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RESEARCH ARTICLE

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Remarks on exact solutions

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Abstract This paper first discusses the historical context of the influential "Exact Solutions" book, which was co-authored by Malcolm MacCallum. It then makes various technical points about such solutions. It is useful to characterize solutions in terms of the properties of matter they contain or the kinds of test particle motions which are possible. It is also interesting to consider which isometries are implied by particular kinds of matter behaviour. Judging the validity of an approximation by the smallness of the tensor components can be misleading. Finally, an example is given of a result which is obvious in Newtonian theory but little understood in General Relativity.

Keywords General relativity · Exact solutions · Test particle · Kerr-Schild form

1 Introduction

First, I should like to thank the Programme Committee for inviting me to this happy occasion. It gives me the opportunity to congratulate Malcolm on his 60th birthday and offer him best wishes for the future. As a former president of the GRG society I want to use the occasion to thank Malcolm for the work he has done and is still doing for our society. The fact that all matters concerning administration, the budget, information about members and events, relations to other organisations and other things are proceeding smoothly we owe primarily to Malcolm's circumspection and dependability. We can only hope that he will continue doing this important service.

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2 Exact solutions: history and generalities

Let me now turn to one of Malcolm's principal research areas, exact solutions. While I was never one of his collaborators, I may consider myself, along with A.Z. Petrov, as a forerunner in trying to find, classify and understand such solutions. The title of my 1958 PhD thesis, done with Pascual Jordan's group at the University of Hamburg, was "Konstruktionen und Charakterisierungen von Lösungen der Einsteinschen Gravitationsfeldgleichungen" [1]. This project was continued in a series of seven publications entitled "Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie", produced by Jordan's students Wolfgang Kundt, Ray Sachs, Manfred Trümper, István Ozsváth and myself [2]. These were published between 1960 and 1965 by the Akademie der Wissenschaften und der Literatur in Mainz. They were widely read, particularly by the then young relativists in London and Cambridge and nicknamed "Mainz bible", perhaps an allusion to the famous Gutenberg-Bibel, printed around 1550 in Mainz. From this activity also emerged the chapter on "Exact Solutions", which Wolfgang Kundt and I contributed to the (still useful) book "Gravitation: An introduction to current research", edited by Louis Witten and published in 1962 [3].

These works had a similar origin to the "Exact Solutions" book, the second edition of which appeared last year. In both cases the researchers wanted to work on solutions and therefore wished to have a survey of what was known. There was a difference, however. In the "old" attempt, attention was mainly given to those few solutions which could in some sense be understood in terms of newly discovered intrinsic, coordinate-independent and physically interpretable properties. In the "new" attempt, which was begun around 1975 by Hans Stephani and Dietrich Kramer, joined soon by Eduard Herlt and a little later by Malcolm, the ambition was to survey all known solutions, whether they were yet "understood" or not. This effort necessitated a big literature search and organisational strategy, in view of the many solutions which had been found and were being steadily added.

The 1962 article mentioned above had 50 pages and about 100 references; the new edition of the Exact Solutions book (which has Cornelius Hoenselaers as an additional author) has more than 700 pages and lists more than 2000 references. The basic tools for classifying solutions were already outlined in the 1962 article, but the variety of methods for obtaining solutions and, consequently, the number of "known" solutions has increased very much since the old days. Thanks to the above-mentioned authors we have now a well-structured gallery, useful not least because of the chapters on methods, concepts and formalisms. The book can, and hopefully will, be used as a basis for research to widen and deepen the understanding of what solutions can contribute to the physical implications of General Relativity. An example of this way of looking at solutions is Jiri Bicak's contribution to Einstein's Field Equations and Their Physical Implications [4].

3 Examples

Next, to justify my presence at this meeting, I wish to make some remarks related to exact solutions. They are technically simple but may illustrate points of general interest, or raise questions.

(a) For classifying solutions in general it is usual to focus primarily on properties of the metric and not on the matter variables (which may even be absent). But sometimes it is of interest, not least since it is matter (including radiation) that is observed, to characterize solutions in terms of the properties of matter.

Consider the class of dust solutions, taking into account a cosmological constant term. Ask for the kinematically simplest solutions where the matter moves rigidly (in Born's sense). Then, if in addition the motion is irrotational, the solution is Einstein's static universe, this being characterized by the above properties. If, alternatively, matter is rotating with constant angular velocity ($\nabla_\alpha \omega^\beta = 0$), it is the Gödel universe, which is characterized by these properties. Note that in both cases the characterization is global if the underlying spacetime is assumed simply-connected and inextensible. It is then implied that spacetime is non-singular, i.e. affinely complete. Note also that while both solutions have large isometry groups, no isometry assumptions enter the characterizing properties; the isometries follow from simple, intuitive, kinematical assumptions. The existence of a metric has to be taken for granted since it enters – so far in any successful physical theory - into the description of the matter and its state; but the metric and its connection are needed only pointwise for stating the characterizing properties. Finally, it is interesting to compare the roles of Λ in the two solutions. In the Einstein universe, a positive Λ term balances the isotropic gravitational attraction; in Gödel's model, a negative Λ-term balances the anisotropic gravitational tidal field and the anisotropic centrifugal field.

(b) Non-static Robertson-Walker spacetimes are the only non-stationary spacetimes which admit an irrotational, geodesic flow of fundamental observers with respect to whom a distribution of free photons appears everywhere isotropic. This characterization may be rephrased for solutions whose matter consists of a non-interacting mixture of expanding or contracting dust and radiation, the latter appearing isotropic with respect to the former. (In the second version, irrotationality is implied). These three examples raise the question: Which isometrics are implied by which kinds of matter behaviour?

(c) Some vacuum spacetimes can similarly be characterized in terms of the particular kinds of test particle motions which are possible within them; examples were given in my 1962 article with Kundt [3]. Petrov type N vacuum fields are the only ones whose effect on clouds of freely falling test particles is strictly transverse. In them, and only in them, there exists a lightlike vector field k such that for any observer with 4-velocity U, relative accelerations of freely falling test particles occur only if their connection vectors are contained in the plane orthogonal to both U and k. I should like to add that a plane sandwich gravitational wave has a permanent effect on a cloud of test particles originally at rest; it causes a shear motion and astigmatic focussing. This was already noticed in 1959 by Bondi, Pirani and Robinson [5] and analyzed in detail by Penrose in 1965 [6]. This is a nonlinear effect, not visible in the linear approximation.

(d) If added to the Minkowski metric, the expressions $A(t-z)(dx^2-dy^2)$ (T) and $\frac{1}{2}(x^2-y^2)A''(t-z)(dt-dz)^2$ (L) both satisfy the vacuum equation to first order in A. They provide gauge-equivalent linearized plane-wave solutions in the transverse (T) and longitudinal (L) gauge, respectively. Naively one would expect (T) to provide a good approximation everywhere provided $|A|$ is small and (L) would seem to give a reliable approximation near the z-axis only if $|A''|$ is small. Actually case (L), being of Kerr-Schild type, gives an exact solution (in accordance with a theorem of G ürses and Gursey (GG), while (T) is only approximate.

This example shows that judging the validity of an approximation by the smallness of the tensor components can be wrong. It also illustrates the GG-theorem: if a "linear solution" can be gauge-transformed into the Kerr-Schild type, then the latter will be an exact solution. As another example of this, the linearized Schwarzschild solution can be gauge-transformed into Kerr-Schild (Eddington-Finkelstein) form, which is also exact. Incidentally, the Eddington-Finkelstein-form of the Schwarzschild metric is linear in m, so the frequently made assertion that the perihelion advance is a "nonlinear" effect in contrast to light-deflection, has no intrinsic meaning, as pointed out in J. L. Synge's textbook on general relativity [7]. The main simplicity of the Kerr-Schild metrics $g - 2Sk \otimes k$ with $g^{-2}(k, k) = 0$ is their simple invertibility. This, of course, also applies if the seed metric g is not flat. An important question is whether the GG-result is generalizable to non-flat g's.

(e) Non-existence theorems are not a topic of exact solutions. In spite of this, I wish to end by calling attention to the fact that it has still not been shown that there is no static, asymptotically flat solution containing two bodies, separated by a totally geodesic surface in 3-space, a result which is obvious in Newtonian theory. This illustrates how little we understand equations of motion in General Relativity.

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