chi &= \left\langle \chi_{\alpha_1}, D\chi_{\alpha_1\alpha_2\dot{\alpha}_1}, D^2 \chi_{\alpha_1\alpha_2\alpha_3\dot{\alpha}_1\dot{\alpha}_2} \cdots \right\rangle,
\end{aligned}
\tag{4.2}
$$

together with their complex conjugate modules.

The product of two of the multiplets (4.2) can be decomposed into an infinite series of irreducible modules labeled by an extra integer j, the conformal spin. This is easily understood in the oscillator representation reviewed in Appendix A. We need the following decompositions (and their conjugates)

$$
\begin{aligned}
\mathbb{V}^f \otimes \mathbb{V}^f &= \sum_{j=-2}^\infty \mathbb{V}_j^{ff}, & \mathbb{V}^f \otimes \mathbb{V}^{\bar f} &= \sum_{j=+2}^\infty \mathbb{V}_j^{f\bar f}, \\
\mathbb{V}^f \otimes \mathbb{V}^\psi &= \sum_{j=-1}^\infty \mathbb{V}_j^{f\psi}, & \mathbb{V}^f \otimes \mathbb{V}^{\bar\chi} &= \sum_{j=+1}^\infty \mathbb{V}_j^{f\bar\chi}, \\
\mathbb{V}^{\bar f} \otimes \mathbb{V}^\psi &= \sum_{j=+1}^\infty \mathbb{V}_j^{\bar f\psi}, & \mathbb{V}^{\bar f} \otimes \mathbb{V}^{\bar\chi} &= \sum_{j=-1}^\infty \mathbb{V}_j^{\bar f\bar\chi}, \\
\mathbb{V}^{\bar\psi} \otimes \mathbb{V}^\psi &= \sum_{j=+1}^\infty \mathbb{V}_j^{\bar\psi\psi}, & \mathbb{V}^\chi \otimes \mathbb{V}^\psi &= \sum_{j=-1}^\infty \mathbb{V}_j^{\chi\psi}.
\end{aligned}
\tag{4.3}
$$

The explicit form of the irreducible modules appearing in the above decompositions is discussed in Appendix A. The importance of these decompositions is that the one-loop link Hamiltonian commutes with the generators of the conformal algebra and thus assumes constant values on each of the irreducible modules. Moreover, each irreducible module appears only once within all the above decompositions and there will be no mixing problem between equivalent modules.[6]

Introducing the projectors $\mathbf{P}_j^{ff}$ etc. projecting on the respective modules we can write for each glue-glue link (not written explicitly)

$$
\mathbf{H}^{\mathrm{FF}} = \sum_{j=-2}^\infty E_j^{ff}(\mathbf{P}_j^{ff} + \mathbf{P}_j^{\bar f\bar f}) + \sum_{j=+2}^\infty (E_j^{f\bar f,\mathbf{I}}\mathbf{I} + E_j^{f\bar f,\mathbf{X}}\mathbf{X})(\mathbf{P}_j^{f\bar f} + \mathbf{P}_j^{\bar f f}),
\tag{4.4}
$$

where the coefficients E_j need to be determined and $\mathbf{I}$ and $\mathbf{X}$ are the identity operator

---

[5]All components of the Lorentz multiplets $D^0 f, D^0 \psi, \ldots$ are conformal *primaries*. The conformal *highest-weight* state is distinguished by being also the highest-weight of the Lorentz multiplet.

[6]This amounts to saying that for each irreducible module there is only a single (or 1×1 matrix of) coefficients instead of an $n \times n$ matrix in the case of an n-fold occurrence.

and the exchange operator, respectively:

$$\begin{aligned}
\mathbf{I} \quad &: \quad (D^n f)(D^m \bar{f}) \rightarrow (D^n f)(D^m \bar{f}), \\
\mathbf{X} \quad &: \quad (D^n f)(D^m \bar{f}) \rightarrow (D^m \bar{f})(D^n f).
\end{aligned} \tag{4.5}$$

Similarly, we write, for the quark-gluon and the gluon-quark link:

$$\mathbf{H}^{\mathrm{qF}} = \sum_{j=-1}^{\infty} E_j^{\chi f}(\mathbf{P}_j^{\chi f} + \mathbf{P}_j^{\bar{\psi}\bar{f}}) + \sum_{j=+1}^{\infty} E_j^{\chi\bar{f}}(\mathbf{P}_j^{\chi\bar{f}} + \mathbf{P}_j^{\bar{\psi}f}),$$

$$\mathbf{H}^{\mathrm{Fq}} = \sum_{j=-1}^{\infty} E_j^{\chi f}(\mathbf{P}_j^{f\psi} + \mathbf{P}_j^{\bar{f}\bar{\chi}}) + \sum_{j=+1}^{\infty} E_j^{\chi\bar{f}}(\mathbf{P}_j^{\bar{f}\psi} + \mathbf{P}_j^{f\bar{\chi}}) \tag{4.6}$$

and for the length two quark-quark Hamiltonian:

$$\mathbf{H}^{\mathrm{qq}} = \sum_{j=-1}^{\infty} E_j^{\chi\psi}(\mathbf{P}_j^{\chi\psi} + \mathbf{P}_j^{\bar{\psi}\bar{\chi}}) + \sum_{j=+1}^{\infty} E_j^{\bar{\psi}\psi}(\mathbf{P}_j^{\bar{\psi}\psi} + \mathbf{P}_j^{\chi\bar{\chi}}). \tag{4.7}$$

In writing the above equations we have already made use of various symmetries that allow us to identify the coefficients of the projectors on different modules. Perhaps one not so obvious symmetry is a "chiral" type of symmetry rotating f and $\bar{f}$ by opposite phases (see [38] for a discussion in the context of $\mathcal{N} = 4$ SYM). This symmetry is responsible for the closure of the chiral sector. We denote by A the quantum numbers of the fields under such symmetry transformation.

The coefficients in equations (4.4,4.6,4.7) can be fixed in two ways, either by truncating the $\mathcal{N} = 4$ results of [29] or by lifting the light cone results of [22–25]. Many terms can also be explicitly computed or tested by considering chains of length two as discussed in Appendix B.

Before giving a derivation of the results we collect below the complete set of coefficients of the two-site Hamiltonian, yielding the full matrix of anomalous dimensions to one-loop ($h(j) = \sum_{k=1}^{j} 1/k$):

$$\begin{aligned}
E_j^{ff} &= 2h(j+2) - \tfrac{11}{6}, \\
E_j^{f\bar{f},\mathbf{I}} &= h(j-2) + h(j+2) - \tfrac{11}{6}, \\
E_j^{f\bar{f},\mathbf{X}} &= (-1)^j \big(h(j-2) - 4h(j-1) + 6h(j) - 4h(j+1) + h(j+2)\big) \\
&= \frac{6(-1)^{j+1}}{(j-1)j(j+1)(j+2)} \\
E_j^{\chi f} &= h(j+2) + h(j+1) - \tfrac{5}{3}, \\
E_j^{\chi\bar{f}} &= h(j+2) + h(j-1) - \tfrac{5}{3}, \\
E_j^{\chi\psi} &= 2h(j+1) - \tfrac{3}{2}, \\
E_j^{\bar{\psi}\psi} &= h(j+1) + h(j-1) - \tfrac{3}{2}.
\end{aligned} \tag{4.8}$$

The spectrum of operators with the lowest classical dimensions can be found in Tables 1,2,3.

D	L	A	(S_1, S_2)	$[p, r, q]$	E^C
4	2	2	$(0,0)$	$[0,-\ 4,0]$	$-\frac{11}{3}^+$
4	2	2	$(2,0)$	$[4,-\ 6,0]$	$\frac{7}{3}^+$
4	2	0	$(1,1)$	$[2,-\ 6,2]$	0^+
5	2	0	$(3/2,3/2)$	$[3,-\ 8,3]$	3^+
6	3	3	$(0,0)$	$[0,-\ 6,0]$	$\frac{1}{2}^+$
6	3	3	$(1,0)$	$[2,-\ 7,0]$	$-\frac{3}{2}^-$
6	3	3	$(3,0)$	$[6,-\ 9,0]$	$\frac{7}{2}^-$
6	2	2	$(3,1)$	$[6,-10,2]$	$\frac{14}{3}^+$
6	3	1	$(1,1)$	$[2,-\ 8,2]$	$\frac{7}{6}^+$
6	3	1	$(0,1)$	$[0,-\ 7,2]$	$-\frac{11}{6}^-$
6	3	1	$(2,1)$	$[4,-\ 9,2]$	$\frac{7}{6}^-$
6	2	0	$(2,2)$	$[4,-10,4]$	$\frac{21}{5}^+$
7	3	3	$(3/2,1/2)$	$[3,-\ 9,1]$	2^+ and 2^-
7	3	3	$(5/2,1/2)$	$[5,-10,1]$	$\frac{17}{6}^+$
7	3	1	$(5/2,1/2)$	$[5,-10,1]$	$\frac{17}{6}^+$
7	3	1	$(1/2,3/2)$	$[1,-\ 9,3]$	$\frac{23}{12}^+$ and $-\frac{1}{12}^-$
7	3	1	$(3/2,3/2)$	$[3,-10,3]$	$\frac{71}{30}^+$ and $\frac{13}{6}^-$
7	3	1	$(5/2,3/2)$	$[5,-11,3]$	$\frac{197}{60}^+$ and $\frac{41}{12}^-$
7	2	0	$(5/2,5/2)$	$[5,-12,5]$	$\frac{26}{5}^+$

Table 1: A complete list of primary states for the purely gluonic sector of QCD up to $D = 7$. The Dynkin labels $[p, r, q]$ refer to the classical representation of the conformal group as explained in the text. The spin labels $(S_1, S_2) \equiv (p/2, q/2)$ are redundant but we include them for clarity. D is the classical dimension, L the length of the operator and the chirality A counts the number of f minus the number of $\bar{f}$. E is the anomalous dimension in units of $\frac{\alpha_s N_c}{2\pi}$. C denotes charge conjugation: $C : F_{\mu\nu} \to -F_{\mu\nu}^{\mathrm{T}}$. In the language of spin chains it corresponds to reversing the orientation of the chain. For each state in the table with $A \neq 0$ there is a corresponding state with opposite chirality, p and q exchanged and with the same anomalous dimension.

D	L	A	(S_1, S_2)	$[p, r, q]$	E
3	2	1	$(0,0)$	$[0,-3,0]$	$-\frac{3}{2}$
3	2	1	$(1,0)$	$[2,-4,0]$	$\frac{1}{2}$
4	2	1	$(3/2,1/2)$	$[3,-6,1]$	$\frac{3}{2}$
5	2	1	$(2,1)$	$[4,-8,2]$	$\frac{13}{6}$
5	3	2	$(0,0)$	$[0,-5,0]$	$-\frac{4}{3}$
5	3	2	$(1,0)$	$[2,-6,0]$	$-\frac{1}{3}$ and $\frac{2}{3}$
5	3	2	$(2,0)$	$[4,-7,0]$	$\frac{5}{3}$
5	3	0	$(0,1)$	$[0,-6,2]$	$\frac{1}{3}$
5	3	0	$(1,1)$	$[2,-7,2]$	$\frac{1}{3}$

Table 2: A complete list of open chains with boundaries $\chi \ldots \psi$ up to $D = 5$.

D	L	A	(S_1, S_2)	$[p, r, q]$	E
3	2	0	$(1/2, 1/2)$	$[1, -4, 1]$	0
4	2	0	$(1, 1)$	$[2, -6, 2]$	$\frac{4}{3}$
5	2	0	$(3/2, 3/2)$	$[3, -8, 3]$	$\frac{25}{12}$
5	3	1	$(1/2, 1/2)$	$[1, -6, 1]$	$-\frac{1}{2}$
5	3	1	$(3/2, 1/2)$	$[3, -7, 1]$	1

Table 3: A complete list of open chains with boundaries $\bar{\psi} \ldots \psi$ up to $D = 5$. For each state with $A \neq 0$ there is a corresponding state with opposite chirality, p and q exchanged and with the same anomalous dimension.

4.1 Reduction from SYM

All the coefficients $E_j^{\times\times}$ can be obtained from $\mathcal{N} = 4$ SYM. The crucial observation is that the set of Feynman diagrams in $\mathcal{N} = 4$ SYM encompasses all the Feynman diagrams of QCD. What is more, at the one-loop level, the additional propagating degrees of freedom of the supersymmetric theory appear only in a very restricted sense in the diagrams that are relevant to QCD.

Let us explain this in more detail, starting with purely gluonic processes, i.e. interactions that couple only to gluons within a local operator and only emit gluons. The diagrams that contribute to the one-loop scaling dimension are of three basic types (see Fig. 3).
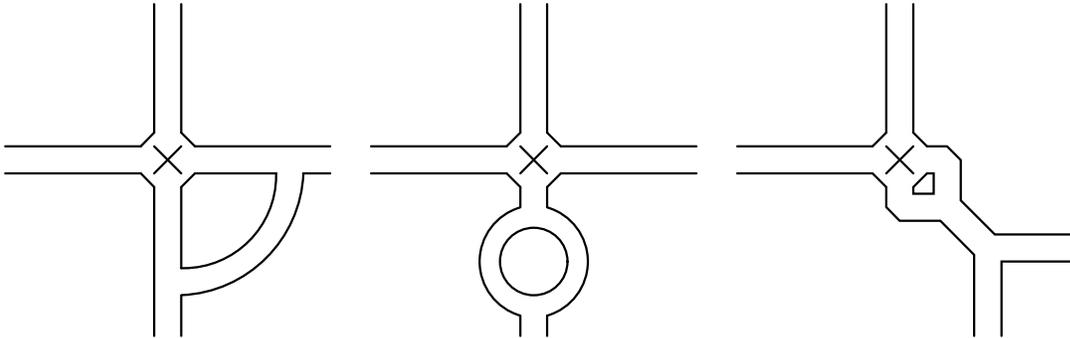

Figure 3: Typical 't Hooft diagrams contributing to the one-loop scaling dimension of a four gluon operator. There are also "degenerate" diagrams where one of the vertices is connected directly to the gluon field appearing in a covariant derivative or a commutator inside the operator.

Consider first the loop-interaction connecting to one field of the operator (Fig. 3, middle). In QCD, the particles in the loop can only be gluons (or the associated ghosts). Fundamental quarks are suppressed in the large N_c limit. In $\mathcal{N} = 4$ SYM, the gluons give precisely the same contribution, but also the adjoint scalars and fermions do couple. Hence we cannot simply read off the value of the gauge loop alone from the final result in $\mathcal{N} = 4$ SYM. We merely know that it gives the same contribution to the scaling dimension for every gluon in the spin chain, but we shall leave the coefficient, E^f, to be

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determined later. Of course it can be computed (and is well-known); at the end of this section we will show a way to derive it from conformal and flavor symmetry. To determine the contribution from the other two types of interactions (Fig. 3, left, right), we note that scalars and fermions in the vertices of $\mathcal{N} = 4$ SYM always come in pairs. This means that for purely gluonic processes they can only contribute within internal loops of the diagram. These two diagrams do not have such loops, hence their contribution is exactly the same for $\mathcal{N} = 4$ SYM as for QCD.

We know that the complete one-loop dilatation generator of $\mathcal{N} = 4$ SYM is given by

$$\mathbf{H}^{\mathcal{N}=4} = \sum_{j=0}^{\infty} E_j^{\mathcal{N}=4} \, \mathbf{P}_j^{\mathcal{N}=4} \qquad \text{with} \qquad E_j^{\mathcal{N}=4} = 2h(j). \tag{4.9}$$

The tricky part is the decomposition of the $\mathcal{N} = 4$ invariant projector into projectors for the $\mathcal{N} = 0$ conformal group. One finds that the module $\mathbb{V}_j^{\mathcal{N}=4}$ contains only chiral combinations of two gluons $\mathbb{V}_{j'}^{ff}, \mathbb{V}_{j'}^{\bar{f}\bar{f}}$ with definite conformal spin $j' = j - 2$. We write

$$\mathbf{P}_j^{\mathcal{N}=4} = \ldots + \mathbf{P}_{j-2}^{ff} + \mathbf{P}_{j-2}^{\bar{f}\bar{f}} + \ldots . \tag{4.10}$$

Consequently, the coefficient E_j^{ff} equals

$$E_j^{ff} = E_{j+2}^{\mathcal{N}=4} + 2E^f = 2h(j+2) + 2E^f \tag{4.11}$$

where E^f is the contribution from the missing loop-diagrams for scalars and fermions. For the non-chiral combination, the determination of $E_j^{f\bar{f}}$ is more involved. The module $\mathbb{V}_j^{\mathcal{N}=4}$ now contains the non-chiral combination $\mathbb{V}_{j'}^{f\bar{f}}$ with conformal spin ranging from $j' = j - 2$ to $j' = j + 2$. A second complication is that the two interacting particles may or may not change place. Finally, in $\mathcal{N} = 4$ SYM they can transform into particles which are not part of QCD but carry the same quantum numbers of the conformal group. While these contributions are clearly dropped for QCD, they are essential for the definition of $\mathbf{P}_j^{\mathcal{N}=4}$ as a projector. We would have to treat all two-particle states of $\mathcal{N} = 4$ with the same quantum numbers as $f\bar{f}$ to find the proper decomposition of projectors. This is rather involved and we just present the final result for the contributions $E_j^{f\bar{f},\mathbf{I}}$ and $E_j^{f\bar{f},\mathbf{X}}$ in (4.8).

The situation for fundamental quarks requires yet another insight, because $\mathcal{N} = 4$ SYM contains only adjoint fields. The point is that the gauge group structures at the one-loop level are very restricted. Again, they can only be of the three general forms depicted in Fig. 3 (forgetting for the moment the double line structure), where a vertex represents the structure constants λ^a or f^{abc}, depending on the type of line it attaches to (fundamental or adjoint). The loop diagram can again not be determined from $\mathcal{N} = 4$ SYM and it gives rise to the unknown coefficient E^ψ. The other two structures can be distinguished by their symmetry under interchange of the particles. This is important, because in $\mathcal{N} = 4$ the second structure can be transformed into the first one by means of a Jacobi identity. This means that for every contribution in the $\mathcal{N} = 4$ Hamiltonian we can derive the corresponding structure. This allows to derive also the terms for the fundamental fermions. Note that the Yukawa-coupling to the scalars in $\mathcal{N} = 4$ involves

two different flavors of fermions. It can therefore be suppressed by considering only one flavor of fermions.

Finally, the unknown constants E^f and E^ψ can be obtained by demanding conservation of the stress energy tensor and flavor currents. We find $E^f = -\frac{11}{12}$ and $E^\psi = -\frac{3}{4}$ in agreement with the β-function of large N_c QCD and results of the next subsection.

The power of the truncation from $\mathcal{N} = 4$ however shows up for integrability and the Bethe ansatz in Sec. 5.4. Here we needed to do detailed computations to obtain the energy coefficients $E_j^{\times\times}$, but the Bethe ansatz for QCD follows from the Bethe ansatz for $\mathcal{N} = 4$ SYM [13] straightforwardly.

4.2   The lift from the light cone

The same results (4.8) can be reached by "lifting" the light cone results [22–25]. The state of the art for this technique is presented in [28] where the full expression for the matrix of anomalous dimensions for the so-called "quasi-partonic" operators is given. This amounts to restricting to the collinear subgroup $SL(2, R)$ of the full conformal group $SO(4, 2)$. Once the problem of classification and enumeration of the operators has been solved, their answer can be lifted to the full conformal group.

Let us show how this works by lifting the expression for the gluon-quark terms which are the ones that cannot be fixed by looking at length-two gauge invariant operators. One works in light-cone coordinates $+, -, \perp$ where $\perp$ denotes the two transverse coordinates. The light cone decomposition is most easily obtained by introducing two light cone vectors n^μ and $\bar{n}^\mu$ satisfying the Lorentz products $n \cdot n = \bar{n} \cdot \bar{n} = 0$ and $n \cdot \bar{n} = 1$. One defines the $+$ and $-$ components by contracting with n and $\bar{n}$ respectively, e.g. $F_{+-} = n^\mu \bar{n}^\nu F_{\mu\nu}$. (The notation is reviewed in [26, 27].)

The collinear group is defined by the following three transformations, forming a subgroup of the full conformal group:

$$x_- \rightarrow \frac{x_-}{1 + 2ax_-}, \quad x_- \rightarrow x_- + c, \quad x_- \rightarrow \lambda x_-. \tag{4.12}$$

It is easy to show that this subgroup is isomorphic to $SL(2, R)$. A generic field, either spinorial or tensorial, can be split into various components Φ carrying a dimension d, a spin projection s (defined as the eigenvalue of the component Σ_{+-} of the spin operator) and a "collinear conformal spin" defined as $\hat{\jmath} = (d + s)/2$, where we used the symbol $\hat{\jmath}$ to distinguish the collinear conformal spin from the conformal counterpart j previously used in the full $SO(4, 2)$ context. We will see now that there is a close relation between these two quantum numbers.

A certain field component Φ can be evaluated on the light cone by introducing a real variable z:

$$\Phi(zn) = \sum_{k=0}^{\infty} \frac{z^k}{k!} D_+^k \Phi(0), \tag{4.13}$$

where the Taylor expansion is a convenient way to keep track of the light-cone derivatives. A generic light-cone composite operator can thus be thought of as a polynomial in the

variables z_i, one for each elementary field. For instance, in [39, 27] was considered the operator:

$$S^+(z_1, z_2, z_3) = \frac{g_{\text{YM}}}{2}\bar{q}(z_1 n)(i\tilde{F}_{\perp+}(z_2 n) + F_{\perp+}(z_2 n)\gamma_5)\gamma_+ q(z_3 n), \qquad (4.14)$$

for which the following Hamiltonian was given (changing slightly their notation to make the comparison with our formulas clearer):

$$\Gamma = \frac{\alpha_{\text{s}} N_{\text{c}}}{2\pi}\left[V_{qF}(\hat{\jmath}_{12}) + U_{Fq}(\hat{\jmath}_{23})\right]. \qquad (4.15)$$

The coefficients V_{qF} and U_{Fq}, to be related to our $E^{\bar{\psi}f}$ and $E^{f\psi}$ respectively, were found to be:

$$\begin{aligned}
V_{qF}(\hat{\jmath}) &= \Psi(\hat{\jmath} + \tfrac{3}{2}) + \Psi(\hat{\jmath} - \tfrac{3}{2}) - 2\Psi(1) - \tfrac{3}{4}, \\
U_{Fq}(\hat{\jmath}) &= \Psi(\hat{\jmath} + \tfrac{1}{2}) + \Psi(\hat{\jmath} - \tfrac{1}{2}) - 2\Psi(1) - \tfrac{3}{4}.
\end{aligned} \qquad (4.16)$$

In (4.16) we have denoted by Ψ the logarithmic derivative of the Gamma function $\Psi(z) = \Gamma'(z)/\Gamma(z)$, related to the harmonic sum by:

$$\Psi(m) = h(m - 1) - \gamma_E, \qquad (4.17)$$

γ_E being the Euler constant that always cancels in the final expression.

We now must relate the collinear $\hat{\jmath}$ to our previous conformal spin j. This can be done by observing that when acting on the rightmost link (i.e. for U_{Fq}) the primary $f_{\alpha\beta}\psi_\gamma$ totally symmetrized has $j = 0$ and contains the light cone component with $\hat{\jmath} = 5/2$ ($\hat{\jmath} = 3/2$ from F and $\hat{\jmath} = 1$ from ψ). Thus in U_{Fq} we should set $\hat{\jmath} = j + 5/2$. Similarly, when acting on the leftmost link, (i.e. for V_{qF}) the relation should be $\hat{\jmath} = j + 3/2$ because now we have $j = 1$.

Finally, in (4.14) there is an explicit factor of g_{YM} in front of the operator. This means that the total anomalous dimensions are shifted by $-11/6$ in units of $\frac{\alpha_{\text{s}} N_{\text{c}}}{2\pi}$ but, since the open chain has two links one must shift the contribution of each link (4.16) by $-11/12$.

Putting all this together yields back the previous results for $E^{\bar{\psi}f}$ and $E^{f\psi}$. In general, all the light-cone Hamiltonian can be lifted in a unique way to the full conformal group.

5 Chiral operators and Bethe ansatz

The conformal group provides a nice and compact bookkeeping tool for local operators in QCD which allowed us to write down the complete planar mixing matrix. The next obvious task is to diagonalize it. Though the complete diagonalization is beyond our reach, it will be possible to compute many interesting physical quantities. In particular, we compute the ground state of the mixing Hamiltonian to leading order[7] in L and analyze the spectrum of small perturbations around it by the Bethe ansatz. Before going into the details of the general Bethe ansatz equations, we find it instructive to discuss a simple reduction of the full mixing problem.

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[7]In fact, one can systematically expand in powers of $1/L$. The coefficients of L^0 and L^{-1} also have interesting physical meaning, being related to the boundary energy and the central charge respectively.

5.1 Chiral sector

There are several types of operators that mix only among themselves and thus can be treated separately. An obvious example is a set of operators with the same quantum numbers such as spin, chirality or classical dimension. Though the length of the chain (the number of fields in an operator) is not a good quantum number, it is also conserved at the one-loop level, simply because producing an extra field requires an extra interaction vertex and is thus suppressed by the QCD coupling. As in [20], we shall first consider operators with minimal classical dimension for a given length. Such operators are composed of the gluon field strength and contain no derivatives:

$$\mathcal{O} = \operatorname{tr} F_{\mu_1 \nu_1} \ldots F_{\mu_L \nu_L}, \tag{5.1}$$

A yet simpler subsector consists of chiral operators which contain only self-dual components of the field strength:

$$\mathcal{O} = \operatorname{tr} f_{\alpha_1 \beta_1} \ldots f_{\alpha_L \beta_L}. \tag{5.2}$$

This relatively simple subset of operators contains the ground state of the full mixing Hamiltonian.

Each entry $f_{\alpha_l \beta_l}$ in the operator represents one site of the chain of length L. Since we are excluding derivatives, the number of degrees of freedom per site is now finite. The three states $f_{\alpha_l \beta_l}$ form the spin-1 representation of $SU(2)_L$. The spin operator is

$$(\mathbf{S}^i f)_{\alpha\beta} = \frac{1}{2} \left(\sigma^i{}_\alpha{}^\gamma \delta_\beta{}^\varepsilon + \delta_\alpha{}^\gamma \sigma^i{}_\beta{}^\varepsilon \right) f_{\gamma\varepsilon}, \tag{5.3}$$

where $\sigma^i{}_\alpha{}^\gamma$, $i = 1, 2, 3$ are the ordinary Pauli matrices. The mixing matrix (2.12) acts pairwise on the adjacent sites of the chain. A pair of spins can be in the spin-0, spin-1 or spin-2 state and the interaction Hamiltonian depends only on the total spin. From (4.4,4.8) we find that

$$\mathbf{H}_{l,l+1} = \begin{cases} -11/6, & \text{if } (\mathbf{S}_l + \mathbf{S}_{l+1})^2 = 0, \\ 1/6, & \text{if } (\mathbf{S}_l + \mathbf{S}_{l+1})^2 = 2, \\ 7/6, & \text{if } (\mathbf{S}_l + \mathbf{S}_{l+1})^2 = 6. \end{cases} \tag{5.4}$$

Using $\mathbf{S}_l^2 = 2$, the Hamiltonian can be also written as [20]

$$\mathbf{H}_{l,l+1} = \tfrac{7}{6}\mathbf{I} + \tfrac{1}{2}\mathbf{S}_l \cdot \mathbf{S}_{l+1} - \tfrac{1}{2}(\mathbf{S}_l \cdot \mathbf{S}_{l+1})^2, \tag{5.5}$$

which is a Hamiltonian of the spin-1 quantum spin chain. Remarkably, this Hamiltonian is integrable [40] and its spectrum can be analyzed by the Bethe ansatz [41–44, 32, 33].

It is convenient to use the basis in which the S_l^3's are diagonal:

$$f_+ = f_{11}, \qquad f_0 = \frac{1}{\sqrt{2}} (f_{12} + f_{21}), \qquad f_- = f_{22}. \tag{5.6}$$

The operator

$$\mathcal{O}_\Omega = \operatorname{tr} f_+^L \tag{5.7}$$

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is an obvious eigenstate of the Hamiltonian (5.5) with anomalous dimension $\gamma_\Omega = 7\alpha_{\rm s} N_{\rm c} L/12\pi$ and spin $S = L$. This state is ferromagnetic (all spins aligned) and in the present case it is the state with the highest possible energy, which is easy to understand from (5.5): anti-alignment of nearest-neighbor spins lowers the energy, so the true ground state is anti-ferromagnetic. Quantum anti-ferromagnets are rather complicated systems even in one dimension, but in the present case the anti-ferromagnetic ground state can be found with the help of the Bethe ansatz [41, 32, 33].

It should be stressed at this point that integrability does not extend to the full Hamiltonian (4.4) due to the presence of the exchange operator $\mathbf{X}$. Clearly, without such term, (4.4) would be the sum of two independent integrable Hamiltonian, one for the chiral and one for the anti-chiral sector. The presence of $\mathbf{X}$ turns the system into a spin ladder by coupling the two sectors. However, it also spoils integrability as can be seen explicitly by constructing the candidate higher order conserved charges and showing that they do not commute with the Hamiltonian. In $\mathcal{N} = 4$ SYM integrability is regained for the entire mixing matrix due to the presence of extra fields. In fact, we interpret the integrability of the chiral sector of QCD as a remnant of $\mathcal{N} = 4$ integrability.

5.2   The Bethe ansatz

The Bethe ansatz describes all eigenstates of the spin chain in terms of elementary excitations around the ferromagnetic (pseudo)vacuum. The excitations close to the pseudo-vacuum (magnons) correspond to replacing some of the f_+'s in (5.7) by f_0 or f_-. The magnons are thus created and annihilated by operators

$$a^\dagger(l) \approx \frac{1}{\sqrt{L}} \sum_{m=1}^{L} {\rm e}^{\,ip(l)m}\, \mathbf{S}_m^-, \qquad a(l) \approx \frac{1}{\sqrt{L}} \sum_{m=1}^{L} {\rm e}^{\,-i\bar{p}(\bar{l})m}\, \mathbf{S}_m^+. \tag{5.8}$$

The momenta here are parameterized by rapidities: ${\rm e}^{\,ip(l)} = (l + i)/(l - i)$, which is standard in the literature on Bethe ansatz. Each magnon reduces the spin by 1, so the state $a^\dagger(l_1) \ldots a^\dagger(l_M)|0\rangle$ has spin $S = L - M$. The rapidities l_i are in general complex because the magnons can form bound states with decaying wave function. The operators (5.8) create eigenstates of the Hamiltonian only asymptotically for very large chains and small number of magnons, when scattering of magnons on each other can be neglected. Nevertheless, the Hamiltonian can be diagonalized in a purely algebraic way if the creation and annihilation operators are appropriately deformed to take into account the scattering. The construction of the spectrum-generating operators, known as the algebraic Bethe ansatz, is rather involved and we refer to the original papers [42, 44] or to the review [33] for the detailed derivation. Here we just quote the results.

The rapidities of magnons should be all different and satisfy a set of algebraic equations:

$$\left(\frac{l_j + i}{l_j - i} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{l_j - l_k + i}{l_j - l_k - i}, \tag{5.9}$$

which is basically the periodicity condition for the multi-magnon wave function. The right hand side of equation (5.9) contains scattering phases. If scattering is neglected

the equation reduces to the quantization condition for the momenta:

$$p(l_j) \equiv \pi - 2 \arctan l_j = \frac{2\pi n_j}{L}. \tag{5.10}$$

Solutions of the Bethe equations $\{l_j, \; j = 1, \ldots, M\}$, $0 \leq M \leq L$ parameterize all eigenstates of the spin-chain Hamiltonian. The eigenvalues are

$$\gamma_{\{l_j\}} = \frac{\alpha_s N_c}{2\pi} \left(\frac{7L}{6} - \sum_{j=1}^{M} \frac{2}{l_j^2 + 1} \right). \tag{5.11}$$

There is a useful relationship between the momentum of a Bethe root and its contribution to the anomalous dimension:

$$\gamma(l) = \frac{\alpha_s N_c}{2\pi} p'(l). \tag{5.12}$$

This relationship holds for arbitrary compounds of Bethe roots and will be very useful in calculations.

The Bethe states which correspond to QCD operators should satisfy an extra condition

$$\prod_{j=1}^{M} \frac{l_j + i}{l_j - i} = 1, \tag{5.13}$$

which guarantees that the state has zero total momentum and hence is translation invariant. This condition reflects the cyclicity of the trace in the operators (5.2).

5.3   Anti-ferromagnetic ground state

We are now in a position to construct the true, anti-ferromagnetic vacuum. Again we omit many details that can be found in the original papers [42–44]. The ground state has spin zero and thus contains L magnons. All their rapidities should be different, so the construction of the anti-ferromagnetic vacuum of the spin chain is similar to the filling of a Fermi sea. As we mentioned earlier, magnons with the same momentum form a bound state, so their rapidities become complex. The left hand side of the Bethe equation (5.9) then is not a pure phase and in the thermodynamic limit $L \to \infty$ it either blows up or goes to zero. This has to be compensated by a zero or a pole in the scattering amplitude on the right hand side. A pole or a zero arises when some of the Bethe roots differ by i. The bound state of k magnons is thus described in the thermodynamic limit by an array of Bethe roots with a common real part and integer or half-integer imaginary parts. Such a compound of Bethe roots is usually called a k-string. The 2-strings play the most important role for the spin-1 chain. They are pairs of roots at

$$l_{2j-1} = \lambda_j + i/2, \qquad l_{2j} = \lambda_j - i/2, \tag{5.14}$$

where the centers of the strings λ_j are real numbers. The ground state of the spin-1 chain is the Fermi sea of 2-strings [42, 44].

To find the distribution of the 2-strings, we first multiply the Bethe equations for both roots and rewrite them in the logarithmic form:

$$p_2(\lambda_j) = \frac{2\pi n_j}{L} + \frac{1}{L}\sum_{k\neq j}\delta_{2,2}(\lambda_j - \lambda_k),\tag{5.15}$$

where $p_2(\lambda)$ is the momentum of the 2-string:

$$e^{ip_2(\lambda)} = \frac{\lambda + i/2}{\lambda - i/2}\frac{\lambda + 3i/2}{\lambda - 3i/2},\tag{5.16}$$

which is equal to

$$p_2(\lambda) = 2\pi - 2\arctan 2\lambda - 2\arctan\frac{2\lambda}{3}.\tag{5.17}$$

The branch of the arctangent is chosen such that $-\pi/2 < \arctan\lambda < \pi/2$. The phase ambiguity in choosing the branch is reflected in the mode numbers n_j which must be different for different strings. For the scattering phase of the two strings we find after some calculations:

$$\delta_{2,2}(\lambda) = 3\pi - 4\arctan\lambda - 2\arctan\frac{\lambda}{2}.\tag{5.18}$$

Since in the vacuum all available one-particle states are occupied, we can put $n_j = j$, $j = 1,\ldots,L/2$ in (5.15).

In the thermodynamic limit, the distribution of Bethe strings can be characterized by a continuous function $\lambda = \lambda(x)$ of the variable $x = j/L$ or by the density

$$\rho(\xi) = -\left.\frac{1}{\lambda'(x)}\right|_{\xi=\lambda(x)} \approx \frac{L}{\lambda_j - \lambda_{j+1}}.\tag{5.19}$$

The thermodynamic limit of the Bethe equations for rapidities is an integral equation for the density:

$$\frac{1}{2}\frac{1}{\lambda^2 + 1/4} + \frac{3}{2}\frac{1}{\lambda^2 + 9/4} = \pi\rho(\lambda)$$
$$+2\int_{-\infty}^{+\infty}d\xi\,\rho(\xi)\left[\frac{1}{(\lambda - \xi)^2 + 1} + \frac{1}{(\lambda - \xi)^2 + 4}\right].\tag{5.20}$$

This equation is derived by subtracting (5.15) for the $(j+1)$-th string from the equation for the j-th string and taking the difference $\lambda_j - \lambda_{j+1}$ to zero. The equation can be easily solved by the Fourier transform:

$$\rho(\lambda) = \int_{-\infty}^{+\infty}\frac{dq}{2\pi}\frac{e^{-iq\lambda}}{2\cosh\frac{q}{2}} = \frac{1}{2\cosh\pi\lambda}.\tag{5.21}$$

The density is normalized as

$$\int_{-\infty}^{+\infty}d\lambda\,\rho(\lambda) = \frac{1}{2},\tag{5.22}$$

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so that the ground state contains precisely $L/2$ 2-strings or L roots and is therefore a spin zero state. The ground-state energy is, to leading order in L:

$$\gamma_0 = \frac{\alpha_s N_c}{2\pi} L \left[\frac{7}{6} - \int_{-\infty}^{+\infty} d\lambda \, \rho(\lambda) \left(\frac{3}{\lambda^2 + 9/4} + \frac{1}{\lambda^2 + 1/4} \right) \right] = -\frac{5\alpha_s N_c L}{12\pi}, \qquad (5.23)$$

which is the lowest possible anomalous dimension for operators of a given length. In the subsector without derivatives, the complete Hamiltonian is also bounded from above by the ferromagnetic vacuum. However, this is an artifact of the truncation – including derivatives it is possible to arbitrarily raise the anomalous dimension of operators of a given length. For instance, in the case of twist-two operators anomalous dimensions grow like $\log S$ where S is the number of derivatives.

5.4   Operators with derivatives

We now consider adding covariant derivatives to the chiral operators discussed above. The resulting set of operators turns out to be the largest integrable sector in the pure Yang-Mills theory. At the same time this sector contains all low-energy excitations around the anti-ferromagnetic vacuum discussed in the previous section. We will show below that the other, non-chiral modes are separated from the vacuum by a gap. The most general operators in the chiral sector have the form

$$\mathcal{O} = \mathrm{tr}(D^{m_1} f) \dots (D^{m_L} f). \qquad (5.24)$$

This sector is closed under renormalization at one-loop and integrability follows from integrability in $\mathcal{N} = 4$ SYM [13] by a simple argument: The operators (5.24) also form a one-loop closed subsector of $\mathcal{N} = 4$ SYM.[8] Therefore not only the mixing matrix for (5.24) is inherited (up to a constant shift), but also its integrability.[9] The mixing matrix can now be diagonalized by a Bethe ansatz. As usual, the Bethe equations are completely fixed by group theory and can be read off from the general result of [45], see also [46].

The mixing of operators (5.24) is described by an $SO(4,2)$ spin chain with spins in the representation whose Dynkin labels are $[p, r, q] = [2, -3, 0]$. The left and right Dynkin indices refer to twice the Lorentz spins $p = 2S_1$ and $q = 2S_2$, whereas the central Dynkin label is given by $r = -D - S_1 - S_2$. The spectrum of the integrable spin chain with these symmetries is characterized by three sets of Bethe roots u_j, l_j and r_j. The

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[8]Similar ideas have been pursued by R. Argurio (private communication).
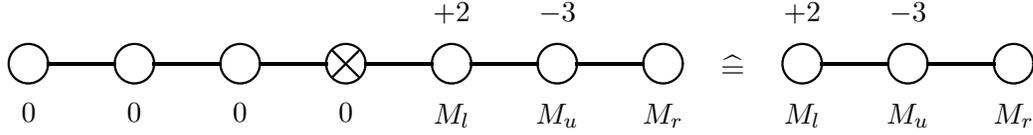[9]In Appendix C we present an independent proof of this statement.

Figure 4: Reduction of the distinguished Dynkin diagram of $SU(4|2,2)$ to $SU(2,2)$. Above the diagrams we have indicated the Dynkin labels of the spin representation and below the excitation numbers of each simple root.

Bethe equations can be inferred from [45]:

$$\left(\frac{l_j + i}{l_j - i}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M_l} \frac{l_j - l_k + i}{l_j - l_k - i} \prod_{k=1}^{M_u} \frac{l_j - u_k - i/2}{l_j - u_k + i/2},$$

$$\left(\frac{u_j - 3i/2}{u_j + 3i/2}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M_u} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{k=1}^{M_l} \frac{u_j - l_k - i/2}{u_j - l_k + i/2} \prod_{k=1}^{M_r} \frac{u_j - r_k - i/2}{u_j - r_k + i/2},$$

$$1 = \prod_{\substack{k=1 \\ k \neq j}}^{M_r} \frac{r_j - r_k + i}{r_j - r_k - i} \prod_{k=1}^{M_u} \frac{r_j - u_k - i/2}{r_j - u_k + i/2}. \tag{5.25}$$

These are precisely the Bethe equations of $\mathcal{N} = 4$ SYM [13] when one truncates the supergroup $SU(2,2|4)$ down to the bosonic part $SU(2,2)$. For this purpose one needs to consider the 'Beast' form in [13] and remove (or simply not excite) the fermionic node along with the three nodes of the internal symmetry $SU(4)$ from the distinguished Dynkin diagram of $SU(2,2|4)$, see Fig. 4. The remaining Cartan matrix for the construction of the Bethe ansatz is the one of $SU(2,2)$ and the spin labels are $[2, -3, 0]$, i.e. (5.25).

If one reduces further by not exciting the u and r roots, one returns to the Bethe equations (5.25) for the $SU(2)_L$ sector. Equivalently, removing l and r roots leads to the $SL(2)$ Bethe ansatz reviewed in [26, 27].

The derivation of the momentum constraint and of the anomalous dimension follows exactly the same route as in the previous section. We simply present the results, which agree with $\mathcal{N} = 4$ SYM [13] up to an overall energy shift proportional to L. The cyclic states are in addition subject to the momentum constraint:

$$\prod_{j=1}^{M_l} \frac{l_j + i}{l_j - i} \prod_{j=1}^{M_u} \frac{u_j - 3i/2}{u_j + 3i/2} = 1. \tag{5.26}$$

The anomalous dimension is

$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(\frac{7L}{6} - \sum_k \frac{2}{l_k^2 + 1} + \sum_k \frac{3}{u_k^2 + 9/4} \right). \tag{5.27}$$

The Bethe state with M_l l-roots, M_u u-roots and M_r r-roots is the highest weight in the $SO(4,2)$ representation with Dynkin labels

$$[2L + M_u - 2M_l, -3L + M_l + M_r - 2M_u, M_u - 2M_r]. \tag{5.28}$$

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D	L	(S_1, S_2)	$[p, r, q]$	E	roots
4	2	$(0,0)$	$[0,-4,0]$	$-11/3$	$l_{1,2} = \pm i/\sqrt{3}$
4	2	$(2,0)$	$[4,-6,0]$	$7/3$	(no roots)
6	3	$(0,0)$	$[0,-6,0]$	$1/2$	$l_1 = 0,\ l_{2,3} = \pm i$ (singular state)
6	3	$(1,0)$	$[2,-7,0]$	$-3/2$	$l_{1,2} = \pm i/\sqrt{5}$
6	3	$(3,0)$	$[6,-9,0]$	$7/2$	(no roots)
6	2	$(3,1)$	$[6,-10,2]$	$14/3$	$u_{1,2} = \pm 3/\sqrt{28}$
7	3	$(3/2, 1/2)$	$[3,-9,1]$	$2,2$	$u_1 = \pm 1/2\sqrt{5/7}$ (paired state)
					$l_1 = \bar{l}_2 = \mp\sqrt{\frac{1+i\sqrt{399}}{30}}$
7	3	$(5/2, 1/2)$	$[5,-10,1]$	$17/6$	$u_1 = l_1 = 0$

Table 4: Anomalous dimensions ($\gamma = \frac{\alpha_s N_c}{2\pi} E$) and Bethe roots for the lowest dimensional chiral closed chains ($A = L$ in Table 1).

In other words it has

$$D = 2L + M_u, \qquad S_1 = L + \tfrac{1}{2}M_u - M_l, \qquad S_2 = \tfrac{1}{2}M_u - M_r. \qquad (5.29)$$

The reference state with no excitations (the pseudo-vacuum) corresponds to the operator (5.7) that was identified in Section 5 as the ferromagnetic vacuum:

$$\mathcal{O}_\Omega = \operatorname{tr} f_{11}^L = \operatorname{tr} f_+^L. \qquad (5.30)$$

Adding a u-root corresponds to adding a derivative $D_{1\dot{1}}$ to the above operator. An l-roots flips one left spin: $1 \to 2$, and an r-root flips one right spin: $\dot{1} \to \dot{2}$. Since the Lorentz spins in (5.29) cannot be negative, the numbers of roots of different types are constrained by

$$M_l \leq L + \tfrac{1}{2}M_u, \qquad M_r \leq \tfrac{1}{2}M_u. \qquad (5.31)$$

Keeping only l-roots brings us back to the Bethe equations of the XXX_1 chain discussed in [20].

We can reproduce from the Bethe equations all anomalous dimensions of the operators in Table 1 that belong to the chiral sector, i.e. those satisfying $A = L$. The results are given in Table 4 for comparison.

Adding the derivatives does not change the ground state, since u-roots give positive contribution to the energy. In Section 6 we will show that adding $\bar{f}$ to a chiral operator raises the energy by substantially larger amount. Thus, the ground state and all the low-energy modes of the mixing matrix are described by an integrable system.

5.5 Open chains

Before moving on to a more detailed discussion of the spectrum of the closed chain, we would like to make some comments on the open chain. (For earlier work on operators of this type we refer the reader to [23, 25, 47].) The open chains with quarks at the ends

are also integrable if the gluon operators in the middle are chiral (see Appendix C). Let us consider operators that have no derivatives:

$$\mathcal{O} = \bar{q} f_{\alpha_2 \beta_2} \dots f_{\alpha_{L-1} \beta_{L-1}} q. \tag{5.32}$$

The mixing matrix for these operators is described by the following open spin–1 chain:

$$\Gamma = \frac{\alpha_s N_c}{2\pi} \left\{ \frac{4 - L_r}{6} + \mathbf{s}_1 \cdot \mathbf{S}_2 + \mathbf{S}_{L-1} \cdot \mathbf{s}_L + \sum_{l=2}^{L-2} \left[\frac{7}{6} \mathbf{I} + \frac{1}{2} \mathbf{S}_l \cdot \mathbf{S}_{l+1} - \frac{1}{2} (\mathbf{S}_l \cdot \mathbf{S}_{l+1})^2 \right] \right\}, \tag{5.33}$$

where $L_r = 0, 1, 2$ counts the number of anti-chiral quarks. Here $\mathbf{s}$ depends on the chirality of the quark (2.3) that is inserted at each end of the spin chain: For chiral quarks we set $\mathbf{s} = \boldsymbol{\sigma}/2$, while anti-chiral ones have no $SU(2)_L$ spin and thus $\mathbf{s} = 0$. Both types of boundary interaction are integrable. In the case with two anti-chiral quarks, we can take either the spin-0 or spin-1 combination of $SU(2)_R$ without affecting the anomalous dimension. The chain with the boundary interaction has been studied recently in [48] (see also [49] for a discussion of open spin chains in a supersymmetric context).

The Bethe ansatz for the system described in (5.33) is

$$\left(\frac{l_j + i}{l_j - i} \right)^{2L - 2 - L_r} = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{l_j - l_k + i}{l_j - l_k - i} \frac{l_j + l_k + i}{l_j + l_k - i} \tag{5.34}$$

and its eigenvalues are given by

$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(\frac{7L - 4L_r - 11}{6} - \sum_{k=1}^{M} \frac{2}{l_k^2 + 1} \right). \tag{5.35}$$

There is no momentum constraint.

We can reproduce from the Bethe equations all anomalous dimensions of the operators in Table 2,3 that belong to the chiral sector, i.e. those satisfying $A = L - 1 - L_r$. The results are given in Table 5 for comparison.

We will not study the thermodynamic limit of the open spin chain here, but we should mention that the anti-ferromagnetic phase of open spin chains exhibit rather unusual behavior under certain circumstances [50].

6 Spectrum of excitations

We can now study the spectrum of excitations around the anti-ferromagnetic ground state of the spin chain. In part, this can be done using techniques that are well known from the literature on one-dimensional anti-ferromagnets. Let us first summarize our findings: There are three types of excitations[10] (Fig. 5):

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[10]In this context, an excitation refers to a distortion of the wave-function which carries an *independent* momentum. Physical states composed from such excitations are subject to constraints: A single excitation may be unphysical, see below.

D	L	L_r	(S_1, S_2)	$[p, r, q]$	E	roots
5	3	0	$(0,0)$	$[0,-5,0]$	$-4/3$	$l_{1,2} = (1 \pm i)/\sqrt[4]{12}$
5	3	0	$(1,0)$	$[2,-6,0]$	$-1/3$	$l_1 = 0$
5	3	0	$(1,0)$	$[2,-6,0]$	$2/3$	$l_1 = 1$
5	3	0	$(2,0)$	$[4,-7,0]$	$5/3$	(no roots)
5	3	1	$(1/2,1/2)$	$[1,-6,1]$	$-1/2$	$l_1 = 1/\sqrt{3}$
5	3	1	$(3/2,1/2)$	$[3,-7,1]$	1	(no roots)
5	3	2	$(1,0)$	$[2,-6,0]$	$1/3$	(no roots)
5	3	2	$(1,1)$	$[2,-7,2]$	$1/3$	(no roots)

Table 5: Anomalous dimensions ($\gamma = \frac{\alpha_s N_c}{2\pi} E$) and Bethe roots for the lowest dimensional open chiral chains ($A = L - 1 - L_r$ in Table 2,3).

- Excitations with spin quantum numbers $(0,1)$ separated from the vacuum by a finite gap: $\varepsilon \sim$ const. These excitations correspond to changing a f into a $\bar{f}$.

- Excitations with spin quantum numbers $(1/2, 0)$ and relativistic dispersion relation: $\varepsilon \sim p \sim 1/L$. These modes are the low-energy states of the anti-ferromagnetic XXX_1 spin chain which are usually called spinons or spin waves. A single spinon has fractional spin and can therefore never appear on its own [32,42,44]. Furthermore, it obeys a non-standard exchange statistics.

- Excitations with spin quantum numbers $(0, 1/2)$ and non-relativistic dispersion relation: $\varepsilon \sim p^2 \sim 1/L^2$. These modes have the lowest possible energy $\sim 1/L^2$ and are turned on by the insertion of a covariant derivative into the operator. The derivative has both a left (chiral) and a right (anti-chiral) index but, since the left spin propagates in the anti-ferromagnetic vacuum, the left and right excitations propagate independently. The left excitation dissolves into the ground state and thus the only new element is the right excitation of classical scaling dimension 1. These excitations are somewhat analogous to magnons in a ferromagnet and can form multi-particle bound states. They are probably the most surprising and potentially interesting result of our analysis.

Before we move on to discuss the nature of each excitation in the next subsections, it must be stressed that, due to the highly non-trivial nature of the anti-ferromagnetic vacuum and to the zero momentum constraint, not all combinations of excitations are allowed. Only certain combinations of the elementary excitations above correspond to physical states that can be identified with QCD operators. This fact is already known in the case of spinons from the literature on anti-ferromagnets and is responsible for their unusual statistic [51]. We shall see more examples of this fact.

6.1 Adding an anti-chiral impurity

We start by showing that replacing one f by an $\bar{f}$ in the chiral spin chain gives rise to a mass gap, i.e. the chains containing such "impurities" have an energy that is larger than the vacuum by a finite amount in the thermodynamic limit $L \to \infty$. For simplicity,
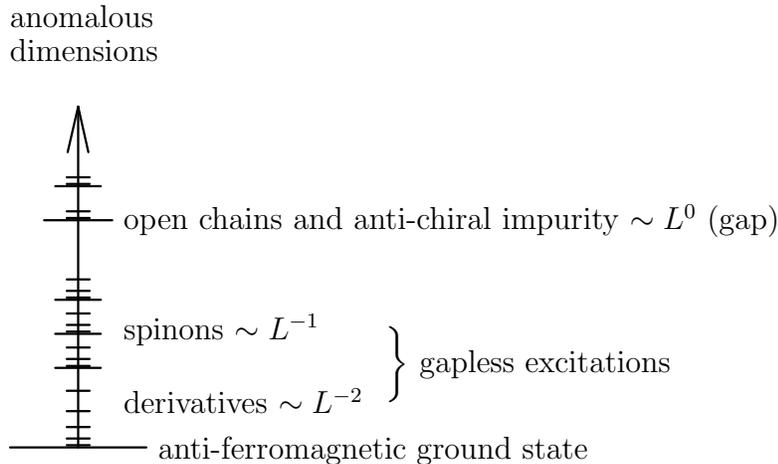
Figure 5: Spectrum of anomalous dimensions constructed over the chiral anti-ferromagnetic ground state. An identical tower of excitations can be constructed over the anti-chiral anti-ferromagnetic ground state.

we will work in the absence of derivatives, but it will be obvious from the discussion in Sec. 6.3 that the presence of derivatives can only increase the gap.

For later use, recall that anti-ferromagnetic closed and open spin chains (with "free" boundary conditions) of length L have a ground state energy behaving as [52]:

$$E_{\text{closed}}(L) = aL + O(1/L), \quad E_{\text{open}}(L) = aL + b + O(1/L). \tag{6.1}$$

The constant a depends on the constant shift in the Hamiltonian. With our normalization it has been fixed in (5.23) to be $a = -\frac{5\alpha_{\text{s}} N_{\text{c}}}{12\pi} < 0$. The constant $b > 0$ is the surface energy (not present for the closed chain with periodic boundary conditions). The terms of order $O(1/L)$ are also of physical interest in many contexts, being related to the central charge of the system, but they will not be needed in the following arguments.

In Section 5 we have shown how to express the Hamiltonian in the chiral sector as a spin chain. In order to study the effect of impurities, we must now write the Hamiltonian containing both f and $\bar{f}$ (but no derivatives) as a spin chain (or better, a spin ladder). This is the Hamiltonian found in [20] to describe the renormalization of operators without derivatives. This Hamiltonian is, of course, the restriction of our present Hamiltonian (2.12) to the subsector without derivatives.

The spin system can be represented as a spin ladder. We have already described the action of $\mathbf{S}$ on f in (5.3). The same operator acts on $\bar{f}$ in the obvious way. But now we also need to describe operators switching f with $\bar{f}$. This can be done by introducing another independent spin operator $\boldsymbol{\tau}$ corresponding to a spin 1/2 representation and acting on each component of f and $\bar{f}$ (thought of as a doublet) as:

$$\boldsymbol{\tau}^3 f = f, \qquad \boldsymbol{\tau}^+ f = 0, \qquad \boldsymbol{\tau}^- f = \bar{f},$$
$$\boldsymbol{\tau}^3 \bar{f} = -\bar{f}, \qquad \boldsymbol{\tau}^+ \bar{f} = f, \qquad \boldsymbol{\tau}^- \bar{f} = 0. \tag{6.2}$$

The two spins $\boldsymbol{\tau}_l$ and $\mathbf{S}_l$ can be visualized as sitting on a spin ladder, two parallel spin

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chains connected by links at each site. The mixing matrix can be written as

$$\Gamma = \frac{\alpha_{\rm s} N_{\rm c}}{2\pi} \sum_{l=1}^{L} \left\{ \frac{1}{2} \left(\mathbf{I} + \boldsymbol{\tau}_l^3 \boldsymbol{\tau}_{l+1}^3 \right) \mathbf{h}_{l,l+1} + \frac{1}{8} \left(\mathbf{I} - \boldsymbol{\tau}_l \cdot \boldsymbol{\tau}_{l+1} \right) \right\} \equiv \frac{\alpha_{\rm s} N_{\rm c}}{2\pi} \, \mathrm{H}, \qquad (6.3)$$

where $\mathbf{h}_{l,l+1}$ is the integrable spin one link Hamiltonian (5.5) found in the previous section:

$$\mathbf{h}_{l,l+1} = \left[\tfrac{7}{6}\mathbf{I} + \tfrac{1}{2}\mathbf{S}_l \cdot \mathbf{S}_{l+1} - \tfrac{1}{2}(\mathbf{S}_l \cdot \mathbf{S}_{l+1})^2 \right]. \qquad (6.4)$$

For chiral states, such as $\mathcal{O}$ in (5.2), $\boldsymbol{\tau}_l^+ \mathcal{O} = 0$, $\boldsymbol{\tau}_l^3 \mathcal{O} = \mathcal{O}$, and the mixing matrix reduces to (5.5).

Denote for convenience by f^A ($A = +, 0, -$) the spin one triplet introduced in (5.6) and write the full wave function as a linear combination

$$\Psi = \sum_{n=1}^{L} \psi_n^{A_1 \ldots A_{n-1} \dot{B} A_{n+1} \ldots A_L} \, \mathrm{tr}(f^{A_1} \ldots f^{A_{n-1}} \bar{f}^{\dot{B}} f^{A_{n+1}} \ldots f^{A_L}), \qquad (6.5)$$

in terms of some coefficients ψ_n. Clearly, one could always bring the impurity to the beginning of the trace, but it is convenient to have it at an arbitrary position.

We want to solve the eigenvalue problem

$$\mathrm{H}\,\Psi = E_{\rm impurity}(L)\Psi, \qquad (6.6)$$

where $E_{\rm impurity}(L)$ denotes the energy of the chiral spin chain with L sites, one of which is replaced by an impurity $\bar{f}$. Decompose

$$\mathrm{tr}(f^{A_1} \ldots f^{A_{n-1}} \bar{f}^{\dot{B}} f^{A_{n+1}} \ldots f^{A_L}) = \sum_{\Omega} C_{\Omega}^{A_{n+1} \ldots A_L, A_1 \ldots A_{n-1}} \, \mathrm{tr}(\bar{f}^{\dot{B}} \Xi_{\Omega}) \qquad (6.7)$$

where Ω and Ξ_{Ω} are respectively the eigenvalues and eigenvectors of the open XXX$_1$ spin chain with $L - 1$ sites and $C_{\Omega}^{A_{n+1} \ldots A_L, A_1 \ldots A_{n-1}}$ the overlap coefficients.

Partially projecting the original coefficients ψ_n onto the new basis:

$$\chi_n^{\dot{B}} = \psi_n^{A_1 \ldots A_{n-1} \dot{B} A_{n+1} \ldots A_L} C_{\Omega}^{A_{n+1} \ldots A_L, A_1 \ldots A_{n-1}} \qquad (6.8)$$

and doing some index manipulations we get the secular equation

$$(3 + \Omega - E_{\rm impurity}(L))\chi_n^{\dot{B}} = \frac{3}{2}(\chi_{n-1}^{\dot{B}} + \chi_{n+1}^{\dot{B}}) \qquad (6.9)$$

which has the usual plane wave solution. Of course, one must pick a direction in the anti-chiral index, say $\dot{B} = \dot{0}$, and the others are obtained by $SU(2)_R$ rotation.

Thus, setting $\chi_n^{\dot{0}} = \mathrm{e}^{ipn}$ we get the dispersion relation

$$E_{\rm impurity}(L) = \Omega + 3(1 - \cos p). \qquad (6.10)$$

The energy of the open chain on the RHS must be evaluated for $L - 1$ sites because removing the impurity not only opens the chain but also lowers the number of sites by one. The lowest value for the energy is thus reached for $p = 0$. The important point

is that we can now use the bounds (c.f. (6.1)) together with the fact that $a < 0 < b$ to show that

$$E_{\text{impurity}}(L) = E_{\text{closed}}(L) + O(L^0), \tag{6.11}$$

thus establishing the presence of a mass gap for these excitations.

Let us move on to consider now the gapless excitations, all to be found in the integrable chiral sector.

6.2 Spinons

Here we review the spectrum of excitations around the anti-ferromagnetic vacuum of the XXX$_1$ model [42]. The excitations correspond to creating holes in the Fermi sea by removing one or more 2-strings[11] from the vacuum and/or to adding k-strings with $k \neq 2$. They obey a non-trivial exchange statistics [51, 33].

We have already obtained the density of 2-strings $\rho(\lambda)$ in the ground state in (5.21). If one of the 2-strings is removed, then instead of (5.20) the density of 2-strings satisfies

$$\frac{1}{2} \frac{1}{\lambda^2 + 1/4} + \frac{3}{2} \frac{1}{\lambda^2 + 9/4} = \pi \rho_{\text{hole}}(\lambda) + \frac{\pi}{L} \delta(\lambda - \mu)$$

$$+ 2 \int_{-\infty}^{+\infty} d\xi \, \rho_{\text{hole}}(\xi) \left[\frac{1}{(\lambda - \xi)^2 + 1} + \frac{1}{(\lambda - \xi)^2 + 4} \right]. \tag{6.12}$$

The density of 2-strings now has the form[12]

$$\rho_{\text{hole}}(\lambda) = \rho(\lambda) + \frac{1}{L} \sigma_{\text{hole}}(\lambda - \mu) \tag{6.13}$$

and can again be found by the Fourier transform:

$$\sigma_{\text{hole}}(\lambda) = - \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \frac{e^{iq\lambda + |q|}}{4 \cosh^2 \frac{q}{2}} = -\delta(\lambda) + \frac{\lambda}{2 \sinh \pi \lambda}$$

$$+ \frac{1}{4\pi} \left(\Psi \left(\frac{2 + i\lambda}{2} \right) + \Psi \left(\frac{2 - i\lambda}{2} \right) - \Psi \left(\frac{1 + i\lambda}{2} \right) - \Psi \left(\frac{1 - i\lambda}{2} \right) \right), \quad \tag{6.14}$$

where $\Psi(\lambda) = \Gamma'(\lambda)/\Gamma(\lambda)$. The energy and the momentum of the hole can be readily calculated:[13]

$$p_{\text{hole}}(\mu) = -2 \int_{-\infty}^{+\infty} d\lambda \, \sigma_{\text{hole}}(\lambda - \mu) \left(\pi - \arctan 2\lambda - \arctan \frac{2\lambda}{3} \right)$$

$$= \frac{\pi}{2} - \arctan \sinh \pi \mu,$$

$$\varepsilon_{\text{hole}}(\mu) = -\frac{1}{2} \int_{-\infty}^{+\infty} d\lambda \, \sigma_{\text{hole}}(\lambda - \mu) \left(\frac{1}{\lambda^2 + 1/4} + \frac{3}{\lambda^2 + 9/4} \right) = \frac{\pi}{2 \cosh \pi \mu}. \tag{6.15}$$

[11] It is instructive not to think of holes as removed 2-strings, but rather as gaps in the sequence of mode numbers (e.g. $n_j = j$, $n_{j+1} = j + 2$) which was defined below (5.18) for the ground state. This explains why the hole excitation carries spin $-1/2$ and not -2.

[12] In order to avoid confusion, we stress that we will always be concerned with the density of 2-strings. By the notation ρ_{hole} we mean the density of two strings in the presence of a hole in the sea of two strings. Similarly for other type of impurities.

[13] The integrals are most easily done by using the Fourier representation for the density and the fact that $p'_{\text{hole}}(\mu) = -2\varepsilon_{\text{hole}}(\mu)$.

Thus the dispersion relation,

$$\varepsilon_{\text{hole}}(p) = \frac{\pi}{2} \sin p, \tag{6.16}$$

is linear for the low-energy states.

We can also compute the quantum numbers of the hole. And here we encounter a surprise [32]. Each individual 2-string contains two Bethe roots and thus has spin 2, but removing a 2-string from the Fermi sea distorts the distribution of the other strings and this distortion partially screens the spin of the hole! Naively, since we have removed a 2-string, we would expect the correction to the density σ_{hole} to be normalized to minus one. Instead we find:

$$\int_{-\infty}^{+\infty} d\lambda \, \sigma_{\text{hole}}(\lambda) = -\frac{1}{4}, \tag{6.17}$$

so the perturbed density contains not $L - 2$, but $L - 1/2$ Bethe roots. Of course a state with a fractional number of roots or even a fractional number of 2-strings does not make sense and the holes in the sea of 2-strings can only be created in sets of four.[14] Yet each hole is an independent excitation of the anti-ferromagnetic spin chain, which has the dispersion relation (6.16) and spin $1/2$ [32]. The Lorentz quantum numbers of the hole are $(1/2, 0)$.

We now want to investigate what happens if, instead of making a hole, we add one k-string with $k \neq 2$. Of course, since the number of l-roots is already maximized in the anti-ferromagnetic vacuum, this by itself will not lead to a physical state but the modification in the density of 2-strings can be investigated independently of this issue to order $O(1/L)$. If a k-string with $k \neq 2$ is added to a distribution of the 2-string, the equation for the density of 2-strings takes the form (now with $\rho_{k-\text{str}}(\lambda) = \rho(\lambda) + \frac{1}{L} \sigma_{k-\text{str}}(\lambda - \mu)$):

$$\frac{1}{2} \frac{1}{\lambda^2 + 1/4} + \frac{3}{2} \frac{1}{\lambda^2 + 9/4} = \pi \rho_{k-\text{str}}(\lambda)$$
$$+ \frac{k-2}{2L} \frac{1}{(\lambda - \mu)^2 + (k-2)^2/4} + \frac{k}{L} \frac{1}{(\lambda - \mu)^2 + k^2/4} + \frac{k+2}{2L} \frac{1}{(\lambda - \mu)^2 + (k+2)^2/4}$$
$$+ 2 \int_{-\infty}^{+\infty} d\xi \, \rho_{k-\text{str}}(\xi) \left[\frac{1}{(\lambda - \xi)^2 + 1} + \frac{1}{(\lambda - \xi)^2 + 4} \right]. \tag{6.18}$$

The middle line comes from the scattering of a 2-string on the k-string whose rapidity μ should be determined self-consistently and depends on the rapidities of other excitations. The middle line can be obtained by taking the derivative of the scattering phase for this process:

$$\delta_{2k}(\lambda) = 2\pi(2 - \delta_{k1}) - 2 \arctan \frac{2\lambda}{k-2} - 4 \arctan \frac{2\lambda}{k} - 2 \arctan \frac{2\lambda}{k+2}. \tag{6.19}$$

For the density perturbation we find:

$$\sigma_{1-\text{str}}(\lambda) = -\frac{1}{2 \cosh \pi \lambda} \tag{6.20}$$

---

[14]We have tacitly assumed that L is even. For a spin chain with odd length L, the filled Fermi sea is not physical as it contains a half-integer number $L/2$ 2-strings. Here, two holes would lead to a physical state.

type	M_l	M_u	M_2	S_1	S_2	D	$\varepsilon(p)$
hole	$-1/2$	0	$1/4$	$1/2$	0	0	$\sim \pi|p|/2$
1-string of l's	0	0	$1/2$	0	0	0	$-$
k-string of l's, $n>2$	$k-2$	0	1	$-k+2$	0	0	$-$
1-string of u's	$1/2$	1	$-1/4$	0	$1/2$	1	$\sim p^2/2$
k-string of u's, $n>1$	1	k	$-1/2$	$k/2-1$	$k/2$	k	$\sim p^2/2k$

Table 6: Chiral excitations and their quantum numbers. M_2 indicates the number of the 2-strings removed from the Fermi sea.

and

$$\sigma_{k-\text{str}}(\lambda) = -\int_{-\infty}^{+\infty} \frac{dq}{2\pi}\, e^{\,iq\lambda-(n-2)|q|/2} = -\frac{k-2}{2\pi}\,\frac{1}{\lambda^2+(k-2)^2/4} \qquad (6.21)$$

for $k>2$. The modification in the density of 2-string in the presence of one such k-string is thus normalized as

$$\int_{-\infty}^{+\infty} d\lambda\, \sigma_{1-\text{str}}(\lambda) = -\frac{1}{2}\,, \qquad (6.22)$$

and

$$\int_{-\infty}^{+\infty} d\lambda\, \sigma_{k-\text{str}}(\lambda) = -1\,, \qquad (6.23)$$

so that effectively the 1-string adds no roots ($-1/2$ of a 2-string plus the single Bethe root) and the k-string adds $k-2$ roots to a Bethe state. However, all k-strings do not contribute to the energy and momentum and thus do not correspond to propagating degrees of freedom. They are responsible for the spin and internal degrees of freedom of the magnons.

More generally, a physical $2N$-magnon state will contain $2N$ holes and some number of k-string with $k \neq 2$. See Table 6 for a summary of chiral excitations (including also the ones discussed below). How many strings of each type can be inserted in a particular state depends on N in a complicated way and requires a careful analysis of the Bethe equations [42, 51]. The dependence on N leads to a non-trivial exchange statistics of magnons so that the Hilbert space of $2N$ magnons is not just $2N$ copies of a single-particle space [51, 33]. An obvious consistency condition on a distribution of holes and k-strings (but not the only one!) is that the total number of 2-strings, $L/2 + M_2$, should be integer and the total number of l-roots should not exceed $L + M_u/2$.

6.3 Adding derivatives and spin separation

Let us finally consider the modes excited by the insertion of a covariant derivative or, in the language of the Bethe ansatz, by the addition of a u-root. Composite QCD operators associated to physical states of this kind contain one or more derivatives on top of the anti-ferromagnetic vacuum of the $SU(2)_L$ spin chain. We will see that these states are similar to BMN operators in $\mathcal{N}=4$ SYM but with important differences which arise because of the anti-ferromagnetic nature of the ground state. The derivative can be thought of as an "impurity" propagating on the background composed of $f_{\alpha\beta}$'s. Since

the anti-ferromagnetic vacuum is a Lorentz scalar, the presence of a derivative adds a chiral and an anti-chiral index to the operator. We write:

$$\mathcal{O}_{\alpha\dot\alpha}^{(n)} = \sum_{l=1}^{L} \mathrm{e}^{\frac{2\pi inl}{L}} \, \mathrm{tr} \, f_{\alpha_1\alpha_2} \ldots D_{\alpha_0\dot\alpha} f_{\alpha_{2l-1}\alpha_{2l}} \ldots f_{\alpha_{2L-1}\alpha_{2L}} \psi_\alpha^{\alpha_0\ldots\alpha_{2L}}, \qquad (6.24)$$

where we have introduced the wave-function $\psi_\alpha^{\alpha_0\ldots\alpha_{2L}}$ (linear combination of products of $\epsilon_{\alpha_i\alpha_j}$) to denote that all but one α_k's are contracted with each other and the remaining $\alpha_k = \alpha$. The operator (6.24) with zero momentum ($n = 0$) and $\psi_\alpha^{\alpha_0\ldots\alpha_{2L}} = \delta_\alpha^{\alpha_0}\psi_{\mathrm{gs}}^{\alpha_1\ldots\alpha_{2L}}$, where $\psi_{\mathrm{gs}}^{\alpha_1\ldots\alpha_{2L}}$ is the ground state wave function, is a descendant:

$$\sum_{l=1}^{L} \mathrm{tr} \, f_{\alpha_1\alpha_2} \ldots D_\mu f_{\alpha_{2l-1}\alpha_{2l}} \ldots f_{\alpha_{2L-1}\alpha_{2L}} \psi_{\mathrm{gs}}^{\alpha_1\ldots\alpha_{2L}} = \partial_\mu \, \mathrm{tr} \, f_{\alpha_1\alpha_2} \ldots f_{\alpha_{2L-1}\alpha_{2L}} \psi_{\mathrm{gs}}^{\alpha_1\ldots\alpha_{2L}}, \quad (6.25)$$

and has the same anomalous dimension as the unperturbed operator. In complete analogy with the BMN states [5], if the momentum of the derivative insertion is non-zero but small, the energy will be small. Of course, the cyclicity of the trace (zero total momentum condition) requires taking $\psi_\alpha^{\alpha_0\ldots\alpha_{2L}}$ different from $\delta_\alpha^{\alpha_0}\psi_{\mathrm{gs}}^{\alpha_1\ldots\alpha_{2L}}$, i.e. requires perturbing the anti-ferromagnetic vacuum.

Thus, we encounter an important difference between (6.24) and BMN operators in $\mathcal{N} = 4$ SYM. Unlike BMN operators, (6.24) contains not one but two elementary excitations! One of them carries momentum $p = 2\pi n/L$ and is associated with the dotted index carried by the derivative. The other is associated with the extra undotted index induced by the derivative, but propagating independently. The reason is that the number of linearly independent states obtained by different ways of contracting indices $\alpha_0, \ldots, \alpha_{2L}$ is enormous for large L and in the overwhelming majority of cases the unpaired index α that is left after all contractions are done is not the undotted index of the derivative. The left spin of the derivative thus looses its individuality and dissolves in the sea of the background spin. The excess of the left spin is carried away by an independent excitation. The mechanism of spin separation just described is a direct analog of the spin-charge separation in [53]. In this particular case the left and right spinors should have opposite momenta because of the trace condition, but in a more general setup with several derivatives different excitations are absolutely independent.

We just saw that states consisting of only one derivative must necessarily contain a spinon and thus their anomalous dimension scales as $1/L$. However, an operator of the following type with two derivatives and all undotted indices contracted

$$\mathcal{O}_{\dot\alpha\dot\beta} = \sum C_{lm} \, \mathrm{tr} \, f_{\alpha_1\alpha_2} \ldots D_{\alpha_{2L+1}\dot\alpha} f_{\alpha_{2l-1}\alpha_{2l}} \ldots D_{\alpha_{2L+2}\dot\beta} f_{\alpha_{2m-1}\alpha_{2m}} \ldots f_{\alpha_{2L-1}\alpha_{2L}} \psi^{\alpha_1\ldots\alpha_{2L+2}}$$
$$(6.26)$$

might have parametrically smaller anomalous dimension $\sim 1/L^2$, since at least the zero momentum condition can be satisfied by making the anti-chiral excitations propagate in opposite directions.

To investigate whether it is really possible to have a physical state with energy scaling like $1/L^2$ over the ground state (i.e. without spinons) we study the distortion in the

distribution of 2-strings caused by the presence of one u-root exactly in the same spirit as in the previous section.

If we add a u-root, the equations for the density of 2-strings is modified by the scattering term. The scattering phase is

$$\delta_{2u}(\lambda) = 2\arctan\lambda - \pi, \tag{6.27}$$

and modifies the equation for the distribution of 2-strings as follows:

$$\frac{1}{2}\frac{1}{\lambda^2+1/4} + \frac{3}{2}\frac{1}{\lambda^2+9/4} = \pi\rho_u(\lambda)$$
$$+2\int_{-\infty}^{+\infty} d\xi\,\rho_u(\xi)\left[\frac{1}{(\lambda-\xi)^2+1} + \frac{1}{(\lambda-\xi)^2+4}\right] - \frac{1}{L}\frac{1}{(\lambda-u)^2+1}, \tag{6.28}$$

The above equation is solved by

$$\rho_u(\lambda) = \rho(\lambda) + \frac{1}{L}\sigma_u(\lambda-u), \tag{6.29}$$

where

$$\sigma_u(\lambda) = \int_{-\infty}^{+\infty}\frac{dq}{2\pi}\frac{e^{-iq\lambda}}{4\cosh^2\frac{q}{2}} = \frac{\lambda}{2\sinh\pi\lambda} \tag{6.30}$$

is the distortion in the sea of 2-strings caused by the scattering off the u-root. The distortion of the density is normalized as

$$\int_{-\infty}^{+\infty} d\lambda\,\sigma_u(\lambda) = \frac{1}{4}\,. \tag{6.31}$$

Thus, the distortion of the Fermi sea caused by the u-root effectively adds an extra $1/2$ of an l-root to the background distribution and the Lorentz spin of an excitation of this type turns out to be $(0,1/2)$. The scattering off the background distribution of spins completely screens the left spin of the derivative, so that the derivative excitation is a right spinor, rather than a vector. Note however that this excitation is not viable on its own due to the fractional l-root. It needs to be accompanied by a fractional excitation of type l or another derivative excitation. A single derivative excitation can only be accompanied by a hole in order for the Fermi sea to be occupied by an integer number of 2-strings. Globally, this state has spin $(1/2,1/2)$ and dimension $L+1$ as expected when adding a derivative. However, it carries not one, but *two* independent momenta, one for the u-excitation and one for the l-distortion.[15]

---

[15]In fact the two momenta are related by the momentum constraint, which would otherwise, in the case of a single excitation, force the momentum to be zero.

We can now compute the energy and the momentum of the derivative excitation:

$$
\begin{aligned}
p_u(u) &= \pi - 2\arctan\frac{2u}{3} \\
&\quad -2\int_{-\infty}^{+\infty} d\lambda\, \sigma_u(\lambda - u)\left(\pi - \arctan 2\lambda - \arctan\frac{2\lambda}{3}\right) \\
&= \frac{\pi}{2} - 2\arctan\frac{2u}{3} + 2\arctan 2u - \arctan\sinh\pi u. \\
\varepsilon_u(u) &= \frac{3/2}{u^2 + 9/4} - \frac{1}{2}\int_{-\infty}^{+\infty} d\lambda\, \sigma_u(\lambda - u)\left(\frac{1}{\lambda^2 + 1/4} + \frac{3}{\lambda^2 + 9/4}\right) \\
&= \frac{3/2}{u^2 + 9/4} - \frac{1/2}{u^2 + 1/4} + \frac{\pi}{2\cosh\pi u}\,,
\end{aligned}
\tag{6.32}
$$

which gives a parametric representation of the dispersion relation $\varepsilon = \varepsilon_u(p)$. Thus, in the limit of small momenta, the contribution of a right spinor with the momentum $p = 2\pi n/L$ to the anomalous dimension is given by a BMN-like formula:

$$
\gamma(n) - \gamma_0 = \frac{\pi\alpha_{\mathrm{s}} N_{\mathrm{c}} n^2}{L^2}\,.
\tag{6.33}
$$

Looking at the quantum number summarized in Table 6 we see that the configuration with 2 u-modes and a 1-string of l-roots over the anti-ferromagnetic vacuum correspond to a physical state with no holes in the 2-string vacuum. Thus there exist operators with two derivatives and anomalous dimension of order $O(1/L^2)$ over the ground state.

For completeness let us mention that the computation for an k-string of u-roots is similar. We obtain the dispersion relation

$$
\begin{aligned}
p_u(u) &= 2\pi - 2\arctan\frac{2u}{k+2} - 2(1 - \delta_{k,2})\arctan\frac{2u}{k-2}\,, \\
\varepsilon_u(u) &= \frac{(k+2)/2}{u^2 + (k+2)^2/4} + \frac{(k-2)/2}{u^2 + (k-2)^2/4}\,,
\end{aligned}
\tag{6.34}
$$

and find that also one l-root is effectively added to the Fermi sea. The k-string describes an k-particle bound state of magnons.

Acknowledgments

A The conformal group and QCD

The conformal algebra $\mathfrak{su}(2,2) = \mathfrak{so}(2,4)$ consists of the Lorentz generators $\mathcal{L}^\alpha_\beta$ and $\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}$, the dilatation operator $\mathcal{D}$, the momentum $\mathcal{P}_{\alpha\dot\beta}$ and the conformal boost $\mathcal{K}^{\alpha\dot\beta}$. They obey the commutation relations

$$[\mathcal{L}^\alpha_\beta, \mathcal{L}^\delta_\gamma] = \delta^\alpha_\gamma \mathcal{L}^\delta_\beta - \delta^\delta_\beta \mathcal{L}^\alpha_\gamma, \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \mathcal{L}^\delta_\gamma] = 0, \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \bar{\mathcal{L}}^{\dot\delta}_{\dot\gamma}] = \delta^{\dot\alpha}_{\dot\gamma} \bar{\mathcal{L}}^{\dot\delta}_{\dot\beta} - \delta^{\dot\delta}_{\dot\beta} \bar{\mathcal{L}}^{\dot\alpha}_{\dot\gamma},$$

$$[\mathcal{D}, \mathcal{L}^\alpha_\beta] = 0, \quad [\mathcal{D}, \bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}] = 0, \quad [\mathcal{D}, \mathcal{P}_{\alpha\dot\beta}] = \mathcal{P}_{\alpha\dot\beta}, \quad [\mathcal{D}, \mathcal{K}^{\alpha\dot\beta}] = -\mathcal{K}^{\alpha\dot\beta},$$

$$[\mathcal{L}^\alpha_\beta, \mathcal{P}_{\gamma\dot\delta}] = \delta^\alpha_\gamma \mathcal{P}_{\beta\dot\delta} - \frac{1}{2}\delta^\alpha_\beta \mathcal{P}_{\gamma\dot\delta}, \quad [\mathcal{L}^\alpha_\beta, \mathcal{K}^{\gamma\dot\delta}] = -\delta^\alpha_\gamma \mathcal{K}^{\beta\dot\delta} + \frac{1}{2}\delta^\alpha_\beta \mathcal{K}^{\gamma\dot\delta},$$

$$[\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \mathcal{P}_{\gamma\dot\delta}] = \delta^{\dot\alpha}_{\dot\delta} \mathcal{P}_{\gamma\dot\beta} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathcal{P}_{\gamma\dot\delta}, \quad [\bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta}, \mathcal{K}^{\gamma\dot\delta}] = -\delta^{\dot\alpha}_{\dot\delta} \mathcal{K}^{\gamma\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathcal{K}^{\gamma\dot\delta},$$

$$[\mathcal{K}^{\alpha\dot\beta}, \mathcal{P}_{\gamma\dot\delta}] = \delta^\alpha_\gamma \bar{\mathcal{L}}^{\dot\beta}_{\dot\delta} + \delta^{\dot\beta}_{\dot\delta} \mathcal{L}^\alpha_\gamma + \delta^\alpha_\gamma \delta^{\dot\beta}_{\dot\delta} \mathcal{D}. \tag{A.1}$$

The conformal algebra has rank three and the Cartan subalgebra can be chosen to be

$$\mathcal{J}_0 = \left\{ \mathcal{L}^1_1, \ \bar{\mathcal{L}}^{\dot1}_{\dot1}, \ \mathcal{D} \right\} \tag{A.2}$$

(recall that $\mathcal{L}^1_1 = -\mathcal{L}^2_2$) whereas raising and lowering operators can be written respectively as

$$\mathcal{J}_+ = \left\{ \mathcal{L}^1_2, \ \bar{\mathcal{L}}^{\dot1}_{\dot2}, \ \mathcal{K}^{\alpha\dot\beta} \right\} \quad \text{and} \quad \mathcal{J}_- = \left\{ \mathcal{L}^2_1, \ \bar{\mathcal{L}}^{\dot2}_{\dot1}, \ \mathcal{P}_{\alpha\dot\beta} \right\}. \tag{A.3}$$

A.1 Multiplets

A highest weight state $|\text{h.w.}\rangle$ is a state that is annihilated by the operators in $\mathcal{J}_+$. By acting with the lowering generators $\mathcal{J}_-$ one can generate all the states of the multiplet (module)

$$\left\{ 1, \mathcal{J}_-, \mathcal{J}_-^2, \dots \right\} |\text{h.w.}\rangle. \tag{A.4}$$

We will give an example directly related to QCD below in App. A.3.

A highest-weight state and thus the corresponding multiplet is characterized by the charge eigenvalues of the Cartan subalgebra $\mathcal{J}_0$. A standard way of expressing these are the Dynkin labels

$$[p, r, q] \tag{A.5}$$

where p and q are *twice* the $SU(2)_L \times SU(2)_R$ spins (S_1, S_2) and r is a negative number related to the scaling dimension D by $r = -D - (p+q)/2$.

A.2 Unitarity and shortening

Generic unitary representations $[p, r, q]$ of the conformal group satisfy the unitarity bound[16]

$$p + r + q \leq -2, \tag{A.6}$$

---

[16]For the compact counterpart $\mathfrak{su}(4) = \mathfrak{so}(6)$, all Dynkin labels would have to be non-negative integers.

but there are also exceptional unitary multiplets whose Dynkin labels are related by

$$p + r = -1, \quad q = 0 \qquad \text{or} \qquad r + q = -1, \quad p = 0. \tag{A.7}$$

In QCD, the elementary fields correspond to the second type while local operators are of the first type. Note that all the unitary multiplets are infinite-dimensional as required by the non-compact nature of the conformal group.

Generically, all combinations of the momentum generator $\mathcal{P}$ generate new states when acting on the highest-weight state. Here, there are four exceptions

$$
\begin{array}{rll}
\text{scalars} & p = 0 = q, & r = -1, \\
\text{chiral fields} & p \neq 0 = q, & r = -1 - p, \\
\text{anti-chiral fields} & p = 0 \neq q, & r = -1 - q, \\
\text{conserved currents} & p \neq 0 \neq q, & r = -2 - q - p.
\end{array}
\tag{A.8}
$$

We know for instance that the scalars[17] satisfy the equations of motion (classically) $\mathcal{P}^2 |\text{h.w.}\rangle = 0$ while the conserved currents are defined by $\mathcal{P}^\mu |\text{h.w.}_\mu\rangle = 0$. The chiral and anti-chiral fields satisfy both types of conditions.

The chiral and anti-chiral fields (A.8) are the central objects in QCD. We will denote these multiplets by

$$\mathbb{V}^{+k} : [k, -1 - k, 0] \qquad \text{and} \qquad \mathbb{V}^{-k} : [0, -1 - k, k], \tag{A.9}$$

where k is restricted to 1 or 2 in the specific case of QCD. The fields also enjoy a number of useful features in terms of representation theory which makes them rather easy to handle despite the fact that they are infinite-dimensional. For instance, they can be represented by means of a set of harmonic oscillators as we shall demonstrate below.

A.3   Oscillator representation

The conformal algebra has a nice representation in terms of a set of bosonic oscillators $[a^\alpha, a^\dagger_\beta] = \delta^\alpha_\beta$ and $[b^{\dot\alpha}, b^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta}$ ($\alpha = 1, 2$, $\dot\alpha = \dot1, \dot2$). The generators take the form:

$$
\begin{aligned}
\mathcal{L}^\alpha_\beta &= a^\dagger_\beta a^\alpha - \tfrac{1}{2}\delta^\alpha_\beta a^\dagger_\gamma a^\gamma, \quad \bar{\mathcal{L}}^{\dot\alpha}_{\dot\beta} = b^\dagger_{\dot\beta} b^{\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}_{\dot\beta} b^\dagger_{\dot\gamma} b^{\dot\gamma} \\
\mathcal{D} &= 1 + \tfrac{1}{2}a^\dagger_\lambda a^\lambda + \tfrac{1}{2}b^\dagger_{\dot\gamma} b^{\dot\gamma}, \quad \mathcal{P}_{\alpha\dot\beta} = a^\dagger_\alpha b^\dagger_{\dot\beta}, \quad \mathcal{K}^{\alpha\dot\beta} = a^\alpha b^{\dot\beta}
\end{aligned}
\tag{A.10}
$$

which can easily be seen to obey the commutation relations (A.1). (See e.g. [54] for the general theory and extension to supergroups.) It is also possible to construct an extra $\mathfrak{u}(1)$ generator, not part of the conformal algebra and commuting with it:

$$\mathcal{A} = \tfrac{1}{2}a^\dagger_\lambda a^\lambda - \tfrac{1}{2}b^\dagger_{\dot\gamma} b^{\dot\gamma}. \tag{A.11}$$

This generator is associated to the "chiral" type symmetry transformation discussed in the text.

---

[17]Of course, there are no elementary scalar fields in QCD but it is just as easy to discuss the general case.

If we choose a vacuum vector $|0\rangle$ annihilated by a^α and $b^{\dot\beta}$, highest weights are of type $(a_2^\dagger)^k|0\rangle$ or $(b_2^\dagger)^k|0\rangle$. However, since the Lorentz structure is the familiar one, we will only use $\mathcal{K}^{\alpha\dot\beta}|\text{h.w.}\rangle = 0$ as the relevant condition and refer to the whole Lorentz multiplets $a_{\alpha_1}^\dagger \ldots a_{\alpha_k}^\dagger|0\rangle$ and $b_{\dot\alpha_1}^\dagger \ldots b_{\dot\alpha_k}^\dagger|0\rangle$ as "primaries". The modules constructed by acting with lowering operators on the above states provide the oscillator realization of the fundamental fields in the Lagrangian. Thus, the modules (A.9) are spanned by:

$$\mathbb{V}^{+k} = \langle a_{\alpha_1}^\dagger \ldots a_{\alpha_k}^\dagger|0\rangle,\ a_{\alpha_1}^\dagger \ldots a_{\alpha_{k+1}}^\dagger b_{\dot\alpha_1}^\dagger|0\rangle,\ a_{\alpha_1}^\dagger \ldots a_{\alpha_{k+2}}^\dagger b_{\dot\alpha_1}^\dagger b_{\dot\alpha_2}^\dagger|0\rangle, \ldots \rangle,$$
$$\mathbb{V}^{-k} = \langle b_{\dot\alpha_1}^\dagger \ldots b_{\dot\alpha_k}^\dagger|0\rangle,\ a_{\alpha_1}^\dagger b_{\dot\alpha_1}^\dagger \ldots b_{\dot\alpha_{k+1}}^\dagger|0\rangle,\ a_{\alpha_1}^\dagger a_{\alpha_2}^\dagger b_{\dot\alpha_1}^\dagger \ldots b_{\dot\alpha_{k+2}}^\dagger|0\rangle, \ldots \rangle. \qquad \text{(A.12)}$$

For convenience, the modules appearing in QCD shall also be denoted as $\mathbb{V}^\Phi$ where $\Phi = f, \bar{f}, \chi, \psi, \bar\chi, \bar\psi$ are the physical primary fields.

Recalling that $\mathcal{P}_{\alpha\dot\beta} = -iD_{\alpha\dot\beta}$ we write (e.g. $f_{\alpha\beta} = a_\alpha^\dagger a_\beta^\dagger|0\rangle$):

$$\mathbb{V}^f = \mathbb{V}^{+2} = \left\langle f_{\alpha_1\alpha_2}, Df_{\alpha_1\alpha_2\alpha_3\dot\alpha_1}, D^2 f_{\alpha_1\alpha_2\alpha_3\alpha_4\dot\alpha_1\dot\alpha_2}\cdots\right\rangle,$$
$$\mathbb{V}^{\bar f} = \mathbb{V}^{-2} = \left\langle \bar f_{\dot\alpha_1\dot\alpha_2}, D\bar f_{\alpha_1\dot\alpha_1\dot\alpha_2\dot\alpha_3}, D^2 \bar f_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}\cdots\right\rangle,$$
$$\mathbb{V}^\psi = \mathbb{V}^{+1} = \left\langle \psi_{\alpha_1}, D\psi_{\alpha_1\alpha_2\dot\alpha_1}, D^2\psi_{\alpha_1\alpha_2\alpha_3\dot\alpha_1\dot\alpha_2}\cdots\right\rangle,$$
$$\mathbb{V}^\chi = \mathbb{V}^{+1} = \left\langle \chi_{\alpha_1}, D\chi_{\alpha_1\alpha_2\dot\alpha_1}, D^2\chi_{\alpha_1\alpha_2\alpha_3\dot\alpha_1\dot\alpha_2}\cdots\right\rangle,$$
$$\mathbb{V}^{\bar\chi} = \mathbb{V}^{-1} = \left\langle \bar\chi_{\dot\alpha_1}, D\bar\chi_{\alpha_1\dot\alpha_1\dot\alpha_2}, D^2\bar\chi_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2\dot\alpha_3}\cdots\right\rangle,$$
$$\mathbb{V}^{\bar\psi} = \mathbb{V}^{-1} = \left\langle \bar\psi_{\dot\alpha_1}, D\bar\psi_{\alpha_1\dot\alpha_1\dot\alpha_2}, D^2\bar\psi_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2\dot\alpha_3}\cdots\right\rangle. \qquad \text{(A.13)}$$

The indices of the same kind (dotted or undotted) in the above modules are always understood as totally symmetrized.

A.4 Tensor products of two fields

In computing the action of the dilatation operator on two adjacent fields in a gauge invariant operator we are naturally led to the problem of decomposing the product of two of the modules (A.13). In the oscillator representation we would have to introduce two commuting sets of oscillators $[a_{(i)}^\alpha, a_{(j),\beta}^\dagger] = \delta_{ij}\delta_\beta^\alpha$ and $[b_{(i)}^{\dot\alpha}, b_{(j),\dot\beta}^\dagger] = \delta_{ij}\delta_{\dot\beta}^{\dot\alpha}$ where we will only be interested in the case $i,j = 1,2$ but in principle one could generalize to the product of an arbitrary number of (A.13). The generators are simply the sums of the generators for each set, in particular the conformal boost for the product of two singletons (A.13) is $\mathcal{K}^{\alpha\dot\beta} = a_{(1)}^\alpha b_{(1)}^{\dot\beta} + a_{(2)}^\alpha b_{(2)}^{\dot\beta}$. Highest weight states in the product can be found by looking for solutions to $\mathcal{K}^{\alpha\dot\beta}|\text{h.w.}\rangle = 0$ amongst all linear combinations of states of the type

$$a_{(1),\alpha_1}^\dagger \ldots a_{(1),\alpha_p}^\dagger b_{(1),\dot\alpha_1}^\dagger \ldots b_{(1),\dot\alpha_q}^\dagger a_{(2),\beta_1}^\dagger \ldots a_{(2),\beta_m}^\dagger b_{(2),\dot\beta_1}^\dagger \ldots b_{(2),\dot\beta_n}^\dagger|0\rangle \qquad \text{(A.14)}$$

where now $|0\rangle$ is annihilated by both sets of $a_{(i)}^\alpha$ and $b_{(i)}^{\dot\alpha}$.

Obviously there are more possibilities to satisfy the highest weight condition and in fact the product of two arbitrary modules (A.13) decomposes into an infinite sum of irreducible components, distinguished by a quantum number j named "conformal spin".

After using the oscillator representation to get some practice, one can recast the results in the more compact language of fields or in the even more compact language of Dynkin labels. Assuming $k \geq |k'|$ we have

$$\mathbb{V}^{\pm k} \otimes \mathbb{V}^{k'} = \sum_{j=\mp k'}^{\infty} \mathbb{V}_j^{\pm k + k'}. \tag{A.15}$$

The Dynkin labels of the modules in the decomposition (A.15) are

$$\mathbb{V}_j^{+s} : \begin{cases} [s+2j, -s-2-j, 0] & \text{for } -s/2 \leq j \leq 0, \\ [s+j, -s-2-2j, j] & \text{for } j \geq 0, \end{cases}$$

$$\mathbb{V}_j^{-s} : \begin{cases} [0, -s-2-j, s+2j] & \text{for } -s/2 \leq j \leq 0, \\ [j, -s-2-2j, s+j] & \text{for } j \geq 0. \end{cases} \tag{A.16}$$

A proof of these identities is sketched in App. A.5.

Let us explain in practice how the decomposition works by considering the product of two modules $\mathbb{V}^f$:

$$\mathbb{V}^f \otimes \mathbb{V}^f = \sum_{j=-2}^{\infty} \mathbb{V}_j^{ff}. \tag{A.17}$$

To understand the structure of the decomposition we present the explicit form of the first few elements in the first few modules. In order to do that we shall introduce the following notation:

- All indices of the same kind written explicitly are meant as *totally symmetrized* even if they belong to different fields.

- *Anti-symmetrization* between two indices by contraction with $\epsilon^{\alpha\beta}$ is denoted by replacing the contacted indices by the symbol $\times$ (or $\dot{\times}$).

For instance $\epsilon^{\gamma\delta} f_{\gamma(\alpha} f_{\beta)\delta}$ is written as $f_{\times\alpha} f_{\beta\times}$. As long as we consider only the products of two modules (A.13), there is no room for confusion since indices can only be anti-symmetrized between distinct fields.

Thus we have:

$$\mathbb{V}_{-2}^{ff} = \langle f_{\times\times} f_{\times\times}, \ Df_{\times\times\alpha\dot{\alpha}} f_{\times\times} + f_{\times\times} Df_{\times\times\alpha\dot{\alpha}}, \ \ldots \rangle,$$
$$\mathbb{V}_{-1}^{ff} = \langle f_{\times\alpha} f_{\times\beta}, \ Df_{\times\alpha\gamma\dot{\gamma}} f_{\times\beta} + f_{\times\alpha} Df_{\times\beta\gamma\dot{\gamma}}, \ Df_{\times\times\alpha\dot{\alpha}} f_{\times\times} - f_{\times\times} Df_{\times\times\alpha\dot{\alpha}}, \ \ldots \rangle,$$
$$\mathbb{V}_0^{ff} = \langle f_{\alpha\beta} f_{\gamma\delta}, \ Df_{\alpha\beta\eta\dot{\eta}} f_{\gamma\delta} + f_{\alpha\beta} Df_{\gamma\delta\eta\dot{\eta}}, \ Df_{\times\alpha\beta\dot{\eta}} f_{\times\gamma} - f_{\times\alpha} Df_{\times\beta\gamma\dot{\eta}}, \ \ldots \rangle,$$
$$\mathbb{V}_1^{ff} = \langle Df_{\alpha\beta\eta\dot{\eta}} f_{\gamma\delta} - f_{\alpha\beta} Df_{\gamma\delta\eta\dot{\eta}}, \ \ldots \rangle \tag{A.18}$$

and so on. The first element in each module is the primary. The descendants are obtained taking derivatives and decomposing the result into irreducible representations. Primaries in the modules $\mathbb{V}_j^{ff}$ for $j \geq 0$ have j covariant derivatives. Similar decompositions hold for all the modules. In particular:

$$\mathbb{V}^f \otimes \mathbb{V}^{\bar{f}} = \sum_{j=+2}^{\infty} \mathbb{V}_j^{f\bar{f}} \tag{A.19}$$

where, for instance,

$$\mathbb{V}_2^{f\bar{f}} = \langle f_{\alpha\beta}\bar{f}_{\dot\alpha\dot\beta}, Df_{\alpha\beta\gamma\dot\alpha}\bar{f}_{\dot\beta\dot\gamma} + f_{\alpha\beta}D\bar{f}_{\gamma\dot\alpha\dot\beta\dot\gamma}, \dots \rangle \tag{A.20}$$

and so on.

Similarly, we shall write $\mathbb{V}^f \otimes \mathbb{V}^\psi = \sum_{j=-1}^{\infty} \mathbb{V}_j^{f\psi}$ and so on for the tensor products involving fermions. The lower bound of the sum can be read off from (A.13,A.15) in each case.

A.5 Polya counting

Using Polya counting[18] we can prove the decomposition of tensor products (A.15). Let us denote a state with weight $[p, r, q]$ of $SU(2,2)$ by the monomial $a^p b^q d^{-r-p/2-q/2}$

$$[p, r, q] \rightarrow a^p b^q d^{-r-p/2-q/2}. \tag{A.21}$$

In other words the exponents of a, b are the third components of spin and the exponent of d is the dimension. A multiplet of states is consequently written as a polynomial, where the integer coefficients indicate the multiplicities of states with fixed quantum numbers. This notation is very convenient for dealing with the infinite dimensional modules that appear for the conformal group. For example, consider the field strength component f_{11} and the set of all descendants using the derivative $D_{1\dot{1}}$

$$\{f_{11}, D_{1\dot{1}}f_{11}, D_{1\dot{1}}D_{1\dot{1}}f_{11}, \dots\}. \tag{A.22}$$

Using the rule (A.21) we can write this as the polynomial

$$a^2 d^2 + a^3 b d^3 + a^4 b^2 d^4 + \dots . \tag{A.23}$$

Of course, this is just a geometric series which sums up as

$$\frac{a^2 d^2}{1 - dab}. \tag{A.24}$$

So we have found a compact way to represent f_{11} along with its $D_{1\dot{1}}$ descendants.

All the states of a long unitary multiplet $[p, r, q]$ of the conformal group are summarized in

$$[p, r, q] = \frac{a^p b^q d^{-r-p/2-q/2}(1 - a^{-2p-2})(1 - b^{-2q-2})}{(1 - dab)(1 - da/b)(1 - db/a)(1 - d/ab)(1 - a^{-2})(1 - b^{-2})}. \tag{A.25}$$

In the denominator one finds the momentum and $SU(2)_L$ and $SU(2)_R$ ladder generators. The two differences in the numerator make $SU(2)$ multiplets finite-dimensional.

Generic unitary multiplets $[p, r, q]$ of the conformal group satisfy the unitarity bound (A.6) but there are also the exceptional multiplets in (A.8) which need special attention. Let us discuss these special cases, where we refer to the long multiplet defined in (A.25)

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[18]See e.g. [55] for an introduction in the context of field theory. Polya theory has also been used in [56] to count single-trace operators in $\mathcal{N} = 4$ SYM.

as $[p, r, q]_{\text{L}}$. The simplest example (not directly related to QCD) is an on-shell scalar, which has Dynkin labels $[0, -1, 0]$. It turns out that in order to remove the terms that are reducible due to the equation of motion $D^2\phi = \dots$ we have to subtract a long multiplet $[0, -3, 0]_{\text{L}}$.[19]

$$[0, -1, 0] = [0, -1, 0]_{\text{L}} - [0, -3, 0]_{\text{L}}. \tag{A.26}$$

For an on-shell chiral field of spin $S_1 = p/2$ with $r = -1 - p$ one finds

$$[p, r, 0] = [p, r, 0]_{\text{L}} - [p - 1, r - 1, 1]_{\text{L}} + [p - 2, r - 1, 0]_{\text{L}} \tag{A.27}$$

and analogously for an anti-chiral field. These multiplets violate the bound (A.6), but due to their tightly restricted charges they are indeed unitary.

A less restricted type of multiplet, a conserved current, sits right at the unitarity bound (A.6). Its state content is reduced by the states $D^\mu J_{\mu\dots} = 0$

$$[p, r, q] = [p, r, q]_{\text{L}} - [p - 1, r, q - 1]_{\text{L}}. \tag{A.28}$$

Using these expressions and sum rules of geometric series, it is straightforward to prove (A.15).

B   Length-two primary states

Let us now discuss the gauge invariant primary states that can be constructed with only two fundamental fields ($L = 2$) and an arbitrary number of derivatives. We do this partly to present a check of our results against the standard literature and partly as an illustration of how to use the Hamiltonians (2.12) and (2.13).

Reducing to $L = 2$ chains amounts to projecting all modules in the decompositions (A.16) on the gauge singlet states. In considering $L = 2$ states and in order to make connection with the classical papers on the subject is more convenient to rewrite the gluon modules in a way that makes parity manifest. To this end, we introduce the following projection operators:

$$\mathbf{P}_j^{f\bar{f} \pm \bar{f}f} = \tfrac{1}{2} \left(\mathbf{I} \pm (-)^j \mathbf{X} \right) \left(\mathbf{P}_j^{f\bar{f}} + \mathbf{P}_j^{\bar{f}f} \right), \tag{B.1}$$

in terms of which the Hamiltonian (4.4) reads

$$\mathbf{H}^{\text{FF}} = \sum_{j=-2}^{\infty} E_j^{ff} \left(\mathbf{P}_j^{ff} + \mathbf{P}_j^{\bar{f}\bar{f}} \right) + \sum_{j=+2}^{\infty} \left(E_j^+ \mathbf{P}_j^{f\bar{f}+\bar{f}f} + E_j^- \mathbf{P}_j^{f\bar{f}-\bar{f}f} \right), \tag{B.2}$$

where

$$E_j^\pm = h(j - 2) + h(j + 2) - \frac{11}{6} \mp \frac{6}{(j - 1)j(j + 1)(j + 2)}. \tag{B.3}$$

---

[19]Not incidentally this is just the weight of $D^2\phi$.

For sake of clarity, we give expression for the primary and the first few descendants of these new modules:

$$\mathbb{V}_2^{f\bar{f}+\bar{f}f} = \big\langle f_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} + \bar{f}_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta},$$
$$Df_{\alpha\beta\gamma\dot{\alpha}}\bar{f}_{\dot{\beta}\dot{\gamma}} + f_{\alpha\beta}D\bar{f}_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} + \bar{f}_{\dot{\alpha}\dot{\beta}}Df_{\alpha\beta\gamma\dot{\gamma}} + D\bar{f}_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\beta\gamma},$$
$$Df_{\alpha\beta\gamma\dot{\times}}\bar{f}_{\dot{\times}\dot{\alpha}} - \bar{f}_{\dot{\times}\dot{\alpha}}Df_{\alpha\beta\gamma\dot{\times}}, \quad f_{\alpha\times}D\bar{f}_{\times\dot{\alpha}\dot{\beta}\dot{\gamma}} - D\bar{f}_{\times\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\alpha\times}, \dots \big\rangle,$$
$$\mathbb{V}_2^{f\bar{f}-\bar{f}f} = \big\langle f_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}} - \bar{f}_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta},$$
$$Df_{\alpha\beta\gamma\dot{\alpha}}\bar{f}_{\dot{\beta}\dot{\gamma}} + f_{\alpha\beta}D\bar{f}_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} - \bar{f}_{\dot{\alpha}\dot{\beta}}Df_{\alpha\beta\gamma\dot{\gamma}} - D\bar{f}_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\beta\gamma},$$
$$Df_{\alpha\beta\gamma\dot{\times}}\bar{f}_{\dot{\times}\dot{\alpha}} + \bar{f}_{\dot{\times}\dot{\alpha}}Df_{\alpha\beta\gamma\dot{\times}}, \quad f_{\alpha\times}D\bar{f}_{\times\dot{\alpha}\dot{\beta}\dot{\gamma}} + D\bar{f}_{\times\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\alpha\times}, \dots \big\rangle,$$
$$\mathbb{V}_3^{f\bar{f}+\bar{f}f} = \big\langle Df_{\alpha\beta\gamma\dot{\alpha}}\bar{f}_{\dot{\beta}\dot{\gamma}} - f_{\alpha\beta}D\bar{f}_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} - \bar{f}_{\dot{\alpha}\dot{\beta}}Df_{\alpha\beta\gamma\dot{\gamma}} + D\bar{f}_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\beta\gamma}, \dots \big\rangle,$$
$$\mathbb{V}_3^{f\bar{f}-\bar{f}f} = \big\langle Df_{\alpha\beta\gamma\dot{\alpha}}\bar{f}_{\dot{\beta}\dot{\gamma}} - f_{\alpha\beta}D\bar{f}_{\gamma\dot{\alpha}\dot{\beta}\dot{\gamma}} + \bar{f}_{\dot{\alpha}\dot{\beta}}Df_{\alpha\beta\gamma\dot{\gamma}} - D\bar{f}_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}f_{\beta\gamma}, \dots \big\rangle,$$

where the conventions about symmetrization and anti-symmetrization are the same as those explained in Appendix A.

For the gluon modules the projection onto gauge singlet states amounts to taking the trace. All the elements in $\mathbb{V}_j^{f\bar{f}-\bar{f}f}$ are traceless (thus projected out) for j even and those in $\mathbb{V}_j^{ff}$, $\mathbb{V}_j^{\bar{f}\bar{f}}$ and $\mathbb{V}_j^{f\bar{f}+\bar{f}f}$ are traceless for j odd. Of course, the quark-gluon modules contain no gauge singlet at all and the quark-quark modules contain singlets for all j's.

The surviving length-two gauge invariant primaries are summarized in Table 7.

Name	Range	(S_1, S_2)	D	$[p,r,q]$	τ	χ	A
$\mathbb{V}_{-2}^{ff}$		$(0,0)$	4	$[0,-4,0]$	4	0	2
$\mathbb{V}_j^{ff}$	$j = 0,2,4\dots$	$(2+j/2, j/2)$	$4+j$	$[4+j, -6-2j, j]$	2	4	2
$\mathbb{V}_{-2}^{\bar{f}\bar{f}}$		$(0,0)$	4	$[0,-4,0]$	4	0	-2
$\mathbb{V}_j^{\bar{f}\bar{f}}$	$j = 0,2,4\dots$	$(j/2, 2+j/2)$	$4+j$	$[j, -6-2j, 4+j]$	2	-4	-2
$\mathbb{V}_j^{f\bar{f}+\bar{f}f}$	$j = 2,4,6\dots$	$(j/2, j/2)$	$2+j$	$[j, -2-2j, j]$	2	0	0
$\mathbb{V}_j^{f\bar{f}-\bar{f}f}$	$j = 3,5,7\dots$	$(j/2, j/2)$	$2+j$	$[j, -2-2j, j]$	2	0	0
$\mathbb{V}_{-1,}^{\chi\psi}$		$(0,0)$	3	$[0,-3,0]$	3	0	1
$\mathbb{V}_j^{\chi\psi}$	$j = 0,1,2\dots$	$(1+j/2, j/2)$	$3+j$	$[2+j, -4-2j, j]$	2	2	1
$\mathbb{V}_{-1,}^{\psi\bar{\chi}}$		$(0,0)$	3	$[0,-3,0]$	3	0	-1
$\mathbb{V}_j^{\psi\bar{\chi}}$	$j = 0,1,2\dots$	$(j/2, 1+j/2)$	$3+j$	$[j, -4-2j, 2+j]$	2	-2	-1
$\mathbb{V}_j^{\psi\psi}$	$j = 1,2,3\dots$	$(j/2, j/2)$	$2+j$	$[j, -2-2j, j]$	2	0	0
$\mathbb{V}_j^{\chi\bar{\chi}}$	$j = 1,2,3\dots$	$(j/2, j/2)$	$2+j$	$[j, -2-2j, j]$	2	0	0

Table 7: Table of all gauge singlets primaries of length two. The indices (S_1, S_2) refer to the Lorentz spins, D is the classical dimension, $[p,r,q]$ are the Dynkin labels and A is the $U(1)$ charge introduced above. The twist is defined, as usual, by $\tau = D - (p+q)/2$. In the presence of "asymmetric" operators, dimension and twist are not enough to uniquely specify the Lorentz representation – we need an extra quantum number, e.g. the "asymmetry" factor $\chi = p - q$. Notice the presence of exceptional cases with negative conformal spin.

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We are particularly interested in computing the one-loop anomalous dimension of the primaries. First of all, in order to disentangle the contribution of descendants carrying the same quantum numbers, one always performs the computation at zero injected momentum, for which total derivative terms simply do not contribute. Because of the $U(1)$ symmetry relating the anomalous dimensions of the related operators, we can restrict our attention to $A \geq 0$. Finally, the anomalous dimensions of the primaries in the last two modules in Table 7 are obviously the same. We are thus led to consider the renormalization of the following operators at zero injected momentum:[20] (the range of the conformal spin j is the same as in Table 7)

$$
\begin{aligned}
\mathbb{V}_{-2}^{ff} &: \quad \mathrm{tr}\ F^{\mu\nu}F_{\mu\nu} \\
\mathbb{V}_{j}^{ff} &: \quad \mathrm{tr}\ F_{\mu(\nu}\overleftrightarrow{D}_{\rho_1}\ldots\overleftrightarrow{D}_{\rho_j}F_{\lambda)\sigma} - \text{traces} \\
\mathbb{V}_{j}^{f\bar{f}+\bar{f}f} &: \quad \mathrm{tr}\ F^{\mu}_{(\rho_1}\overleftrightarrow{D}_{\rho_2}\ldots\overleftrightarrow{D}_{\rho_{j-1}}F_{\rho_j)\mu} - \text{traces} \\
\mathbb{V}_{j}^{f\bar{f}-\bar{f}f} &: \quad \mathrm{tr}\ \tilde{F}^{\mu}_{(\rho_1}\overleftrightarrow{D}_{\rho_2}\ldots\overleftrightarrow{D}_{\rho_{j-1}}F_{\rho_j)\mu} - \text{traces} \\
\mathbb{V}_{-1}^{\chi\psi} &: \quad \bar{q}q \\
\mathbb{V}_{j}^{\chi\psi} &: \quad \bar{q}[\gamma_\mu, \gamma_{(\nu]}\overleftrightarrow{D}_{\rho_1}\ldots\overleftrightarrow{D}_{\rho_j)}q - \text{traces} \\
\mathbb{V}_{j}^{\bar{\psi}\psi} &: \quad \bar{q}\gamma_{(\rho_1}\overleftrightarrow{D}_{\rho_2}\ldots\overleftrightarrow{D}_{\rho_j)}q - \text{traces}.
\end{aligned}
\tag{B.4}
$$

Indices inside round brackets are totally symmetrized and by "$-$traces" we mean that the tensor structure is irreducible, i.e. tracing over any two indices yields zero. This reduction will always be understood in the following. The operator $\mathrm{tr}\ F^{\mu\nu}F_{\mu\nu}$ (the Lagrangian) contains both the selfdual and anti-selfdual part but both terms renormalize the same way due to the $U(1)$ symmetry. Its anomalous dimension is well known:

$$
\mathrm{tr}\ F^{\mu\nu}F_{\mu\nu} : \quad \gamma = -\frac{11}{3}\frac{\alpha_s N_c}{2\pi} \equiv 2\frac{\alpha_s N_c}{2\pi}E_{-2}^{ff}.
\tag{B.5}
$$

This is the first check. The factor of two comes because we must sum the (equal) contributions of $\mathbf{H}_{1,2}^{\mathrm{FF}}$ and $\mathbf{H}_{2,1}^{\mathrm{FF}}$. Similarly, the quark mass term $\bar{q}q$ contains both the chiral ($\chi\psi$) and anti-chiral ($\bar{\chi}\bar{\psi}$) and its anomalous dimension is also well known:

$$
\bar{q}q : \quad \gamma = -\frac{3}{2}\frac{\alpha_s N_c}{2\pi} \equiv \frac{\alpha_s N_c}{2\pi}E_{-1}^{\chi\psi}.
\tag{B.6}
$$

(This time there is no factor of two because we are considering an open chain.)

The anomalous dimensions of the primary operators of $\mathbb{V}_{j}^{f\bar{f}+\bar{f}f}$ and $\mathbb{V}_{j}^{\bar{\psi}\psi}$ are also well known from the classical work of [1]. We do not write explicitly the flavor structure but even for flavor singlets, in the large N_c limit, the mixing between gluonic and quark operators is suppressed. In particular, in this limit their matrix of anomalous dimension is upper triangular and the eigenvalues can be read off from the diagonal entries. It is even possible to rescale the quark kinetic energy in the Lagrangian by a factor $\sqrt{N_c}$ and

---

[20]The exact conformal operators contain total derivatives in particular combinations fixed by conformal symmetry [37].

obtain a diagonal matrix:

$$\operatorname{tr} F^\mu_{(\rho_1} \overset{\leftrightarrow}{D}_{\rho_2} \dots \overset{\leftrightarrow}{D}_{\rho_{j-1}} F_{\rho_j)\mu} :$$
$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(-\frac{4}{j(j-1)} - \frac{4}{(j+1)(j+2)} + 4h(j) - \frac{11}{3} \right) \equiv 2\frac{\alpha_s N_c}{2\pi} E_j^+,$$
$$\bar{q}\gamma_{(\rho_1} \overset{\leftrightarrow}{D}_{\rho_2} \dots \overset{\leftrightarrow}{D}_{\rho_j)} q :$$
$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(h(j+1) + h(j-1) - 3/2 \right) \equiv \frac{\alpha_s N_c}{2\pi} E_j^{\bar{\psi}\psi}. \tag{B.7}$$

In the same way, the anomalous dimension of the primary operator of $\mathbb{V}_j^{f\bar{f}-\bar{f}f}$ can be read off from the work of [57]:

$$\operatorname{tr} \tilde{F}^\mu_{(\rho_1} \overset{\leftrightarrow}{D}_{\rho_2} \dots \overset{\leftrightarrow}{D}_{\rho_{j-1}} F_{\rho_j)\mu} :$$
$$\gamma = \frac{\alpha_s N_c}{2\pi} \left(-\frac{8}{j(j+1)} + 4h(j) - \frac{11}{3} \right) \equiv 2\frac{\alpha_s N_c}{2\pi} E_j^-. \tag{B.8}$$

The work [57] contains also the matrix of anomalous dimensions of a second group of operators but the gluonic term appears to be a descendant, casting some doubts on the relevance of this second set of anomalous dimensions.

We are left with the anomalous dimensions of the primaries for $\mathbb{V}_j^{\chi\psi}$ and $\mathbb{V}_j^{ff}$. We worked out their anomalous dimensions by diagrammatic techniques, employing the previous trick of zero injected momentum and the following additional ones (also well known, see e.g. [58]):

- The symmetrization of the indices was accomplished by contracting each of them with a constant vector Δ_{ρ_i} etc.

- The tracelessness condition was enforced by dropping in the evaluation of the Feynman diagrams all terms containing $(\Delta)^2$, Δ^μ (and for the second case also Δ^σ).

- Finally, to ensure that the anti-symmetrization of any three indices had been removed, we evaluated a particular component that is manifestly free of such contribution, namely we set (at the end!): $\Delta_\rho = \delta_{1\rho}$, $\mu = 2$, (and for the second case also $\sigma = 2$).

The resulting anomalous dimensions are:

$$\bar{q}[\gamma_\mu, \gamma_{(\nu}] \overset{\leftrightarrow}{D}_{\rho_1} \dots \overset{\leftrightarrow}{D}_{\rho_j)} q : \qquad \gamma = \frac{\alpha_s N_c}{2\pi} \left(2h(j+1) - \frac{3}{2} \right) \equiv \frac{\alpha_s N_c}{2\pi} E_j^{\chi\psi},$$
$$\operatorname{tr} F_{\mu(\nu} \overset{\leftrightarrow}{D}_{\rho_1} \dots \overset{\leftrightarrow}{D}_{\rho_j} F_{\lambda)\sigma} : \qquad \gamma = \frac{\alpha_s N_c}{2\pi} \left(4h(j+2) - \frac{11}{3} \right) \equiv 2\frac{\alpha_s N_c}{2\pi} E_j^{ff}, \tag{B.9}$$

confirming the corresponding entries in the Hamiltonian. The last two operators are "asymmetric". They cannot be generated perturbatively in the operator product of two currents because such product does not change chirality. They might however be interesting for comparing with lattice calculations [58].

C R-matrix

Formalism. Let us give a brief introduction into the R-matrix formalism. We define several vector spaces which we label by $k, l, \ldots$ These are (not necessarily equal) modules $\mathbb{V}^{k,l,\cdots}$ of a symmetry algebra and they are thus called 'spins'. In our case the symmetry algebra is the conformal algebra and a 'spin' will be a chiral field strength multiplet. To each spin we associate a spectral parameter $u_{k,l,\ldots}$. The scattering of two spins k, l is assumed to be elastic: Neither the spin modules $\mathbb{V}^{k,l}$ nor the spectral parameters $u_{k,l}$ are modified, there is merely a phase-shift when two spins are interchanged. The phase shift is described by the R-matrix $\mathbf{R}_{kl}(u_k - u_l)$, which is a unitary bi-linear operator acting on the two spins

$$\mathbf{R}_{kl}(u_k - u_l) : \mathbb{V}^k \times \mathbb{V}^l \to \mathbb{V}^k \times \mathbb{V}^l. \tag{C.1}$$

In this formalism, integrability means that the Yang-Baxter equation is satisfied. It ensures that the order in which spins scatter does not matter

$$\mathbf{R}_{ik}(u_{ik})\mathbf{R}_{il}(u_{il})\mathbf{R}_{kl}(u_{kl}) = \mathbf{R}_{kl}(u_{kl})\mathbf{R}_{il}(u_{il})\mathbf{R}_{ik}(u_{ik}), \tag{C.2}$$

where we have defined

$$u_{kl} = u_k - u_l. \tag{C.3}$$

The form of the R-matrix is in general very difficult, but it can be constructed for arbitrary semi-simple algebras and arbitrary irreducible representations. In our case the 'spins' will all be of the type of oscillator representations. These have special properties that allow us to compute the R-matrix rather conveniently.

R-matrix for spin oscillators. First of all, we introduce the compact notation $\mathcal{J}^A{}_B$ for the generators of $\mathfrak{su}(2,2)$. Here, an index A combines a chiral and an anti-chiral index α and $\dot{\alpha}$. We can now introduce a combined oscillator $A^A = \{a^\alpha, b^\dagger_{\dot{\beta}}\}$ which behaves like a four-component version of a^α. The combined generators are now written as

$$\mathcal{J}^A{}_B = A^\dagger_B A^A - \tfrac{1}{4}\delta^A_B A^\dagger_C A^C, \qquad \mathcal{A} = \tfrac{1}{2}A^\dagger_A A^A + 1 \tag{C.4}$$

and satisfy the commutation relation

$$[\mathcal{J}^A{}_B, \mathcal{J}^C{}_D] = \delta^A_D \mathcal{J}^C{}_B - \delta^C_B \mathcal{J}^A{}_D, \qquad [\mathcal{J}^A{}_B, \mathcal{A}] = 0. \tag{C.5}$$

Let us now consider the R-matrix acting on the two oscillator modules $\mathbb{V}^k$ and $\mathbb{V}^l$. We know that the R-matrix is invariant under conformal symmetry therefore it is useful to consider the decomposition of the tensor product $\mathbb{V}^k \otimes \mathbb{V}^l$ into irreducible components, (A.15). As each component $\mathbb{V}^{kl}_j$ appears with multiplicity one, the R-matrix can merely assign one eigenvalue to each:

$$\mathbf{R}_{kl}(u) = \sum_j \mathbf{P}^{kl}_j R^{kl}_j(u). \tag{C.6}$$

We will show below that these eigenvalues obey the recursion formula

$$R^{kl}_j(u_{kl}) = \frac{u_{kl} + \tfrac{i}{4}\left(J^2_{kl,j} - J^2_{kl,j-1}\right)}{u_{kl} - \tfrac{i}{4}\left(J^2_{kl,j} - J^2_{kl,j-1}\right)} R^{kl}_{j-1}(u_{kl}). \tag{C.7}$$

Here $J^2_{kl,j}$ denotes the eigenvalue of the quadratic Casimir

$$\mathcal{J}^2 = \mathcal{J}^A{}_B \mathcal{J}^B{}_A \tag{C.8}$$

on $\mathbb{V}^{kl}_j$. For a highest-weight representation with Dynkin labels $[p, r, q]$ it is given by[21]

$$J^2_{[p,r,q]} = \tfrac{1}{2}p(p+2) + \tfrac{1}{4}(2r+p+q)(2r+p+q+8) + \tfrac{1}{2}q(q+2). \tag{C.9}$$

This result for the R-matrix is completely general, it holds for any two oscillator representations, even fermionic oscillators, for any unitary group.

Closed Chiral Chain. Let us now consider only spins which are chiral field strengths, i.e. only modules of the type $\mathbb{V}^f = \mathbb{V}_{[2,-3,0]}$. The monodromy matrix of this system is given by

$$\Omega_k(u) = \mathbf{R}_{k,1}(u)\mathbf{R}_{k,2}(u)\ldots\mathbf{R}_{k,L}(u) \tag{C.10}$$

where the auxiliary spin labeled by k transforms like a field strength $\mathbb{V}^f$. The transfer matrix is

$$\mathrm{T}(u) = \mathrm{tr}_k \, \Omega_k(u) \tag{C.11}$$

The nearest-neighbor Hamiltonian of an integrable system is given by the logarithmic derivative of the R-matrix at $u = 0$

$$\mathbf{H}^{ff} = i\mathbf{R}^{-1}_{ff}(0) \, \frac{\partial \mathbf{R}_{ff}}{\partial u}(0). \tag{C.12}$$

The recursion formula for the eigenvalues E^{ff}_j of the two-site Hamiltonian is thus

$$E^{ff}_j = \frac{8}{J^2_j - J^2_{j-1}} + E^{ff}_{j-1}. \tag{C.13}$$

In our case the quadratic Casimir (C.9) of $\mathbb{V}^{ff}_j$ is given by $J^2_j = 2(j+2)(j+3)$. The recursion formula for the energy eigenvalues is thus

$$E^{ff}_j = \frac{2}{j+2} + E^{ff}_{j-1} = 2h(j+2) + 2E^f. \tag{C.14}$$

where E^f is an energy shift that does not follow from integrability. This agrees precisely with the result from planar QCD and thus the sector of chiral field strengths is integrable.

Open Chiral Chain. The integrable open chiral chain has spins $\mathbb{V}^f$ at all sites except 1 and L, where the spin is $\mathbb{V}^\chi = \mathbb{V}^\psi = \mathbb{V}_{[1,-2,0]}$. This system is characterized by the monodromy matrix [48][22] [23]

$$\Omega_k(u) = \mathbf{R}_{k,1}(u \pm \tfrac{i}{2})\mathbf{R}_{k,2}(u)\ldots\mathbf{R}_{k,L-1}(u)\mathbf{R}_{k,L}(u \pm \tfrac{i}{2}) \tag{C.15}$$

---

[21]A curious observation is that the quadratic Casimir vanishes precisely for $[0,0,0]$ (trivial representation), $[2,-3,0]$, $[0,-3,2]$ (field strength components), $[1,-4,1]$ (conserved flavor currents), $[0,-4,0]$ (Lagrangian) and $[1,-2,1]$.

[22]In [48] the system of $SU(2)$ spin $+1$ in the bulk and $SU(2)$ spin $+1/2$ at the boundary was investigated. It arises in the $SU(2)_L$ sector of QCD and the $SU(2,2)$ system is a straightforward generalization.

[23]For the chain of length $L = 2$ we should choose $\Omega_k(u) = \mathbf{R}_{k,1}(u)\mathbf{R}_{k,2}(u)$ instead where the auxiliary spin is $\mathbb{V}^\chi = \mathbb{V}^\psi$.

where the auxiliary spin labeled by k transforms like a field strength $\mathbb{V}^f$. The open chain transfer matrix with $SU(2,2)$-preserving boundary condition ($\mathbf{K} = 1$) is

$$T(u) = \text{tr}_k \, \Omega_k(u)\Omega_k^{-1}(-u). \tag{C.16}$$

The Hamiltonian in the bulk is given by the same expression as before. At the boundaries the Hamiltonian is modified due to the shift in the definition of $\Omega_k(u)$

$$\mathbf{H}_{\chi f} = i\mathbf{R}_{\chi f}(\pm i/2)^{-1}\frac{\partial \mathbf{R}_{\chi f}}{\partial u}(\pm i/2) \tag{C.17}$$

consequently, the energy eigenvalues obey

$$E_j^{\chi f} = \frac{4}{J_j^2 - J_{j-1}^2 + 2} + \frac{4}{J_j^2 - J_{j-1}^2 - 2} + E_{j-1}^{\chi f}. \tag{C.18}$$

The eigenvalue of the second Casimir for $\mathbb{V}_j^{\chi f}$ is $J_{\chi f,j}^2 = 2j^2 + 8j + \frac{15}{4}$; when we substitute this we obtain

$$E_j^{\chi f} = \frac{1}{j+1} + \frac{1}{j+2} + E_{j-1}^{\chi f} = h(j+2) + h(j+1) + E^\chi + E^f \tag{C.19}$$

where $E^\chi + E^f$ is a constant we cannot determine here.

Also the chains with one or two anti-chiral fermions at the ends, but all chiral field strengths are integrable. We simply replace first/last R-matrix in Ω by $\mathbf{R}_{k,1/L}(u \pm \frac{3i}{2})$. The eigenvalue of the second Casimir for $\mathbb{V}_j^{\bar{\chi}f}$ is $J_{\chi f,j}^2 = 2j^2 + 4j - \frac{9}{4}$ and we obtain

$$E_j^{\bar{\chi}f} = \frac{1}{j+2} + \frac{1}{j-1} + E_{j-1}^{\chi f} = h(j+2) + h(j-1) + E^\chi + E^f. \tag{C.20}$$

In this case an anti-chiral field is not an obstacle to integrability. The reason why an anti-chiral field strength breaks integrability is that it may propagate while the anti-chiral quark is fixed at the end of the chain. In other words, integrability breaks down for mixed field strengths due to $E_j^{f\bar{f},\mathbf{X}} \neq 0$.

Proof. Here we will present a proof of the Yang-Baxter equation for the R-matrix given above. It is similar to the one for $\mathcal{N} = 4$ given in [16]. The three spin modules for the YBE will be of oscillator type and one of them will be fundamental. First of all, consider the R-matrix of an oscillator spin k and a fundamental spin i. The tensor product $\mathbb{V}^k \otimes \mathbb{V}_{\text{fund}}$ has two irreducible components which we shall denote by $\mathbb{V}_\pm^{ki}$. The projectors are given by

$$\mathbf{P}_\pm^{ki} = \frac{1}{2} \pm \frac{\mathcal{J}_{ki} + 1 - \frac{1}{2}A_k}{2A_k - 1} \tag{C.21}$$

where A_k is the value of $\mathcal{A}$ on $\mathbb{V}^k$ and $\mathcal{J}_{ki}$ is the algebra generator acting on $\mathbb{V}^k$ whose components $\mathcal{J}_k{}^A{}_B$ specify the map $\mathbb{V}_{\text{fund}} \to \mathbb{V}_{\text{fund}}$. The difference of eigenvalues of the quadratic Casimir is

$$J_{ki,+}^2 - J_{ki,-}^2 = 2(2A_k - 1). \tag{C.22}$$

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The R-matrix (C.6,C.7) therefore reads

$$\mathbf{R}_{ki}(u_{ki}) = \mathbf{P}_+^{ki} + \frac{u_{ki} - \frac{i}{4}(J_{ki,+}^2 - J_{ki,-}^2)}{u_{ki} + \frac{i}{4}(J_{ki,+}^2 - J_{ki,-}^2)} \mathbf{P}_-^{ki} = \frac{u_{ki} + i - \frac{i}{2}A_k + i\mathcal{J}_{ki}}{u_{ki} + \frac{i}{2}(2A_k - 1)}. \tag{C.23}$$

Now consider the YBE for two oscillator spins k, l and a fundamental spin i and expand the R-matrices $\mathbf{R}_{ik}(u_{ik}), \mathbf{R}_{il}(u_{il})$ involving a fundamental spin

$$0 \overset{!}{=} \mathbf{R}_{ik}(u_{ik})\mathbf{R}_{il}(u_{il})\mathbf{R}_{kl}(u_{kl}) - \mathbf{R}_{kl}(u_{kl})\mathbf{R}_{il}(u_{il})\mathbf{R}_{ik}(u_{ik})$$

$$= \frac{i(u_{ik} + u_{il} + 2i - \frac{i}{2}A_k - \frac{i}{2}A_l)}{2\left(u_{ik} + \frac{i}{2}(2A_k - 1)\right)\left(u_{il} + \frac{i}{2}(2A_l - 1)\right)} \left[\mathcal{J}_{ki} + \mathcal{J}_{li}, \mathbf{R}_{kl}(u_{kl})\right]$$

$$+ \frac{i(u_{kl} + \frac{i}{2}A_k - \frac{i}{2}A_l)}{2\left(u_{ik} + \frac{i}{2}(2A_k - 1)\right)\left(u_{il} + \frac{i}{2}(2A_l - 1)\right)} \left[\mathcal{J}_{ki} - \mathcal{J}_{li}, \mathbf{R}_{kl}(u_{kl})\right]$$

$$- \frac{1}{2\left(u_{ik} + \frac{i}{2}(2A_k - 1)\right)\left(u_{il} + \frac{i}{2}(2A_l - 1)\right)} \left\{[\mathcal{J}_{ki}, \mathcal{J}_{li}], \mathbf{R}_{kl}(u_{kl})\right\}$$

$$- \frac{1}{2\left(u_{ik} + \frac{i}{2}(2A_k - 1)\right)\left(u_{il} + \frac{i}{2}(2A_l - 1)\right)} \left[\{\mathcal{J}_{ki}, \mathcal{J}_{li}\}, \mathbf{R}_{kl}(u_{kl})\right]. \tag{C.24}$$

The first term vanishes because

$$\mathcal{J}_{ki} + \mathcal{J}_{li} = \mathcal{J}_{kl,i} \tag{C.25}$$

is the rotation generator acting on $\mathbb{V}^k \otimes \mathbb{V}^l$ which clearly commutes with the covariant R-matrix. Let us now introduce the symbol $\mathbf{C}_{kl,i}$ for the combination of generators in the second term

$$\mathcal{J}_{ki} - \mathcal{J}_{li} = \mathbf{C}_{kl,i}. \tag{C.26}$$

Then we can express the commutator in the third term in terms of $\mathbf{C}_{kl,i}$

$$[\mathcal{J}_{ki}, \mathcal{J}_{li}] = \tfrac{1}{4}[\mathcal{J}_{kl}^2, \mathbf{C}_{kl,i}] \tag{C.27}$$

and the quadratic Casimir $\mathcal{J}_{kl}^2$ on $\mathbb{V}^k \times \mathbb{V}^l$. Finally, we can write the anti-commutator in the fourth term using squared generators[24]

$$\{\mathcal{J}_{ki}, \mathcal{J}_{li}\} = (\mathcal{J}_{kl,i})^2 - (\mathcal{J}_{k,i})^2 - (\mathcal{J}_{l,i})^2. \tag{C.28}$$

Using the rules (C.4,C.5) for oscillator representations, we can compute the contracted product of two generators

$$\mathcal{J}^A{}_B \mathcal{J}^B{}_C = (\mathcal{A} - 2)\mathcal{J}^A{}_C + \tfrac{3}{4}(\mathcal{A}^2 - 1)\delta_C^A. \tag{C.29}$$

It is a special property of oscillator representations that $\mathcal{J}^A{}_B\mathcal{J}^B{}_C$ can be written as a linear combination of $\mathcal{J}^A{}_C$ and δ_C^A.[25] We can now simplify the anti-commutator (C.28)

$$\{\mathcal{J}_{ki}, \mathcal{J}_{li}\} = (\mathcal{J}_{kl,i})^2 - (A_k - 2)\mathcal{J}_{k,i} - (A_l - 2)\mathcal{J}_{l,i} - \tfrac{3}{4}(A_k^2 + A_l^2 - 2) \tag{C.30}$$

$$= (\mathcal{J}_{kl,i})^2 - \tfrac{1}{2}(A_k + A_l - 4)\mathcal{J}_{kl,i} - \tfrac{3}{4}(A_k^2 + A_l^2 - 2) - \tfrac{1}{2}(A_k - A_l)\mathbf{C}_{kl,i}.$$

---

[24]Note that we have to distinguish between $(\mathcal{J}_i)^2 = \mathcal{J}_i\mathcal{J}_i$ and the quadratic Casimir $\mathcal{J}^2 = \text{tr}_i \mathcal{J}_i\mathcal{J}_i$.

[25]For generic representations only a four-fold product of generators of the above sort can be written as something simpler.

Again all terms except the one proportional to $\mathbf{C}_{kl,i}$ drop out in commutators with $\mathbf{R}_{kl}$.

We can now put everything together and find

$$
\begin{aligned}
0 &\overset{!}{=} \mathbf{R}_{ik}(u_{ik})\mathbf{R}_{il}(u_{il})\mathbf{R}_{kc}(u_{kl}) - \mathbf{R}_{kl}(u_{kl})\mathbf{R}_{il}(u_{il})\mathbf{R}_{ik}(u_{ik}) \\
&= i\,\frac{u_{kl}\big[\mathbf{C}_{kl,i},\mathbf{R}_{kl}(u_{kl})\big] + \tfrac{i}{4}\big[\mathcal{J}_{kl}^2,\{\mathbf{C}_{kl,i},\mathbf{R}_{kl}(u_{kl})\}\big]}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{li}+\tfrac{i}{2}(2A_l-1)\big)}
\end{aligned}
\tag{C.31}
$$

To proceed further, let us investigate the form of the operator $\mathbf{C}_{kl,i}$. It is a map $\mathbb{V}^k \otimes \mathbb{V}^l \otimes \mathbb{V}^i \to \mathbb{V}^k \otimes \mathbb{V}^l \otimes \mathbb{V}^i$. By construction it is invariant under simultaneous rotation of all modules by the same amount. When we combine the (oscillator) modules into irreducible components, we can write

$$
\mathbf{C}_{kl,i}: \mathbb{V}_j^{kl} \otimes \mathbb{V}_{\text{fund}} \to \sum_{j'} \mathbb{V}_{j'}^{kl} \otimes \mathbb{V}_{\text{fund}}.
\tag{C.32}
$$

By comparing the highest weight vectors of $\mathbb{V}_j^{kl}$ and $\mathbb{V}_{j'}^{kl}$ we see that $\mathbf{C}_{kl,i}$ can only be invariant when $j' \in \{j-1, j, j+1\}$. Furthermore, $\mathbf{C}_{kl,i}$ has negative parity under interchange of k and l. As the parity is alternating with j, the cases $j' = j \pm 1$ are allowed while $j' = j$ is not. In conclusion we can write

$$
\mathbf{C}_{kl,i}: \mathbb{V}_j^{kl} \otimes \mathbb{V}_{\text{fund}} \to \mathbb{V}_{j-1}^{kl} \otimes \mathbb{V}_{\text{fund}} \ \oplus \ \mathbb{V}_{j+1}^{kl} \otimes \mathbb{V}_{\text{fund}}
\tag{C.33}
$$

This is another generic feature of tensor products of oscillator representations. The recursion relation (C.7)

$$
R_j^{kl}(u_{kl}) = \frac{u_{kl} + \tfrac{i}{4}\big(J_{kl,j}^2 - J_{kl,j-1}^2\big)}{u_{kl} - \tfrac{i}{4}\big(J_{kl,j}^2 - J_{kl,j-1}^2\big)}\, R_{j-1}^{kl}(u_{kl})
\tag{C.34}
$$

follows from projecting (C.31) to the various irreducible components. Similarly, for a different ordering of the three spins we obtain the same relation from the YBE

$$
\begin{aligned}
0 &\overset{!}{=} \mathbf{R}_{ki}(u_{ki})\mathbf{R}_{kl}(u_{kl})\mathbf{R}_{il}(u_{il}) - \mathbf{R}_{il}(u_{il})\mathbf{R}_{kl}(u_{kl})\mathbf{R}_{ki}(u_{ki}) \\
&= \frac{i\big(u_{il}-u_{ki}-\tfrac{i}{2}A_l+\tfrac{i}{2}A_k\big)}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}\big[\mathcal{J}_{ki}+\mathcal{J}_{li},\mathbf{R}_{kl}(u_{kl})\big] \\
&\quad + \frac{i\big(u_{kl}+2i-\tfrac{i}{2}A_k-\tfrac{i}{2}A_l\big)}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}\big[\mathcal{J}_{ki}-\mathcal{J}_{li},\mathbf{R}_{kl}(u_{kl})\big] \\
&\quad + \frac{1}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}\big\{\mathcal{J}_{ki}-\mathcal{J}_{li},\big[\mathcal{J}_{ki}+\mathcal{J}_{li},\mathbf{R}_{kl}(u_{kl})\big]\big\} \\
&\quad - \frac{1}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}\big[\mathcal{J}_{ki}\mathcal{J}_{ki}-\mathcal{J}_{li}\mathcal{J}_{li},\mathbf{R}_{kl}(u_{kl})\big] \\
&\quad - \frac{1}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}\big\{[\mathcal{J}_{ki},\mathcal{J}_{li}],\mathbf{R}_{kl}(u_{kl})\big\} \\
&= i\,\frac{u_{kl}\big[\mathbf{C}_{kl,i},\mathbf{R}_{kl}(u_{kl})\big] + \tfrac{i}{4}\big\{[\mathcal{J}_{kl}^2,\mathbf{C}_{kl,i}],\mathbf{R}_{kl}(u_{kl})\big\}}{2\big(u_{ki}+\tfrac{i}{2}(2A_k-1)\big)\big(u_{il}+\tfrac{i}{2}(2A_l-1)\big)}.
\end{aligned}
\tag{C.35}
$$

References

[1] H. Georgi and H. D. Politzer, *"Electroproduction scaling in an asymptotically free theory of strong interactions"*, Phys. Rev. D9, 416 (1974). ● D. J. Gross and F. Wilczek, *"Asymptotically free gauge theories. 2"*, Phys. Rev. D9, 980 (1974).

[2] G. 't Hooft, *"A Planar Diagram Theory for Strong Interactions"*, Nucl. Phys. B72, 461 (1974). ● A. M. Polyakov, *"Gauge fields and strings"*, Harwood (1987), Chur, Switzerland, 301p, Contemporary Concepts in Physics, 3.

[3] J. M. Maldacena, *"The large N limit of superconformal field theories and supergravity"*, Adv. Theor. Math. Phys. 2, 231 (1998), `hep-th/9711200`.

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"Gauge theory correlators from non-critical string theory"*, Phys. Lett. B428, 105 (1998), `hep-th/9802109`. ● E. Witten, *"Anti-de Sitter space and holography"*, Adv. Theor. Math. Phys. 2, 253 (1998), `hep-th/9802150`.

[5] D. Berenstein, J. M. Maldacena and H. Nastase, *"Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills"*, JHEP 0204, 013 (2002), `hep-th/0202021`.

[6] A. Santambrogio and D. Zanon, *"Exact anomalous dimensions of $\mathcal{N} = 4$ Yang-Mills operators with large R charge"*, Phys. Lett. B545, 425 (2002), `hep-th/0206079`.

[7] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"A semi-classical limit of the gauge/string correspondence"*, Nucl. Phys. B636, 99 (2002), `hep-th/0204051`.

[8] J. M. Maldacena, *"Wilson loops in large N field theories"*, Phys. Rev. Lett. 80, 4859 (1998), `hep-th/9803002`. ● S.-J. Rey and J.-T. Yee, *"Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity"*, Eur. Phys. J. C22, 379 (2001), `hep-th/9803001`. ● J. K. Erickson, G. W. Semenoff and K. Zarembo, *"Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory"*, Nucl. Phys. B582, 155 (2000), `hep-th/0003055`. ● N. Drukker and D. J. Gross, *"An exact prediction of $\mathcal{N} = 4$ SUSYM theory for string theory"*, J. Math. Phys. 42, 2896 (2001), `hep-th/0010274`.

[9] S. Frolov and A. A. Tseytlin, *"Multi-spin string solutions in $AdS_5 \times S^5$"*, Nucl. Phys. B668, 77 (2003), `hep-th/0304255`. ● N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, *"Stringing Spins and Spinning Strings"*, JHEP 0309, 010 (2003), `hep-th/0306139`. ● S. Frolov and A. A. Tseytlin, *"Quantizing three-spin string solution in $AdS_5 \times S^5$"*, JHEP 0307, 016 (2003), `hep-th/0306130`. ● N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, *"Precision Spectroscopy of AdS/CFT"*, JHEP 0310, 037 (2003), `hep-th/0308117`. ● J. Engquist, J. A. Minahan and K. Zarembo, *"Yang-Mills Duals for Semiclassical Strings"*, JHEP 0311, 063 (2003), `hep-th/0310188`. ● G. Arutyunov and M. Staudacher, *"Matching Higher Conserved Charges for Strings and Spins"*, JHEP 0403, 004 (2004), `hep-th/0310182`. ● A. A. Tseytlin, *"Spinning strings and AdS/CFT duality"*, `hep-th/0311139`. ● S. A. Frolov, I. Y. Park and A. A. Tseytlin, *"On one-loop correction to energy of spinning strings in S^5"*, `hep-th/0408187`.

[10] D. Serban and M. Staudacher, *"Planar $\mathcal{N} = 4$ gauge theory and the Inozemtsev long range spin chain"*, JHEP 0406, 001 (2004), `hep-th/0401057`.

49

[11] C. Kristjansen, *"Three-spin strings on $AdS_5 \times S^5$ from $\mathcal{N} = 4$ SYM"*, Phys. Lett. B586, 106 (2004), `hep-th/0402033`. • V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, *"Classical/quantum integrability in AdS/CFT"*, JHEP 0405, 024 (2004), `hep-th/0402207`.

[12] J. A. Minahan and K. Zarembo, *"The Bethe-ansatz for $\mathcal{N} = 4$ super Yang-Mills"*, JHEP 0303, 013 (2003), `hep-th/0212208`.

[13] N. Beisert and M. Staudacher, *"The $\mathcal{N} = 4$ SYM Integrable Super Spin Chain"*, Nucl. Phys. B670, 439 (2003), `hep-th/0307042`.

[14] N. Beisert, C. Kristjansen and M. Staudacher, *"The dilatation operator of $\mathcal{N} = 4$ conformal super Yang-Mills theory"*, Nucl. Phys. B664, 131 (2003), `hep-th/0303060`. • N. Beisert, *"The su(2/3) dynamic spin chain"*, Nucl. Phys. B682, 487 (2004), `hep-th/0310252`.

[15] N. Beisert, V. Dippel and M. Staudacher, *"A Novel Long Range Spin Chain and Planar $\mathcal{N} = 4$ Super Yang-Mills"*, JHEP 0407, 075 (2004), `hep-th/0405001`.

[16] N. Beisert, *"The Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory and Integrability"*, Phys. Rept. 405, 1 (2005), `hep-th/0407277`.

[17] K. Okuyama and L.-S. Tseng, *"Three-point functions in $\mathcal{N} = 4$ SYM theory at one-loop"*, JHEP 0408, 055 (2004), `hep-th/0404190`. • R. Roiban and A. Volovich, *"Yang-Mills Correlation Functions from Integrable Spin Chains"*, JHEP 0409, 032 (2004), `hep-th/0407140`.

[18] M. Kruczenski, *"Spin chains and string theory"*, Phys. Rev. Lett. 93, 161602 (2004), `hep-th/0311203`. • M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, *"Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains"*, Nucl. Phys. B692, 3 (2004), `hep-th/0403120`.

[19] R. Hernandez and E. Lopez, *"The SU(3) spin chain sigma model and string theory"*, `hep-th/0403139`. • J. Stefanski, B. and A. A. Tseytlin, *"Large spin limits of AdS/CFT and generalized Landau- Lifshitz equations"*, JHEP 0405, 042 (2004), `hep-th/0404133`. • A. Agarwal and S. G. Rajeev, *"The dilatation operator of $\mathcal{N} = 4$ SYM and classical limits of spin chains and matrix models"*, `hep-th/0405116`. • K. Ideguchi, *"Semiclassical strings on $AdS_5 \times S^5/Z_M$ and operators in orbifold field theories"*, JHEP 0409, 008 (2004), `hep-th/0408014`. • M. Smedbäck, *"Pulsating strings on $AdS_5 \times S^5$"*, JHEP 0407, 004 (2004), `hep-th/0405102`. • S. Ryang, *"Circular and folded multi-spin strings in spin chain sigma models"*, JHEP 0410, 059 (2004), `hep-th/0409217`. • Y. Susaki, Y. Takayama and K. Yoshida, *"Open semiclassical strings and long defect operators in AdS/dCFT correspondence"*, `hep-th/0410139`.

[20] G. Ferretti, R. Heise and K. Zarembo, *"New Integrable Structures in Large-N QCD"*, Phys. Rev. D70, 074024 (2004), `hep-th/0404187`.

[21] A. P. Bukhvostov, G. V. Frolov, L. N. Lipatov and E. A. Kuraev, *"Evolution equations for quasi-partonic operators"*, Nucl. Phys. B258, 601 (1985).

[22] L. N. Lipatov, *"High-energy asymptotics of multicolor QCD and exactly solvable lattice models"*, JETP Lett. 59, 596 (1994), `hep-th/9311037`. • L. D. Faddeev and G. P. Korchemsky, *"High-energy QCD as a completely integrable model"*, Phys. Lett. B342, 311 (1995), `hep-th/9404173`.

[23] V. M. Braun, S. E. Derkachov and A. N. Manashov, *"Integrability of three-particle evolution equations in QCD"*, Phys. Rev. Lett. 81, 2020 (1998), `hep-ph/9805225`.

[24] V. M. Braun, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, *"Baryon distribution amplitudes in QCD"*, Nucl. Phys. B553, 355 (1999), `hep-ph/9902375`.

[25] A. V. Belitsky, *"Renormalization of twist-three operators and integrable lattice models"*, Nucl. Phys. B574, 407 (2000), `hep-ph/9907420`.

[26] V. M. Braun, G. P. Korchemsky and D. Muller, *"The uses of conformal symmetry in QCD"*, Prog. Part. Nucl. Phys. 51, 311 (2003), `hep-ph/0306057`.

[27] A. V. Belitsky, V. M. Braun, A. S. Gorsky and G. P. Korchemsky, *"Integrability in QCD and beyond"*, Int. J. Mod. Phys. A19, 4715 (2004), `hep-th/0407232`.

[28] A. V. Belitsky, S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, *"Dilatation operator in (super-)Yang-Mills theories on the light-cone"*, `hep-th/0409120`.

[29] N. Beisert, *"The Complete One-Loop Dilatation Operator of $\mathcal{N} = 4$ Super Yang-Mills Theory"*, Nucl. Phys. B676, 3 (2004), `hep-th/0307015`.

[30] A. Y. Morozov, *"Matrix of mixing of scalar and vector mesons of dimension $D \leq 8$ in QCD."*, Sov. J. Nucl. Phys. 40, 505 (1984), in Russian. • J. A. Gracey, *"Classification and one loop renormalization of dimension six and eight operators in quantum gluodynamics"*, Nucl. Phys. B634, 192 (2002), `hep-ph/0204266`.

[31] P. Di Vecchia and A. Tanzini, *"$\mathcal{N} = 2$ super Yang-Mills and the XXZ spin chain"*, `hep-th/0405262`.

[32] L. D. Faddeev and L. A. Takhtajan, *"What is the spin of a spin wave?"*, Phys. Lett. A85, 375 (1981).

[33] L. D. Faddeev, *"How Algebraic Bethe Ansatz works for integrable model"*, `hep-th/9605187`.

[34] E. Witten, *"Perturbative gauge theory as a string theory in twistor space"*, Commun. Math. Phys. 252, 189 (2004), `hep-th/0312171`.

[35] M. E. Peskin and D. V. Schroeder, *"An introduction to quantum field theory"*, Addison-Wesley (1995), Reading, USA, 842p.

[36] H. Kluberg-Stern and J. B. Zuber, *"Ward identities and some clues to the renormalization of gauge invariant operators"*, Phys. Rev. D12, 467 (1975).

[37] Y. M. Makeenko, *"Conformal operators in quantum chromodynamics"*, Sov. J. Nucl. Phys. 33, 440 (1981). • T. Ohrndorf, *"Constraints from conformal covariance on the mixing of operators of lowest twist"*, Nucl. Phys. B198, 26 (1982).

[38] K. A. Intriligator, *"Bonus symmetries of $\mathcal{N} = 4$ super-Yang-Mills correlation functions via AdS duality"*, Nucl. Phys. B551, 575 (1999), `hep-th/9811047`.

[39] E. V. Shuryak and A. I. Vainshtein, *"Theory of power corrections to deep inelastic scattering in quantum chromodynamics. 1. Q^2 effects"*, Nucl. Phys. B199, 451 (1982). • E. V. Shuryak and A. I. Vainshtein, *"Theory of power corrections to deep inelastic scattering in quantum chromodynamics. 2. Q^4 effects: Polarized target"*, Nucl. Phys. B201, 141 (1982).

[40] A. B. Zamolodchikov and V. A. Fateev, *"Model factorized S matrix and an integrable Heisenberg chain with spin 1"*, Yad. Fiz. 32, 581 (1980), in Russian. • P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, *"Yang-Baxter equation and representation theory: I"*, Lett. Math. Phys. 5, 393 (1981). • N. Y. Reshetikhin, *"Integrable models of quantum one-dimensional magnets with O(N) and Sp(2K) symmetry"*, Theor. Math. Phys. 63, 555 (1985).

[41] H. Bethe, *"On the theory of metals. 1. Eigenvalues and eigenfunctions for the linear atomic chain"*, Z. Phys. 71, 205 (1931).

[42] L. A. Takhtajan, *"The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins"*, Phys. Lett. A87, 479 (1981).

[43] H. M. Babujian, *"Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spin S"*, Phys. Lett. A90, 479 (1982).

[44] H. M. Babujian, *"Exact solution of the isotropic Heisenberg chain with arbitrary spins: Thermodynamics of the model"*, Nucl. Phys. B215, 317 (1983).

[45] E. Ogievetsky and P. Wiegmann, *"Factorized S matrix and the Bethe ansatz for simple Lie groups"*, Phys. Lett. B168, 360 (1986).

[46] H. Johannesson, *"The structure of low lying excitations in a new integrable quantum chain model"*, Nucl. Phys. B270, 235 (1986).

[47] S. E. Derkachov, G. P. Korchemsky and A. N. Manashov, *"Evolution equations for quark gluon distributions in multi-color QCD and open spin chains"*, Nucl. Phys. B566, 203 (2000), `hep-ph/9909539`.

[48] D. Arnaudon, N. Crampé, A. Doikou, F. Luc and E. Ragoucy, *"Analytical Bethe Ansatz for closed and open gl(n)-spin chains in any representation"*, `math-ph/0411021`.

[49] B. Chen, X.-J. Wang and Y.-S. Wu, *"Integrable open spin chain in super Yang-Mills and the plane-wave / SYM duality"*, JHEP 0402, 029 (2004), `hep-th/0401016`. • B. Chen, X.-J. Wang and Y.-S. Wu, *"Open spin chain and open spinning string"*, Phys. Lett. B591, 170 (2004), `hep-th/0403004`.

[50] A. Doikou, *"The XXX spin s quantum chain and the alternating s^1, s^2 chain with boundaries"*, Nucl. Phys. B634, 591 (2002), `hep-th/0201008`.

[51] N. Reshetikhin, *"S matrices in integrable models of isotropical magnetic chains. 1"*, J. Phys. A24, 3299 (1991).

[52] H. W. J. Blöte, J. L. Cardy and M. P. Nightingale, *"Conformal invariance, the central charge and universal finite size amplitudes at criticality"*, Phys. Rev. Lett. 56, 742 (1986). • I. Affleck, *"Universal term in the free energy at a critical point and the conformal anomaly"*, Phys. Rev. Lett. 56, 746 (1986).

[53] P. W. Anderson, *"The Resonating valence bond state in La-2 CuO-4 and superconductivity"*, Science 235, 1196 (1987).

[54] I. Bars and M. Gunaydin, *"Unitary representations of noncompact supergroups"*, Commun. Math. Phys. 91, 31 (1983).

[55] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *"The Hagedorn / deconfinement phase transition in weakly coupled large N gauge theories"*, `hep-th/0310285`. • M. Bianchi, J. F. Morales and H. Samtleben, *"On stringy $AdS_5 \times S^5$*

and higher spin holography*”, JHEP 0307, 062 (2003), `hep-th/0305052`. • N. Beisert, M. Bianchi, J. F. Morales and H. Samtleben, *“On the spectrum of AdS/CFT beyond supergravity”*, JHEP 0402, 001 (2004), `hep-th/0310292`.

[56] B. Sundborg, *“The Hagedorn transition, deconfinement and $\mathcal{N} = 4$ SYM theory”*, Nucl. Phys. B573, 349 (2000), `hep-th/9908001`.

[57] M. A. Ahmed and G. G. Ross, *“Polarized lepton - hadron scattering in asymptotically free gauge theories”*, Nucl. Phys. B111, 441 (1976).

[58] D. G. Robertson and F. Wilczek, *“Anomalous dimensions of anisotropic gauge theory operators”*, Phys. Lett. B251, 434 (1990).