$K(E_{10})$, supergravity and fermions

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ABSTRACT: We study the fermionic extension of the $E_{10}/K(E_{10})$ coset model and its relation to eleven-dimensional supergravity. Finite-dimensional spinor representations of the compact subgroup $K(E_{10})$ of $E_{10}(\mathbb{R})$ are studied and the supergravity equations are rewritten using the resulting algebraic variables. The canonical bosonic and fermionic constraints are also analysed in this way, and the compatibility of supersymmetry with local $K(E_{10})$ is investigated. We find that all structures involving A_9 levels $\ell = 0, 1$ and 2 nicely agree with expectations, and provide many non-trivial consistency checks of the existence of a supersymmetric extension of the $E_{10}/K(E_{10})$ coset model, as well as a new derivation of the 'bosonic dictionary' between supergravity and coset variables. However, there are also definite discrepancies in some terms involving level $\ell = 3$, which suggest the need for an extension of the model to infinite-dimensional faithful representations of the fermionic degrees of freedom.

KEYWORDS: Supergravity Models, Gauge Symmetry, Global Symmetries.

Contents

1. Introduction

The symmetry structures of eleven-dimensional supergravity [1] are widely believed to be instrumental, if not crucial, for finding a non-perturbative and background independent formulation of M-Theory. Starting with the work of [2, 3] the chain of exceptional symmetry groups $E_n(\mathbb{R})$ has been a recurring theme in analyses of these symmetry structures [4 – 7]. In particular, the emergence of the hyperbolic Kac-Moody algebra E_{10} in the reduction to one dimension had already been conjectured in [8], see also [9, 10]. More recently, it was argued that the bosonic supergravity field equations in eleven space-time dimensions correspond to a non-linear realization of the indefinite Kac-Moody group E_{11} (jointly with the conformal group) [11, 12], and these considerations were extended also to the (massive) IIA and IIB supergravity theories [13 – 15] by using the same groups.

Independent evidence for the relevance of E_{10} came from a study of the dynamical behaviour of the bosonic fields near a space-like singularity, which showed that the chaotic oscillations of BKL type [16] near the singularity can be effectively described in terms of a 'cosmological billiard' involving the E_{10} Weyl chamber [17, 18]. The cosmological billiard description was subsequently extended to the conjecture of a 'correspondence' between eleven-dimensional supergravity (and M-theory) and a one-dimensional σ-model on the infinite-dimensional $E_{10}/K(E_{10})$ coset space [19, 20]. This coset model has a non-linearly realised E_{10} symmetry and rephrases the dynamical evolution as a null geodesic motion on the $E_{10}/K(E_{10})$ coset space. A truncation of this model was shown to be dynamically equivalent to a truncation of the bosonic equations of eleven-dimensional supergravity [19, 20], and also to truncations of (massive) IIA and IIB supergravity [21, 22]. In yet another development, a one-dimensional geodesic model based on E_{11} was introduced in [23, 24], merging some features of the E_{11} proposal of [12] with [19].

In this paper we study the extension of the one-dimensional $E_{10}/K(E_{10})$ σ-model of [19] to include fermionic degrees of freedom. Some of our results have already been announced in [25], see also [26 – 28]. The resulting model describes a spinning massless particle on $E_{10}/K(E_{10})$, where the fermionic degrees of freedom are assigned to spinor representations of $K(E_{10})$, in analogy with the finite-dimensional hidden symmetries. Since the maximal compact subgroup $K(E_{10})$ of E_{10} is not of Kac-Moody (or any other classified) type [29] (see [30] for related studies in $K(E_9)$), an important part of the present paper (namely, section 2) is devoted to the study of the basic structure of the infinite-dimensional $K(E_{10})$. In particular, we will need to develop some representation theory below in order to describe the spin degrees of freedom. Here we will be mostly concerned with finite-dimensional, i.e. *unfaithful* representations. It will turn out that this unfaithfulness leads (beyond the first two A_9 levels in a level decomposition) to a conflict between full $K(E_{10})$ covariance and local supersymmetry. Our main conclusion is therefore that, in order to arrive at an extension of the bosonic $E_{10}/K(E_{10})$ model reconciling these two requirements, it will be necessary to replace the unfaithful spinor representation by a faithful infinite-dimensional one.

Nevertheless, the unfaithful spinor representations of $K(E_{10})$ will allow us to study many aspects of $D = 11$ supergravity (to lowest fermion order), for instance admitting an

Supergravity equation	Coset model equation
$\mathcal{G}_{ab} = 0$	$\mathcal{D}^{(0)}P^{(0)} = P^{(1)}P^{(1)} + P^{(2)}P^{(2)} + P^{(3)}P^{(3)}$
$\mathcal{M}_{abc} = 0$	$\mathcal{D}^{(0)}P^{(1)} = P^{(0)}P^{(1)} + P^{(1)}P^{(2)} + P^{(2)}P^{(3)}$
$D_{[0}F_{a_1 a_2 a_3 a_4]} = 0$	$\mathcal{D}^{(0)}P^{(2)} = P^{(0)}P^{(2)} + P^{(1)}P^{(3)}$
$D_{[0}\Omega_{ab]\,c} = 0$	$\mathcal{D}^{(0)}P^{(3)} = P^{(0)}P^{(3)}$
$\mathcal{E}_a = 0$	$\mathcal{D}^{(0)}\Psi = P^{(1)}\Psi + P^{(2)}\Psi + P^{(3)}\Psi$
$\mathcal{G}_{00} = 0$	$\mathcal{C}^{(0)} = P^{(0)}P^{(0)} + P^{(1)}P^{(1)} + P^{(2)}P^{(2)} + P^{(3)}P^{(3)} \approx 0$
$\mathcal{G}_{0a} = 0$	$\mathcal{C}^{(3)} = P^{(3)}P^{(0)} + P^{(2)}P^{(1)} \approx 0$
$\mathcal{M}_{0ab} = 0$	$\mathcal{C}^{(4)} = P^{(3)}P^{(1)} + P^{(2)}P^{(2)} \approx 0$
$D_{[a_1}F_{a_2 a_3 a_4 a_5]} = 0$	$\mathcal{C}^{(5)} = P^{(3)}P^{(2)} \approx 0$
$D_{[a_1}\Omega_{a_2 a_3\,a_4]} = 0$	$\mathcal{C}^{(6)} = P^{(3)}P^{(3)} \approx 0$
$\mathcal{E}_0 = 0$	$\mathcal{S} = P^{(0)}\Psi + P^{(1)}\Psi + P^{(2)}\Psi + P^{(3)}\Psi \approx 0$

Table 1: List of corresponding equations with indications of the A_9 level structure on the coset side. The horizontal line shows the division into coset equations of motion and constraint equations. The equations on both sides of the correspondence have to be truncated in order to make the dynamical correspondence exact. The precise correspondence depends on the 'dictionary' between supergravity and coset variables. Our current knowledge of this dictionary will be detailed in eqs. (5.1) and (5.6) below.

independent re-derivation of the bosonic 'dictionary' required for the dynamical equivalence in [19, 20]. The methods used in this derivation rest on an analysis of the supersymmetry variations by techniques developed already long ago in studies of the hidden 'R-symmetries' $K(E_7) \equiv \mathrm{SU}(8)$ and $K(E_8) \equiv \mathrm{Spin}(16)/Z_2$ [31 – 33].

We will also explore the canonical structure of the one-dimensional model and study the bosonic and supersymmetry constraints and parts of the constraint algebra and show how these relate to the supergravity equations (for the special case of homogeneous cosmological solutions of $D = 11$ supergravity, the bosonic constraints were already given in [34]). Our present results allow for the first time for a unified treatment of all bosonic and fermionic equations of supergravity in an E_{10} context.[1]

For the reader's convenience and for later reference we list in table 1 the correspondences between the equations of supergravity and those of the $E_{10}/K(E_{10})$ model. To this end we denote the components of the Einstein equation (A.18) by $\mathcal{G}_{AB}$, the components of the matter equation (A.19) by $\mathcal{M}^{ABC}$ and the components of the gravitino equation (A.24) by $\mathcal{E}_A$ (see appendix A for details). The components of the bosonic Bianchi identities (A.21) and (A.22) are written out fully. The flat space-time index range is $A, B = 0, 1, \ldots, 10$, while small Latin letters $a, b = 1, \ldots, 10$ are (flat) spatial indices. As explained in more detail below the $E_{10}/K(E_{10})$ model gives rise to certain 'currents'

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[1]A full treatment of all bosonic equations of motions and constraints in the case of type I supergravity and DE_{10} has been given in [35].

$\mathcal{P} = (P^{(0)}, P^{(1)}, P^{(2)}, P^{(3)}, \ldots)$ where the superscripts indicate the level in the A_9 level decomposition. The constraints on the coset side are denoted by $\mathcal{C}^{(\ell)}$, while $\mathcal{S}$ is the supersymmetry constraint expressed in coset variables ('$\approx$' means 'weakly zero'). The explicit expressions will be derived in section 6.

Table 1 is very schematic (see appendix A.3 for an explanation of the supergravity objects appearing in the left column). When decomposed according to level ℓ, the bosonic equations of motion are of the general structure

$$\mathcal{D}^{(0)} P^{(\ell)} = \sum_{m \geq 0} P^{(m)} * P^{(\ell+m)} \tag{1.1}$$

where the symbol $*$ stands for a sum over all representations at the indicated levels, and where we (crucially) make use of the triangular gauge, as explained in [20]. The sum on the right hand side (r.h.s.) of this equation in principle involves an infinite number of terms, but can be consistently truncated to any finite level (by setting $P^{(\ell)} = 0$ for $\ell > \ell_0$). The bosonic coset constraints, on the other hand, as they follow from supergravity, take the form (for $\ell \geq 3$)

$$\mathcal{C}^{(\ell)} = \sum_{m=0}^{\ell} P^{(m)} * P^{(\ell-m)} \approx 0. \tag{1.2}$$

when expressed in terms of coset variables, and hence only contain a finite number of terms on the r.h.s.. The scalar constraint $\mathcal{C}^{(0)}$, corresponding to the Hamiltonian constraint of the gravity system, plays a special rôle, as it is currently taken to be a $K(E_{10})$ singlet, whereas one would expect the remaining constraints to transform in some representation of $K(E_{10})$. As we will show, the bosonic constraints all follow from the canonical bracket $\{\mathcal{S}_\alpha, \mathcal{S}_\beta\}$ of the supersymmetry constraint. This may obviate the necessity to impose them by introducing extra (bosonic) Lagrange multipliers in the (partially) supersymmetric coset model. The presence of constraints usually signals the presence of gauge symmetries — as is obviously true for the $D = 11$ supergravity constraints —, but their origin is less clear in the present context. The tensor structure of our constraints is reminiscent of the structure of the 'central charge representation' $L(\Lambda_1)$ of E_{11} (which is of highest weight type) first considered in [36], and proposed there to explain the emergence of space-time. By contrast, we here focus on the compact $K(E_{10})$ since the fermionic supersymmetry constraint $\mathcal{S}_\alpha$ can at most be a representation of $K(E_{10})$ and not of E_{10} (see also [37] for a discussion of the link between central charges and hidden symmetries). In addition, preliminary calculations indicate that our bosonic constraints $\mathcal{C}$ do not properly transform as an E_{10} representation.

In summary, the correspondence of table 1 works beautifully, but only up to a point. The correspondence between supergravity and the E_{10} model, as presently known, requires a truncation, where, on the supergravity side, one retains only first-order spatial gradients of the bosonic fields while discarding the spatial gradients of the fermionic fields, and where, on the σ-model side, one neglects all bosonic level $\ell > 3$ degrees of freedom, and restricts attention to *unfaithful* spinor representations of $K(E_{10})$. While there is thus perfect agreement of all quantities up to $\ell \leq 2$, and partial agreement at level $\ell = 3$ (and,

as the present work shows, this agreement extends substantially beyond the equations of motion), the following discrepancies appear at level $\ell = 3$:

- the Hamiltonian (scalar) constraint as computed from supergravity, or equivalently (as explicitly shown in section 6) from the canonical bracket of the supersymmetry constraint, differs from the one obtained from the standard bilinear invariant form $(= \langle \mathcal{P} | \mathcal{P} \rangle)$ at $\ell = 3$, cf. eqs. (6.7) and (6.6);

- the supersymmetry constraint (a Dirac-type spinor with 32 real components) fails to transform covariantly under $K(E_{10})$ beyond level $\ell = 2$; likewise, it appears impossible to manufacture an exact $K(E_{10})$ invariant from $\mathcal{P}$, the supersymmetry parameter ϵ (the **32** spinor of $K(E_{10})$) and the 'gravitino' Ψ (the **320** vector spinor of $K(E_{10})$);

- while one would expect the bosonic constraints $\mathcal{C}^{(\ell)}$ (for $\ell \geq 3$) to fit into a multiplet of $K(E_{10})$, we here find that, with the presently known 'dictionary', the constraints studied below transform only partly in a $K(E_{10})$ covariant manner.

The first of these disagreements was already suggested by the fact that the positivity of the E_{10} invariant bilinear form 'away' from the Cartan subalgebra seems difficult to reconcile with the fact that the 'potential' (essentially, minus the scalar curvature of the spatial hypersurface) in the scalar constraint of canonical gravity can become negative [18] (for instance for spatially homogeneous spaces of constant positive curvature). The second and third are more subtle, but may go to the root of the problem we are trying to address, namely the question of how to embed the full higher-dimensional field theory into a one-dimensional σ-model. Indeed, one cannot expect to be able to realize full supersymmetry in a context where there are infinitely many bosonic degrees of freedom, but only finitely many fermionic ones, and this expectation is confirmed by the fact that our model does not admit full $K(E_{10})$ invariance and supersymmetry simultaneously. We offer some speculations on how to solve this problem in the conclusions. The key question is therefore whether (and how) it is possible to extend the known unfaithful finite-dimensional spinor representations of $K(E_{10})$ constructed in [26, 25, 27] to faithful infinite-dimensional ones. The necessity of faithful representations is also suggested by the gradient conjecture of [19] according to which the higher order spatial gradients of the bosonic fields are encoded into certain higher level 'gradient representations' of the infinite-dimensional Lie algebra. The finite-dimensional unfaithful spinor representation obviously does not allow for an analogous conjecture; in order to accommodate spatial dependence, one evidently needs infinitely many fermionic components as well.

The main purpose of this paper is thus two-fold: (i) to give a detailed account of the agreements between supergravity and the one-dimensional $E_{10}/K(E_{10})$ σ-model so far established, in particular concerning the fermionic sector; and (ii) to exhibit in as clear as possible a fashion the remaining discrepancies that need to be resolved in order to arrive at a fully compatible description incorporating both supersymmetry as well as full E_{10} and $K(E_{10})$ symmetry.

This paper is structured as follows. In section 2, we study the structure and representation theory of $K(E_{10})$ in purely mathematical terms, emphasizing notably the existence

and structure of *ideals* of Lie($K(E_{10})$). The $D = 11$ supergravity equations and variations are rewritten in redefined variables in section 3. In section 4, we present a fermionic extension of the one-dimensional $E_{10}/K(E_{10})$ coset model and derive its basic equations of motion and constraints. In section 5 we establish a (partial) correspondence with the coset equations of section 4 and the supergravity equations of section 3. The canonical structure of the constraints are studied in section 6. Concluding remarks are offered in section 7. Appendix A contains a number of conventions used for $D = 11$ supergravity and, appendix B a proof of a theorem on the consistency of unfaithful representations stated in section 2.

2. Structure and representations of $K(E_{10})$

Let us briefly recall the definition of the hyperbolic Kac-Moody algebra $\mathfrak{e}_{10}$ via the Chevalley-Serre presentation (see [38] for further details). The basic data are the set of generators $\{(e_i, f_i, h_i) \,|\, i = 1, \ldots, 10\}$ and the E_{10} Cartan matrix a_{ij} corresponding to the Dynkin diagram in figure 1, which also displays our numbering conventions for the simple roots. These generators are subject to the defining relations

$$
\begin{aligned}
[h_i, e_j] &= a_{ij} e_j \ , \\
[h_i, f_j] &= -a_{ij} f_j \ , \\
[e_i, f_j] &= \delta_{ij} h_j \ ,
\end{aligned}
\tag{2.1}
$$

where h_i span a Cartan subalgebra: $[h_i, h_j] = 0$. In addition, the generators obey the multilinear Serre relations

$$
\begin{aligned}
(\operatorname{ad} e_i)^{1-a_{ij}} e_j &= 0, \\
(\operatorname{ad} f_i)^{1-a_{ij}} f_j &= 0.
\end{aligned}
\tag{2.2}
$$

These are the relations which have to be imposed on the free Lie algebras generated by the e_i (for the positive, strictly upper triangular half of $\mathfrak{e}_{10}$) and the f_i (for the negative, strictly lower triangular half of $\mathfrak{e}_{10}$) in order to obtain the Kac-Moody algebra $\mathfrak{e}_{10}$.

By definition, the maximal compact subgroup $K(E_{10})$ is the subgroup of E_{10} whose Lie algebra consists of the fixed point set under the Chevalley involution θ (such subalgebras are also referred to as 'involutory subalgebras'). The latter is defined to act by

$$
\theta(e_i) = -f_i, \quad \theta(f_i) = -e_i, \quad \theta(h_i) = -h_i \qquad (i = 1, \ldots, 10)
\tag{2.3}
$$

on the simple Chevalley generators of E_{10}, and extends to all of $\mathfrak{e}_{10}$ by the invariance property $\theta([x, y]) = [\theta(x), \theta(y)]$. The associated invariant subalgebra will be designated by

$$
\mathfrak{k} \equiv \mathfrak{k}\mathfrak{e}_{10} \equiv \operatorname{Lie}(K(E_{10})).
\tag{2.4}
$$

It is sometimes convenient to define a generalized 'transposition' by
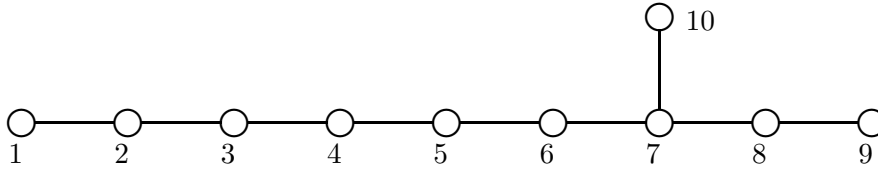
$$
x^T := -\theta(x).
\tag{2.5}
$$

Figure 1: Dynkin diagram of E_{10} with numbering of nodes.

In terms of this transposition, the Lie algebra $\mathfrak{k}$ consists of all 'antisymmetric' elements of $\mathfrak{e}_{10}$. Similarly, the group $K(E_{10})$ consists of all the 'orthogonal' elements of $k \in E_{10}$: $k\,k^T = k^T\,k = \mathbf{1}$. Lastly, as we shall see in more detail below, the rank of $\mathfrak{k}$ is *nine*,[2] i.e. strictly smaller than the rank (ten) of E_{10}. This contrasts with the finite-dimensional exceptional hidden symmetry groups, for which $rank\, E_n = rank\, K(E_n)$ [2] (for $n = 6, 7, 8$).

2.1 $K(E_{10})$ at low levels

In order to determine the structure of the fixed point set under θ we first require some results about the structure of the Lie algebra $\mathfrak{e}_{10}$. The low level structure of the latter can be conveniently described in terms of a 'level decomposition' in terms of irreducible representations of the $\mathrm{SL}(10) \equiv A_9$ subgroup of E_{10}; see e.g. [20] for our conventions and the relevant low level commutators, and [39] for a table of higher level representations. The generators for A_9 levels $\ell = 0, \ldots, 3$ are[3]

$$
\begin{array}{c|c|c|c}
\ell = 0 & \ell = 1 & \ell = 2 & \ell = 3 \\
\hline
K^a{}_b & E^{a_1 a_2 a_3} & E^{a_1 \ldots a_6} & E^{a_0 | a_1 \ldots a_8}
\end{array}
\tag{2.6}
$$

where small latin indices from the beginning of the alphabet take the values $1, \ldots, 10$ and are to be thought of as *flat* indices w.r.t. the spatial Lorentz group $\mathrm{SO}(10)$. The simple positive (raising) generators in terms of (2.6) are

$$
e_a = K^a{}_{a+1} \quad \text{(for } a = 1, \ldots, 9) \quad , \qquad e_{10} = E^{8\,9\,10}. \tag{2.7}
$$

Similarly for the simple negative (lowering) generators $f_a = K^{a+1}{}_a$, $f_{10} = F_{8\,9\,10}$, where $F_{a_1 a_2 a_3}$, of level $\ell = -1$, is a transposed generator introduced below.

The level-0 elements $K^a{}_b$ generate the general linear group $\mathrm{GL}(10)$, the rigid subgroup of the group of purely spatial diffeomorphisms acting on the spatial slices in eleven spacetime dimensions. The trace generator $K \equiv \sum_{a=1}^{10} K^a{}_a$ here arises because the exceptional Cartan generator h_{10} is included in the level $\ell = 0$ sector, in addition to the nine traceless $\mathrm{SL}(10)$ generators h_i $(i = 1, \ldots, 9)$. The remaining tensors are irreducible $\mathrm{SL}(10)$ tensors with defining symmetries (using (anti-)symmetrizers of strength one)

$$
E^{a_1 a_2 a_3} = E^{[a_1 a_2 a_3]},
$$

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[2]V. Kac, private communication.

[3]The corresponding low level representations for the finite-dimensional E_n groups were already given in [6], and for E_{11} in [40].

$$E^{a_1...a_6} = E^{[a_1...a_6]},$$
$$E^{a_0|a_1...a_8} = E^{a_0|[a_1...a_8]}, \qquad E^{[a_0|a_1...a_8]} = 0. \tag{2.8}$$

Under the Chevalley involution we have

$$K^a{}_b = (K^b{}_a)^T := -\theta(K^b{}_a). \tag{2.9}$$

The remaining negative level $\ell \geq -3$ generators are given by

$$F_{a_1a_2a_3} := (E^{a_1a_2a_3})^T := -\theta(E^{a_1a_2a_3}),$$
$$F_{a_1...a_6} := (E^{a_1...a_6})^T := -\theta(E^{a_1...a_6}),$$
$$F_{a_0|a_1...a_8} := (E^{a_0|a_1...a_8})^T := -\theta(E^{a_0|a_1...a_8}). \tag{2.10}$$

The generators of $K(E_{10})$, will always be normalized as the anti-symmetric combinations $J = E - F$.[4] Explicitly, we have, up to $\ell = 3$, and putting a 'level' subscript on the generators,

$$J_{(0)}^{ab} = K^a{}_b - K^b{}_a,$$
$$J_{(1)}^{a_1a_2a_3} = E^{a_1a_2a_3} - F_{a_1a_2a_3},$$
$$J_{(2)}^{a_1...a_6} = E^{a_1...a_6} - F_{a_1...a_6},$$
$$J_{(3)}^{a_0|a_1...a_8} = E^{a_0|a_1...a_8} - F_{a_0|a_1...a_8}. \tag{2.11}$$

Similarly, we define the 'symmetric' elements

$$S_{(0)}^{ab} = K^a{}_b + K^b{}_a,$$
$$S_{(1)}^{a_1a_2a_3} = E^{a_1a_2a_3} + F_{a_1a_2a_3},$$
$$S_{(2)}^{a_1...a_6} = E^{a_1...a_6} + F_{a_1...a_6},$$
$$S_{(3)}^{a_0|a_1...a_8} = E^{a_0|a_1...a_8} + F_{a_0|a_1...a_8}, \tag{2.12}$$

which span the level $\ell \leq 3$ sector of the algebra coset space $\mathfrak{e}_{10} \ominus \mathfrak{k}$. This coset forms an infinite-dimensional representation of $\mathfrak{k}$. With regard to its SL(10) representation content, it differs from $\mathfrak{k}$ only in the level $\ell = 0$ sector, and 'outnumbers' $\mathfrak{k}$ only by the ten Cartan subalgebra generators. (For this reason the split real form is sometimes denoted as $E_{10(10)}$.)

From the commutation relations given in [20] we deduce

$$\left[J_{(0)}^{ab}, J_{(0)}^{cd} \right] = \delta^{bc} J_{(0)}^{ad} + \delta^{ad} J_{(0)}^{bc} - \delta^{ac} J_{(0)}^{bd} - \delta^{bd} J_{(0)}^{ac} \equiv 4\delta^{bc} J_{(0)}^{ad}$$

$$\left[J_{(1)}^{a_1a_2a_3}, J_{(1)}^{b_1b_2b_3} \right] = J_{(2)}^{a_1a_2a_3b_1b_2b_3} - 18\delta^{a_1b_1}\delta^{a_2b_2} J_{(0)}^{a_3b_3}$$

$$\left[J_{(1)}^{a_1a_2a_3}, J_{(2)}^{b_1...b_6} \right] = J_{(3)}^{[a_1|a_2a_3]b_1...b_6} - 5!\,\delta^{a_1b_1}\delta^{a_2b_2}\delta^{a_3b_3} J_{(1)}^{b_4b_5b_6}$$

$$\left[J_{(2)}^{a_1...a_6}, J_{(2)}^{b_1...b_6} \right] = -6 \cdot 6!\,\delta^{a_1b_1} \cdots \delta^{a_5b_5} J_{(0)}^{a_6b_6} + \cdots$$

$$\left[J_{(1)}^{a_1a_2a_3}, J_{(3)}^{b_0|b_1...b_8} \right] = -336 \left(\delta^{b_0b_1b_2}_{a_1a_2a_3} J_{(2)}^{b_3...b_8} - \delta^{b_1b_2b_3}_{a_1a_2a_3} J_{(2)}^{b_4...b_8b_0} \right) + \cdots$$

---

[4] This convention differs by a factor 2 from [20], where $J = \frac{1}{2}(E - F)$ for $\ell \geq 1$.

$$\left[J_{(2)}^{a_1...a_6}, J_{(3)}^{b_0|b_1...b_8} \right] = -8! \left(\delta_{a_1...a_6}^{b_0 b_1...b_5} J_{(1)}^{b_6 b_7 b_8} - \delta_{a_1...a_6}^{b_1...b_6} J_{(1)}^{b_7 b_8 b_0} \right) + \cdots$$

$$\left[J_{(3)}^{a_0|a_1...a_8}, J_{(3)}^{b_0|b_1...b_8} \right] = -8 \cdot 8! \left(\delta_{b_1...b_8}^{a_1...a_8} J_{(0)}^{a_0 b_0} - \delta_{b_0 b_1...b_7}^{a_1...a_8} J_{(0)}^{a_0 b_8} - \delta_{b_1...b_8}^{a_0 a_1...a_7} J_{(0)}^{a_8 b_0} \right.$$
$$\left. + 8 \delta_{b_0}^{a_0} \delta_{b_1...b_7}^{a_1...a_7} J_{(0)}^{a_8 b_8} + 7 \delta_{b_0}^{a_1} \delta_{b_1...b_7}^{a_0 a_2...a_7} J_{(0)}^{a_8 b_8} \right) + \cdots . \qquad (2.13)$$

Here, and throughout this paper, we will make use of the following convention in writing these equations: *there is always an implicit anti-symmetrization (with unit weight) on the r.h.s. according to the anti-symmetries of the l.h.s.* — as exemplified in the first equation in (2.13). Under the SO(10) generators $J_{(0)}^{ab}$ all other generators rotate in the standard fashion. The ellipses denote contributions from higher level generators not computed here. From the above relations, it is straightforward to check that the following nine (mutually commuting) elements provide a basis of a Cartan subalgebra of $\mathfrak{k}$

$$J_{(0)}^{34} , \ J_{(0)}^{56} , \ J_{(0)}^{78} , \ J_{(0)}^{9\,10} , \ J_{(2)}^{345678} , \ J_{(2)}^{34569\,10} , \ J_{(2)}^{34789\,10} , \ J_{(2)}^{56789\,10} , \ J_{(0)}^{12} . \qquad (2.14)$$

To see that the level-two elements $J_{(2)}$ in this list cannot generate any level-four elements $J_{(4)}$ (and hence no higher level elements either), one simply observes that the first eight of these elements by themselves constitute a basis of a Cartan subalgebra of SO(16) $\subset E_{8(8)}$.

From the second line in (2.13) we also see that the commutator of two 'level one' generators is schematically $[J_{(1)}, J_{(1)}] = J_{(2)} + J_{(0)}$; consequently, $K(E_{10})$ does not inherit the graded structure present in a level decomposition of $\mathfrak{e}_{10}$. We can nevertheless decompose the algebra $\mathfrak{k}$ according to

$$\mathfrak{k} = \bigoplus_{\ell=0}^{\infty} \mathfrak{k}_{(\ell)} \qquad (2.15)$$

where $\mathfrak{k}_{(0)} \equiv \mathfrak{so}(10)$, and $\mathfrak{k}_{(\ell)}$ is the linear span of all antisymmetric elements in $\mathfrak{e}_{10}^{(\ell)} \oplus \mathfrak{e}_{10}^{(-\ell)}$ (for $\ell \geq 1$). We will thus continue to refer to (2.15) as a 'level decomposition', always keeping in mind that the term 'level' here is to be taken *cum grano salis*. The general structure on $\mathfrak{k}$ is then

$$[\mathfrak{k}_{(\ell)}, \mathfrak{k}_{(\ell')}] \subset \mathfrak{k}_{(\ell+\ell')} \oplus \mathfrak{k}_{(|\ell-\ell'|)}. \qquad (2.16)$$

Hence, $\mathfrak{k}$ is not a Kac-Moody algebra [29], nor is it an integer graded Lie algebra, since, according to (2.15) commutators 'go up *and* down in level'. Note, however, the fact, apparent in eq. (2.16), that the commutators of $\mathfrak{k}$ have a 'filtered structure', i.e. that they are 'graded modulo lower level contributions'.

The first line in (2.13) is the standard $\mathfrak{so}(10)$ algebra. With the transition from E_{10} to the compact subgroup $K(E_{10})$, the SL(10) tensors appearing in the level decomposition now become tensors of its compact subgroup SO(10) = $K($SL(10)$)$ (the subgroup of spatial rotations within the Lorentz group SO(1, 10)), and hence can be reducible in general. The SL(10) irreducible tensor $J_{(3)}^{a_0|a_1...a_8}$ is SO(10) reducible. We decompose it into irreducible pieces $\bar{J}$ and $\hat{J}$ by defining

$$J_{(3)}^{a_0|a_1...a_8} = \bar{J}_{(3)}^{a_0|a_1...a_8} + \frac{8}{3} \delta^{a_0[a_1} \hat{J}_{(3)}^{a_2...a_8]}, \qquad \hat{J}_{(3)}^{a_2...a_8} = \delta_{a_0 a_1} J_{(3)}^{a_0|a_1 a_2...a_8}. \qquad (2.17)$$

The relations of (2.13) for these two generators can also be written separately as

$$\left[J_{(1)}^{a_1 a_2 a_3}, \hat{J}_{(3)}^{b_1 \ldots b_7} \right] = 378\, \delta^{a_1 b_1} \delta^{a_2 b_2} J_{(2)}^{b_3 \ldots b_7 a_3} + \cdots$$

$$\left[J_{(2)}^{a_1 \ldots a_6}, \hat{J}_{(3)}^{b_1 \ldots b_7} \right] = -9 \cdot 7!\, \delta^{a_1 b_1} \cdots \delta^{a_5 b_5} J_{(1)}^{b_6 b_7 a_6} + \cdots$$

$$\left[\hat{J}_{(3)}^{a_1 \ldots a_7}, \hat{J}_{(3)}^{b_1 \ldots b_7} \right] = -21 \cdot 9 \cdot 7!\, \delta^{a_1 b_1} \cdots \delta^{a_6 b_6} J_{(0)}^{a_7 b_7} + \cdots$$

$$\left[J_{(1)}^{a_1 a_2 a_3}, \bar{J}_{(3)}^{b_0 | b_1 \ldots b_8} \right] = -336 \left(\delta_{a_1 a_2 a_3}^{b_0 b_1 b_2} J_{(2)}^{b_3 \ldots b_8} - \delta_{a_1 a_2 a_3}^{b_1 b_2 b_3} J_{(2)}^{b_4 \ldots b_8 b_0} \right)$$
$$\qquad\qquad - 3 \cdot 336\, \delta_{b_0 a_1 a_2}^{b_1 b_2 b_3} J_{(2)}^{b_4 \ldots b_8 a_3} + \cdots$$

$$\left[J_{(2)}^{a_1 \ldots a_6}, \bar{J}_{(3)}^{b_0 | b_1 \ldots b_8} \right] = -8! \left(\delta_{a_1 \ldots a_6}^{b_0 b_1 \ldots b_5} J_{(1)}^{b_6 b_7 b_8} - \delta_{a_1 \ldots a_6}^{b_1 \ldots b_6} J_{(1)}^{b_7 b_8 b_0} \right)$$
$$\qquad\qquad + 3 \cdot 8!\, \delta_{b_0 a_1 \ldots a_5}^{b_1 \ldots b_6} J_{(1)}^{b_7 b_8 a_3} + \cdots$$

$$\left[\bar{J}_{(3)}^{a_0 | a_1 \ldots a_8}, \hat{J}_{(3)}^{b_1 \ldots b_7} \right] = 0 + \cdots$$

$$\left[\bar{J}_{(3)}^{a_0 | a_1 \ldots a_8}, \bar{J}_{(3)}^{b_0 | b_1 \ldots b_8} \right] = -8 \cdot 8! \left(\delta_{b_1 \ldots b_8}^{a_1 \ldots a_8} J_{(0)}^{a_0 b_0} - \delta_{b_0 b_1 \ldots b_7}^{a_1 \ldots a_8} J_{(0)}^{a_0 b_8} - \delta_{b_1 \ldots b_8}^{a_0 a_1 \ldots a_7} J_{(0)}^{a_8 b_0} \right.$$
$$\qquad\qquad \left. + 8\, \delta_{b_0}^{a_0} \delta_{b_1 \ldots b_7}^{a_1 \ldots a_7} J_{(0)}^{a_8 b_8} + 7 \delta_{b_0}^{a_1} \delta_{b_1 \ldots b_7}^{a_0 a_2 \ldots a_7} J_{(0)}^{a_8 b_8} \right)$$
$$\qquad\qquad + 3 \cdot 8 \cdot 7 \cdot 8!\, \delta^{b_0 b_1} \delta_{a_0 b_2 \ldots b_7}^{a_1 \ldots a_7} J_{(0)}^{a_8 b_9} + \cdots \qquad (2.18)$$

Neglecting the mixed representations $\bar{J}_{(3)}^{a_0 | a_1 \ldots a_8}$, the corresponding commutators for $K(E_{11})$ and the fully antisymmetric tensors at levels $\ell \leq 3$ were already given in [36].

2.2 Serre-like relations for $K(E_{10})$

The compact subalgebra $\mathfrak{k}$ admits, thanks to its filtered structure, a 'Chevalley-Serre-like' presentation very similar to the one used to define Kac-Moody algebras [41]. Namely, by transferring the Serre relations (2.2) to the compact subalgebra $\mathfrak{k}$ one arrives at a presentation of $\mathfrak{k}$ as a quotient of a free Lie algebra $\tilde{\mathfrak{k}}$ by some defining relations. More explicitly, let

$$x_i := e_i - f_i, \qquad \mathfrak{k}_1 := \langle x_i : i = 1, \ldots, n \rangle, \qquad (2.19)$$

(where $\langle \ldots \rangle$ denotes the linear span) and let $\tilde{\mathfrak{k}}$ be the free Lie algebra over $\mathfrak{k}_1$. The relations identifying $\mathfrak{k}$ within $\tilde{\mathfrak{k}}$ can now be directly obtained from the standard Serre relations (2.2) and use of the definition (2.19); they read

$$\sum_{m=0}^{1-a_{ij}} C_{ij}^{(m)} (\operatorname{ad} x_i)^m x_j = 0 \qquad (2.20)$$

where the coefficients $C_{ij}^{(m)}$ can be expressed in terms of the Cartan matrix a_{ij}. As shown in [41], the converse is also true: the involutory subalgebra is completely characterized by the relations (2.20) (obviously, there is no analog of the bilinear relations (2.1)).

In the case at hand, that is for $K(E_{10})$, eq. (2.20) yields the following non-trivial Serre-like relation involving the exceptional node (cf. figure 1)

$$\left[x_{10}, [x_{10}, x_7] \right] + x_7 = 0 \qquad (2.21)$$

which in the A_9 basis (2.11), using (2.7), reads

$$\left[J_{(1)}^{8\,9\,10}, \left[J_{(1)}^{8\,9\,10}, J_{(0)}^{7\,8} \right] \right] + J_{(0)}^{7\,8} = 0. \tag{2.22}$$

Remarkably, all other such relations are automatically satisfied if one uses an SO(10) covariant formalism — that is, in order to ensure consistency of a representation we need to verify *only one* relation involving the 'exceptional' node. This simple observation enables us to determine the consistency of any given set of transformation rules of a tentative $K(E_{10})$ representation: supposing that all objects are written in an SO(10) covariant form, all one requires is that relation (2.22) be satisfied on all elements of the representation space. In particular, the consistency is completely 'localized' within the transformation rules under the 'levels' $\ell = 0$ and $\ell = 1$. A formal proof of this statement can be found in appendix B.

This property shows that, if the consistency condition (2.22) is satisfied, the knowledge of the $K(E_{10})$ transformations of 'level' zero and one is sufficient, in principle, to determine *all* the $K(E_{10})$ transformations (see e.g. (2.24) below). As a warning to the reader, however, we note that the analysis of supergravity below gives expressions for more than just the lowest two levels. In such a case, the comparison of the supergravity-derived $\ell \geq 2$ transformation rules with those induced from the abstract mathematical theory will provide further checks of the supergravity/coset correspondence. In some cases we will find disagreement; however, this does not by any means invalidate the reasoning of this section, but rather indicates the need for an appropriate modification of the $E_{10}/K(E_{10})$ model (or of the 'dictionary') we introduce below.

As an application let us consider the two non-faithful representations of $K(E_{10})$ constructed recently [26, 25, 27] (whose realization in the context of the supersymmetric $E_{10}/K(E_{10})$ model will be discussed in much more detail in the following sections). There, the following Dirac-spinor transformation rules at levels $\ell = 0, 1$ were deduced:

$$J_{(0)}^{ab} \cdot \epsilon = \frac{1}{2} \Gamma^{ab} \epsilon \quad , \qquad J_{(1)}^{abc} \cdot \epsilon = \frac{1}{2} \Gamma^{abc} \epsilon. \tag{2.23}$$

Here, ϵ is a 32-dimensional spinor which, as an SO(10) representation, is a 32-component Majorana spinor of SO(10). Now it is straightforward to check from (2.26), by using Γ algebra, that the sufficient consistency condition (2.22) is satisfied. Therefore, there is a unique consistent way of extending the above transformation rules to all of $K(E_{10})$. The higher-level transformations are then *defined* through the filtered algebra structure. For example, the 'level' $\ell = 2$ transformation must be (cf. the second equation in (2.13))

$$J_{(2)}^{a_1 \ldots a_6} \cdot \epsilon := \left(\left[J_{(1)}^{a_1 a_2 a_3}, J_{(1)}^{a_4 a_5 a_6} \right] + 18 \delta^{a_1 a_4} \delta^{a_2 a_5} J_{(0)}^{a_3 a_6} \right) \cdot \epsilon = \frac{1}{2} \Gamma^{a_1 \ldots a_6} \epsilon. \tag{2.24}$$

Continuing in this manner, we obtain, at the next level,

$$\hat{J}_{(3)}^{a_1 \ldots a_7} \cdot \epsilon = \frac{9}{2} \Gamma^{a_1 \ldots a_7} \epsilon \quad , \qquad \bar{J}_{(3)}^{a_0 | a_1 \ldots a_8} \cdot \epsilon = 0 \tag{2.25}$$

The latter relation shows in particular, that the mixed symmetry generator $\bar{J}_{(3)}^{a_0 | a_1 \ldots a_8}$ is trivially represented on the Dirac spinor — in agreement with the fact that one can only

build fully anti-symmetric tensors from Γ-matrices. By repeating (2.24), we could now in principle work out the action of all $K(E_{10})$ elements on ϵ. Going up in level in this way, there will be an (exponentially) increasing number of $K(E_{10})$ elements that are represented trivially like $\bar{J}_{(3)}^{a_0|a_1\dots a_8}$, see also section 2.4.

Similarly, a 'vector-spinor' representation was defined in [25, 27] with $\ell = 0, 1$ transformation rules given by

$$\left(J_{(0)}^{ab} \cdot \Psi\right)_c = \frac{1}{2}\Gamma^{ab}\Psi_c + 2\delta_c^{[a}\Psi^{b]},$$
$$\left(J_{(1)}^{abc} \cdot \Psi\right)_d = \frac{1}{2}\Gamma^{abc}\Psi_d + 4\delta_d^{[a}\Gamma^b\Psi^{c]} - \Gamma_d{}^{[ab}\Psi^{c]}. \tag{2.26}$$

Here, Ψ_a is a 320-dimensional object which, as an SO(10) representation, is the tensor product of the 32-component Majorana spinor of SO(10) with the 10-component vector. In [25] we wrote these transformations using

$$J_\Lambda^{(0)} = \frac{1}{2}\Lambda_{ab}^{(0)} J_{(0)}^{ab}, \qquad J_\Lambda^{(1)} = \frac{1}{3!}\Lambda_{a_1a_2a_3}^{(1)} J_{(1)}^{a_1a_2a_3}, \quad \text{etc.} \tag{2.27}$$

The representation (2.26) will be discussed in more detail below, so let us just record here that it is again straight-forward to check from (2.26) that the consistency condition (2.22) is satisfied. Let us also note that, as an SO(10) representation, Ψ^a is not irreducible since one can isolate a Γ-trace. However, it is irreducible as a representation of $K(E_{10})$: assuming $\Gamma^d\Psi_d = 0$, we find

$$\Gamma^d\left(J_{(1)}^{abc} \cdot \Psi\right)_d = -\Gamma^{[ab}\Psi^{c]} \neq 0 \tag{2.28}$$

whence Γ-tracelessness is not preserved by the level one transformation of $\mathfrak{k}$. For arbitrary spatial dimension Δ, the corresponding result is[5]

$$\Gamma^d\left(J_{(1)}^{abc} \cdot \Psi\right)_d = (9 - \Delta)\Gamma^{[ab}\Psi^{c]} \tag{2.29}$$

Thus, the removal of a Γ-trace is only possible for $\Delta = 9$ (that is, $K(E_9)$), in agreement with the results of [30].

The transformation rules for Ψ_a on 'levels' two and three implied by (2.26) are

$$\left(J_{(2)}^{a_1\dots a_6} \cdot \Psi\right)_b = \frac{1}{2}\Gamma^{a_1\dots a_6}\Psi_b - 10\delta_b^{[a_1}\Gamma^{a_2\dots a_5}\Psi^{a_6]} + 4\Gamma_b{}^{[a_1\dots a_5}\Psi^{a_6]},$$
$$\left(J_{(3)}^{a_0|a_1\dots a_8} \cdot \Psi\right)_b = +\frac{16}{3}\left(\Gamma_b{}^{a_1\dots a_8}\Psi^{a_0} - \Gamma_b{}^{a_0[a_1\dots a_7}\Psi^{a_8]}\right) + 12\delta^{a_0[a_1}\Gamma^{a_2\dots a_8]}\Psi_b$$
$$-168\delta^{a_0[a_1}\Gamma_b{}^{a_2\dots a_7}\Psi^{a_8]} - \frac{16}{3}\Big(-8\delta_b^{a_0}\Gamma^{[a_1\dots a_7}\Psi^{a_8]}$$
$$+\delta_b^{[a_1}\Gamma^{a_2\dots a_8]}\Psi^{a_0} - 7\delta_b^{[a_1}\Gamma_{a_0}{}^{a_2\dots a_7}\Psi^{a_8]}\Big). \tag{2.30}$$

When contracted with a transformation parameter $\Lambda_{a_0|a_1\dots a_8}^{(3)}$ the last relation simplifies to

$$\left(\frac{1}{9!}\Lambda_{a_0|a_1\dots a_8}^{(3)} J_{(3)}^{a_0|a_1\dots a_8} \cdot \Psi\right)_b = +\frac{2}{3 \cdot 8!}\Big(\Lambda_{a_0|a_1\dots a_8}^{(3)}\Gamma_b{}^{a_1\dots a_8}\Psi^{a_0} \tag{2.31}$$

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[5]By contrast, checking the consistency relation (2.22) for 'spinor' or 'vector-spinor' representations does not depend on the spatial dimension Δ; see also [28].

$$+8\Lambda^{(3)}_{b|a_1...a_8}\Gamma^{a_1...a_7}\Psi^{a_8} + 2\Lambda^{(3)}_{c|ca_1...a_7}\Gamma^{a_1...a_7}\Psi_b - 28\Lambda^{(3)}_{c|ca_1...a_7}\Gamma_b{}^{a_1...a_6}\Psi^{a_7}\Big).$$

in agreement with the formulas given in [25, 27].

As shown in [28], by restricting the action of $K(E_{10})$ via the formulas (2.23) and (2.26) to its subgroups appropriate to IIA and IIB supergravity, respectively, one sees that the unfaithful **32** and **320** representations of $K(E_{10})$ give rise simultaneously to both the (vectorlike) IIA and (chiral) IIB spinors.

2.3 Invariant bilinear forms for $K(E_{10})$ spinors

It is possible to define invariant symmetric bilinear forms on the unfaithful **32** and **320** spinor representations of $K(E_{10})$ that can be used to define an action for these fields. We denote these by $(\cdot|\cdot)_s$ for the **32** Dirac-spinor and by $(\cdot|\cdot)_{vs}$ for the **320** vector-spinor.[6] These forms are defined by

$$(\varphi|\chi)_s = \varphi^T\chi \tag{2.32}$$

for Dirac-spinors φ, χ, and by

$$(\Psi|\Phi)_{vs} = \Psi_a^T\Gamma^{ab}\Phi_b \tag{2.33}$$

for vector-spinors $\Psi = (\Psi_a)$ and $\Phi = (\Phi_a)$.

These expressions are known to be invariant under Lorentz transformations, but we also need to verify invariance under the additional $K(E_{10})$ transformations on higher levels. The general relation to check is

$$(x \cdot \varphi|\chi)_s + (\varphi|x \cdot \chi)_s = 0 \tag{2.34}$$

for all $x \in \mathfrak{k}$, and for all φ and χ in the representation space, and similarly for the vector-spinor. Due to the filtered structure of $\mathfrak{k}$ (and the associated recursive definition of the representations) it is sufficient to verify invariance under a transformation with the level-one generator $J^{a_1a_2a_3}_{(1)}$. The invariance of the Dirac-spinor invariant form follows immediately from the fact that $J^{a_1a_2a_3}_{(1)}$ is represented as $\frac{1}{2}\Gamma^{a_1a_2a_3}$ (see eq. (2.23)) and that this is an *anti-symmetric* Γ-matrix, cf. appendix A, like all the other matrices listed in (2.24) and (2.25).

For the vector-spinor the condition (2.34) for $J^{a_1a_2a_3}_{(1)}$ reduces to the evaluation of the expression

$$\left[\Gamma^{cd}, \Gamma^{a_1a_2a_3}\right] + 8\delta^{da_3}\Gamma^{ca_1}\Gamma^{a_2} - 2\delta^{da_3}\Gamma^{cb}\Gamma_b{}^{a_1a_2} - 8\delta^{ca_3}\Gamma^{a_1}\Gamma^{a_2d} + 2\delta^{ca_3}\Gamma_b{}^{a_1a_2}\Gamma^{bd}$$

which needs to vanish for all a_1, a_2, a_3, c, d. (Here, anti-symmetrization over (a_1, a_2, a_3) is implicit.) Doing the calculation in order to check (2.34) shows that *the invariance is dimension dependent*. Whereas (2.33) defines an invariant form on vector spinors for any $SO(\Delta)$, the transformations (2.26) under $J^{c_1c_2c_3}_{(1)}$ is compatible with (2.34) only if $\Delta = 10$, which is the only value for which the above combination vanishes.[7]

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[6]These forms become *anti*-symmetric when evaluated on *anti*-commuting Grassmann variables; in particular, $(\varphi|\varphi)_s = 0$ for anti-commuting fermions φ.

[7]Note that $\Delta = 10$ corresponds to a *one-dimensional* (reduced) dynamics, so that the existence of an action in this case is in agreement with the folklore that invariant actions for hidden symmetries only exist in *odd* space-time dimensions.

2.4 Ideals of $K(E_{10})$

Both the representations (2.23) and (2.26) are *finite-dimensional* representations of the infinite-dimensional algebra $\mathfrak{k}$. Therefore they are necessarily *unfaithful*: this means that there exist generators (or combinations of generators) of $K(E_{10})$ which are mapped to the zero matrix acting on the representation space. The existence of unfaithful representations has a number of consequences which we now discuss. Most importantly, $\mathfrak{k}$ is *not simple* (in the sense of the classification of Lie algebras). The hidden information about $\mathfrak{k}$ (and $\mathfrak{e}_{10}$ itself!) implicit in this result remains to be exploited for its full worth.

Let V be an (unfaithful) representation space of $\mathfrak{k}$ and define the following subset of $\mathfrak{k}$

$$\mathfrak{i}_V := \{ x \in \mathfrak{k} : x \cdot v = 0 \quad \text{for all } v \in V \}, \tag{2.35}$$

i.e. the kernel of the representation map $\rho_V : \mathfrak{k} \rightarrow End(V)$. It is easily checked that the space $\mathfrak{i}_V$ is an *ideal* (under the Lie bracket) of $\mathfrak{k}$. By definition, a representation V is unfaithful if $\mathfrak{i}_V \neq \{0\}$, and the existence of the unfaithful representations above implies the existence of non-trivial ideals in $\mathfrak{k}$. In technical terms, this means that $\mathfrak{k}$ is not simple. The Kac-Moody algebra $\mathfrak{e}_{10}$, by contrast, *is* simple (since its Dynkin diagram is connected).[8] The existence of an ideal (for example implied by an unfaithful representation) also provides us with a new, usually infinite-dimensional representation of $\mathfrak{k}$, namely the ideal $\mathfrak{i}_V$ itself.

Given two ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of $\mathfrak{k}$ one can form new ideals of $\mathfrak{k}$ in a number of ways: the direct sum $\mathfrak{i}_1 \oplus \mathfrak{i}_2$, the commutator $[\mathfrak{i}_1, \mathfrak{i}_2]$, the intersection $\mathfrak{i}_1 \cap \mathfrak{i}_2$ and the quotient $\mathfrak{i}_1 : \mathfrak{i}_2$ are all ideals of $\mathfrak{k}$. In addition, using the (invariant) symmetric bilinear form of $\mathfrak{k}$, it is easily checked that the orthogonal complement $\mathfrak{i}^{\perp}$ of any ideal $\mathfrak{i}$ is a new ideal. These constructions of new ideals are, however, not linked in any obvious way to operations on unfaithful representations. We should also stress that there is no one-to-one correspondence between ideals and unfaithful representations. An important question we leave unanswered here is what the maximal solvable ideal (a.k.a. the radical) of $\mathfrak{k}$ is — its associated quotient would describe the 'semi-simple part' of $\mathfrak{k}$.

For the case of the unfaithful Dirac-spinor representation defined in (2.23), the ideal $\mathfrak{i}_V \equiv \mathfrak{i}_{\text{Dirac}}$ has the following (schematic) structure. As already noted below (2.25), the generator $\bar{J}_{(3)}^{a_0|a_1 \ldots a_8}$ is represented trivially on the Dirac-spinor. Hence,

$$\mathfrak{i}_{\text{Dirac}} = \left\langle \bar{J}_{(3)}^{a_0|a_1 \ldots a_8}, \ldots \right\rangle = G_{(3)} \oplus G_{(4)} \oplus \ldots \tag{2.36}$$

where $G_{(\ell)}$ is the linear span of the generators of the ideal at level ℓ. The spaces $G_{(\ell)}$ contain at least all the elements obtained from lower level $G_{(m)}$ by commuting with $K(E_{10})$ generators $J_{(\ell-m)}$ (for $m < \ell$) in all possible ways. For example, one must have

$$\left[J_{(1)}, G_{(3)} \right] \subset G_{(4)}, \qquad \text{etc.} \tag{2.37}$$

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[8]Some finite-dimensional simple Lie algebras for which a similar phenomenon occurs are $\mathfrak{sl}(4)$ with maximal compact subalgebra $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, and $\mathfrak{e}_{5(5)} \equiv \mathfrak{so}(5,5)$ with maximal compact subalgebra $\mathfrak{so}(5) \oplus \mathfrak{so}(5)$ (the latter being the symmetry of maximal supergravity in six dimensions). In both examples the lack of simplicity of the maximal compact subalgebra is reflected in the existence of unfaithful representations on which one of the summands acts trivially (although it is not always the case that an algebra with ideals splits into a *direct* sum). These examples also show that one and the same ideal may be associated with (in fact, infinitely) many unfaithful representations.

We do not know if repeated commutation exhausts all of the ideal. If this were the case, the Dirac ideal would be a *Hauptideal* generated by a certain lowest weight element among $\bar{J}_{(3)}^{a_0|a_1\ldots a_8}$. However, in order to decide this question we would need to know all the relevant structure constants.

On level $\ell = 4$, $K(E_{10})$ contains four different SO(10) representations which we denote by $\bar{J}_{(4)}^{(ab)}$, $\hat{J}_{(4)}$, $\bar{J}_{(4)}^{a_1\ldots a_9|b_1b_2b_3}$ and $\hat{J}_{(4)}^{[a_1\ldots a_6]}$ (as these will appear nowhere else in this paper there is no need to be more specific here). Since only generators with anti-symmetric SO(10) indices can occur in the Γ algebra, all generators which are not anti-symmetric will belong to the ideal (as is also true for all higher levels). The singlet generator $\hat{J}$ is represented by Γ_0, and the anti-symmetric six index generator $\hat{J}_{(4)}^{a_1\ldots a_6}$ is represented by a $\Gamma^{(6)}$-matrix, just like $J_{(2)}^{a_1\ldots a_6}$. Hence, the generators of relations for the Dirac ideal $\mathfrak{i}_{\mathrm{Dirac}}$ on $\ell = 4$ are[9]

$$G_{(4)} = \left\langle \bar{J}_{(4)}^{ab} ,\ \bar{J}_{(4)}^{a_1\ldots a_9|b_1b_2b_3} ,\ (\hat{J}_{(4)}^{a_1\ldots a_6} - J_{(2)}^{a_1\ldots a_6}) \right\rangle , \tag{2.38}$$

and it is clear at least in principle how to continue in this way to determine the higher level sectors $G_{(5)}, \ldots$ of the ideal. However, this inductive procedure involves also computing the $K(E_{10})$ structure constants to higher and higher level. This is a hard problem computationally [42]. At any rate, it seems intuitively clear from these arguments that the existence of non-trivial ideals in $\mathfrak{k}$ hinges very much on the fact that $\mathfrak{k}$ is *not* a graded Lie algebra, that is, on the existence of the second term on the r.h.s. of (2.16).

For any ideal $\mathfrak{i}_V$ we can define the *quotient Lie algebra*

$$\mathfrak{q}_V := \mathfrak{k}/\mathfrak{i}_V, \tag{2.39}$$

which by general arguments is isomorphic (as a Lie algebra) to the image of $\mathfrak{k}$ under the representation map ρ_V

$$\mathfrak{q}_V \cong \mathrm{Im}\rho_V \subset End(V), \tag{2.40}$$

and so is a Lie subalgebra of the Lie algebra of endomorphisms of the representation space V. Associated with the ideal $\mathfrak{i}_V$ is the orthogonal complement

$$\mathfrak{i}_V^\perp = \{x \in \mathfrak{k} \,|\, \langle x|\mathfrak{i}_V \rangle = 0\}. \tag{2.41}$$

If $\mathfrak{k}$ were finite-dimensional, $\mathfrak{i}_V^\perp$ would be a subalgebra of $\mathfrak{k}$, and, in fact, the same as the quotient Lie algebra $\mathfrak{q}_V$. However, in the infinite-dimensional case, the situation is much more subtle, as formally divergent sums may appear. For this reason, the study of orthogonal complements necessitates extra analytic categories (in additional to the purely algebraic ones considered so far). More specifically, one can make the Lie algebra $\mathfrak{k}$ into a Hilbert space by means of the scalar product

$$(x, y) := -\langle x|y \rangle \tag{2.42}$$

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[9]Fixing a convenient normalisation for $\hat{J}_{(4)}^{a_1\ldots a_6}$.

which is the restriction of the 'almost positive' Hermitean form $-\langle x|\theta(y)\rangle$ on $\mathfrak{e}_{10}$ to $\mathfrak{k}$ (cf. section 2.7 in [38]), and positive definite on $\mathfrak{k}$. The Hilbert space $\mathcal{H}$ is then defined as

$$\mathcal{H} := \big\{ x \in \mathfrak{k} \,|\, (x,x) < \infty \big\} \tag{2.43}$$

A study of the affine case[10] now suggests that, for finite-dimensional representation spaces V, the elements of $\mathfrak{i}_V^\perp$ are generically *not* normalisable w.r.t. the norm (2.42), hence do not belong to $\mathcal{H}$. In other words, the orthogonal complement $\mathfrak{i}_V^\perp$ consists of *distributional* objects.

Let us make these abstract statements a little more concrete for the unfaithful Dirac spinor representation. In this case, the image of the representation map appearing in (2.40) coincides with the set of anti-symmetric (32×32)-matrices since all $J_{(0)}$ and $J_{(1)}$ are represented by anti-symmetric Γ-matrices, and hence their commutators are also anti-symmetric 32×32-matrices. By Γ-matrix completeness, the quotient Lie algebra is therefore isomorphic to $\mathfrak{so}(32)$:

$$\mathfrak{k}/\mathfrak{i}_{\mathrm{Dirac}} \cong \mathfrak{so}(32). \tag{2.44}$$

For the reasons already explained above this relation does *not* mean that the quotient algebra $\mathfrak{k}/\mathfrak{i}_{\mathrm{Dirac}}$ can be explicitly written (in finite terms) as an $\mathfrak{so}(32)$ subalgebra of $\mathfrak{k}$ (as would have been the case for a finite-dimensional Lie algebra). Namely, if it were, one would have to identify a set of elements of $\mathfrak{k}$ obeying the $\mathfrak{so}(32)$ commutation relations. This, in turn, would require solving the relations of the ideal, for example relating the 'level zero' element $J_{(0)}^{[ab]}$ to infinitely many other anti-symmetric two index $\mathfrak{so}(10)$ representations contained in $\mathfrak{k}$. Consequently, the resulting expression would be a formally infinite series in $\mathfrak{k}$ elements, such that the commutator of two such elements would not be a priori defined even in the sense of formal power series (in the affine case leading to a product of δ-functions at coincident arguments). However, this does not necessarily preclude the possibility to regularise the divergence in a physically meaningful way, for instance in terms of an 'evaluation map' as in the affine case [30].

For this reason the correct statement is that $\mathfrak{so}(32)$ is contained in $\mathfrak{k}$ as a quotient, but not as a subalgebra. Let us also note that $\mathfrak{so}(32)$ appears here somewhat accidentally, and is in fact not tied to studying $D = 11$ supergravity, nor to the presence of a three-form potential $A_{M_1 M_2 M_3}$ in this theory. Namely, repeating the same analysis for *pure gravity* (governed by AE_{10}) one finds that the unfaithful Dirac-spinor of $K(AE_{10})$ has as its quotient the very same algebra $\mathfrak{so}(32)$. More importantly, the gravitino (vector spinor) **320** anyway does not fit into a linear representation of SO(32). For the unfaithful vector-spinor the corresponding ideal $\mathfrak{i}_{\mathrm{vs}}$ gives rise to a quotient Lie algebra

$$\mathfrak{k}/\mathfrak{i}_{\mathrm{vs}} \subset \mathfrak{gl}(320) \tag{2.45}$$

but we have not determined which one.

The possible physical significance of SO(32) and of SL(32) had already been investigated in previous work. The relevance of SL(32) (or GL(32)) had first been pointed out

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[10]Which we will discuss elsewhere.

in a study of the dynamics of five-branes [43]. The possible role of these groups as 'generalised holonomy groups' was explored in studies of (partially) supersymmetric solutions of $D = 11$ supergravity [44 – 46]. In [47] it was shown that there are global obstructions to implementing SO(32) (or SL(32)) as symmetry groups of M-theory, as these groups do not possess representations that reduce to the required spinor (double-valued) representations w.r.t. the groups of spatial and space-time rotations in eleven dimensions. This problem is altogether avoided here.

3. Reduction of $D = 11$ supergravity

After the mathematical preliminaries we now turn to supergravity and the E_{10} model in order to see how the structures discussed above are realised in supergravity. A related analysis of massive IIA supergravity (for which the relevant subgroup of E_{10} is $D_9 \equiv$ SO(9,9)) had already been carried out in [21].

3.1 Redefinitions and gauge choices

We decompose the elfbein in pseudo-Gaussian gauge as[11]

$$E_M{}^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix}. \tag{3.1}$$

Small Latin indices again run over the spatial directions $a = 1, \ldots, 10$. Curved indices are $M = 0, 1, \ldots, 10$ and $m = 1, \ldots, 10$. As in [20, 21], we define a rescaled lapse n ($= \tilde{N}$ of [18]) by

$$n := N g^{-1/2}, \tag{3.2}$$

where $\sqrt{g} \equiv \det e_m{}^a$. Anticipating on the supergravity-coset dictionary detailed below, we shall identify the rescaled supergravity lapse (3.2) with the coset 'einbein', used to convert flat (V_0) into curved (V_t) one-dimensional coset indices, that is, we set $V_t = nV_0$, etc., where t denotes the time-parameter used in the coset model (see below for more examples). The redefinition (3.2) is also in accord with the rescaling required in Kaluza-Klein theories to convert the reduced action to Einstein frame (although the relevant formula fails for $d = 2$, remarkably, it does work again in $d = 1$). In the remainder, we will also assume that the following trace of the spatial spin connection vanishes

$$\omega_{a\,ab} = 0. \tag{3.3}$$

The redefined supergravity fermions are denoted by small Greek letter ψ_M and ε, and are related as follows to the 'old' fermionic variables of appendix A by

$$\psi_0 = g^{1/4} \left(\psi_0^{(11)} - \Gamma_0 \Gamma^a \psi_a^{(11)} \right),$$
$$\psi_a = g^{1/4} \psi_a^{(11)},$$

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[11]Our signature is mostly plus. All our other conventions are detailed in appendix A.

$$\varepsilon = g^{-1/4}\varepsilon^{(11)}. \tag{3.4}$$

Here, we have re-instated the superscript (11) also used in [25] to denote the standard $D = 11$ fermions (which for ease of notation was suppressed in app. A). Our convention here is such that we use capital letters for $K(E_{10})$ spinors and small letters for redefined $D = 11$ spinors. Therefore the correspondence with the notation in [25] is $\psi_a \equiv \psi_a^{(10)}$.

The one-dimensional gravitino with lower world index t is then

$$\psi_t = n\psi_0 = ng^{1/4}\left(\psi_0^{(11)} - \Gamma_0\Gamma^a\psi_a^{(11)}\right). \tag{3.5}$$

These redefinitions imply for the supersymmetry variation of n (as in [21])

$$\delta n = i\bar\varepsilon\Gamma^0\psi_t\,. \tag{3.6}$$

This is the one-dimensional analog of the standard vielbein variation in supergravity, and shows that the einbein n and the redefined time-component ψ_t of the gravitino in (3.5) are superpartners.

3.2 Fermion Variations

For the variation of the gravitino component ψ_t we find, with all the redefinitions (3.4), and to linear order in the fermions,

$$\delta_\varepsilon\psi_t = \partial_t\varepsilon + \frac{1}{4}N\Omega_{0[ab]}\Gamma^{ab}\varepsilon - \frac{1}{12}NF_{0abc}\Gamma^{abc}\varepsilon + \frac{N}{48}F_{abcd}\Gamma_0\Gamma^{abcd}\varepsilon \tag{3.7}$$
$$-\frac{1}{8}N\Omega_{[abc]}\Gamma_0\Gamma^{abc}\varepsilon + \frac{1}{2}N\Omega_{0a0}\Gamma^a\Gamma^0\varepsilon - N\Gamma_0\Gamma^a\left(\partial_a\varepsilon + \frac{1}{4}g^{-1}\partial_ag\varepsilon\right),$$

where we made use of (3.3), $\partial_a \equiv e_a{}^m\partial_m$ and where

$$N\Omega_{0ab} = e_a{}^m\partial_t e_{bm}, \qquad (\Rightarrow \omega_{ab0} = \Omega_{0(ab)}, \quad \omega_{0ab} = \Omega_{0[ab]})$$
$$N\Omega_{0a0} = e_a{}^m\partial_m N,$$
$$NF_{0abc} = e_a{}^m e_b{}^n e_c{}^p F_{tmnp}. \tag{3.8}$$

In (3.7), we have already grouped the first three 'connection terms' on the r.h.s. in the level order that will be seen to emerge on the σ-model side. We should like to emphasize that no truncations have been made so far, and the above formula is thus still completely equivalent to the original gravitino variation of $D = 11$ supergravity. In particular, it still contains contributions, namely the last two terms in (3.7) involving spatial gradients of the lapse N and the supersymmetry parameter ε, which are not understood so far in the framework of the $E_{10}/K(E_{10})$ σ-model.

For the 'internal' (redefined) gravitino components we obtain

$$\delta_\varepsilon\psi_a = g^{1/2}\left(\partial_a + \frac{1}{4}g^{-1}\partial_ag\right)\varepsilon + N^{-1}g^{1/2}\Big[\frac{1}{2}N\Omega_{0(ab)}\Gamma^b\Gamma^0\varepsilon$$
$$+\frac{1}{8}N(\Omega_{abc} + \Omega_{cab} - \Omega_{bca})\Gamma^{bc}\varepsilon - \frac{1}{36}NF_{0bcd}\Gamma^0\big(\Gamma_a{}^{bcd} - 6\delta_a^b\Gamma^{cd}\big)\varepsilon$$

$$+\frac{1}{144}NF_{bcde}\left(\Gamma_a{}^{bcde}-8\delta_a^b\Gamma^{cde}\right)\varepsilon\bigg]. \tag{3.9}$$

Let us emphasize once more the importance of using *flat* indices in (3.7) and (3.9), as this will facilitate the comparison with the $K(E_{10})$ covariant quantities to be introduced in the following section. Note also that, by virtue of (3.2), the prefactor of the square bracket in (3.9) is simply n^{-1}. This ensures that, when we rewrite these relations in terms of $K(E_{10})$ covariant objects below, it is always the einbein n, rather than N, which appears in the proper places to make (3.9) a world-line scalar (whereas ψ_t is to be regarded as a 'world line vector').

3.3 Fermion equation of motion

In this section we will adopt the supersymmetry gauge

$$\psi_t = 0 \quad \Longleftrightarrow \quad \psi_0^{(11)} = \Gamma_0\Gamma^a\psi_a^{(11)}. \tag{3.10}$$

With this gauge choice, the local supersymmetry manifests itself only via the supersymmetry constraint. This is analogous to the bosonic sector, where after fixing diffeomorphism and gauge invariances, one is left only with the corresponding constraints on the initial data. As is well-known, the supersymmetry constraint is the time component of the Rarita-Schwinger equation (A.24)

$$\tilde{\mathcal{S}} := \mathcal{E}_0 = \Gamma^{ab}\hat{D}_a(\omega, F)\psi_b^{(11)} = 0. \tag{3.11}$$

Writing out this constraint, we obtain

$$\tilde{\mathcal{S}} = \Gamma^{ab}\bigg[\partial_a\psi_b^{(11)} + \frac{1}{4}\omega_{a\,cd}\Gamma^{cd}\psi_b^{(11)} + \omega_{a\,bc}\psi_c^{(11)} + \frac{1}{2}\omega_{a\,c0}\Gamma^c\Gamma^0\psi_b^{(11)}\bigg]$$
$$+\frac{1}{4}F_{0abc}\Gamma^0\Gamma^{ab}\psi_c^{(11)} + \frac{1}{48}F_{abcd}\Gamma^{abcde}\psi_e^{(11)}. \tag{3.12}$$

The terms involving the spatial spin connection can be further simplified upon use of the tracelessness condition (3.3)

$$\Gamma^{ab}\Big(\frac{1}{4}\omega_{a\,cd}\Gamma^{cd}\psi_b^{(11)} + \omega_{a\,bc}\psi_c^{(11)}\Big) = \frac{1}{8}\Omega_{[ab\,c]}\Gamma^{abcd}\psi_d^{(11)} + \frac{1}{4}\Omega_{ab\,c}\Gamma^{ab}\psi_c^{(11)}. \tag{3.13}$$

To write the remaining ten components of (A.24), we define

$$\tilde{\mathcal{E}}_a := \Gamma^0\Gamma^B\big(\hat{D}_a\psi_B^{(11)} - \hat{D}_B\psi_a^{(11)}\big) = 0. \tag{3.14}$$

When working them out we must not forget to replace $\psi_0^{(11)}$ everywhere by $\Gamma_0\Gamma^a\psi_a^{(11)}$ according to (3.10). Switching to the redefined fermionic variables (3.4), and setting $\mathcal{E}_a := Ng^{1/4}\tilde{\mathcal{E}}_a$, the complete expression is (because sums are now with the Euclidean metric δ_{ab} the position of spatial indices does not really matter anymore, so we put them as convenient, whereas the position of '0' does matter)

$$\mathcal{E}_a = \partial_t\psi_a + N\omega_{0\,ab}\psi_b + \frac{1}{4}N\omega_{0\,cd}\Gamma^{cd}\psi_a \tag{3.15}$$

$$-\frac{1}{12}NF_{0bcd}\Gamma^{bcd}\psi_a \ - \ \frac{2}{3}NF_{0abc}\Gamma^b\psi^c \ + \ \frac{1}{6}NF_{0bcd}\Gamma_a{}^{bc}\psi^d$$

$$+\frac{1}{144}NF_{bcde}\Gamma^0\Gamma^{bcde}\psi_a \ + \ \frac{1}{9}NF_{abcd}\Gamma^0\Gamma^{bcde}\psi_e \ - \ \frac{1}{72}NF^{bcde}\Gamma^0\Gamma_{abcdef}\psi_f$$

$$+N(\omega_{abc} - \omega_{bac})\Gamma^0\Gamma^b\psi^c \ + \ \frac{1}{2}N\omega_{abc}\Gamma^0\Gamma^{bcd}\psi_d \ - \ \frac{1}{4}N\omega_{bcd}\Gamma^0\Gamma^{bcd}\psi_a$$

$$+Ng^{1/4}\Gamma^0\Gamma^b\Big(2\partial_a\psi_b^{(11)} - \partial_b\psi_a^{(11)} - \frac{1}{2}\omega_{c\,cb}\psi_a^{(11)} - \omega_{0\,0a}\psi_b^{(11)} + \frac{1}{2}\omega_{0\,0b}\psi_a^{(11)}\Big).$$

Like (3.7) this expression is completely equivalent to the original version, and thus again contains terms not yet accounted for in the $E_{10}/K(E_{10})$ σ-model. More specifically, in the last line we have collected all the terms which are not understood (involving spatial gradients) or can be eliminated by gauge choice (3.3); the factor of 2 in front of $\partial_a\psi_b^{(11)}$ comes from the extra contribution $\propto \partial_a\psi_0^{(11)}$.

In section 5 we will translate equations (3.7), (3.9), (3.12) and (3.15) into $K(E_{10})$ covariant objects as far as possible.

4. E_{10}-model with fermions

We here briefly summarize previous results. The bosonic degrees of freedom of the $E_{10}/K(E_{10})$ σ-model are contained in a 'matrix' $\mathcal{V}(t) \in E_{10}$ depending on an affine (time) parameter t, in terms of which the trajectory in the $E_{10}/K(E_{10})$ coset space is parametrized. The associated $\mathfrak{e}_{10}$-valued Cartan form can be decomposed as

$$\partial_t\mathcal{V}\mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P} \quad , \qquad \mathcal{Q} \in \mathfrak{k} \ , \ \mathcal{P} \in \mathfrak{e}_{10} \ominus \mathfrak{k}. \tag{4.1}$$

Alternatively, we can write this as

$$\mathcal{D}\mathcal{V}\mathcal{V}^{-1} = \mathcal{P} \tag{4.2}$$

where $\mathcal{D}$ denotes the $K(E_{10})$ covariant derivative

$$\mathcal{D} := \partial_t - \mathcal{Q}, \tag{4.3}$$

involving the $K(E_{10})$ connection $\mathcal{Q}$. Making use of the decomposition of E_{10} into antisymmetric and symmetric elements (cf. (2.11) and (2.12)), we write

$$\mathcal{P} = \frac{1}{2}P_{ab}^{(0)}S_{(0)}^{ab} + \frac{1}{3!}P_{abc}^{(1)}S_{(1)}^{abc} + \frac{1}{6!}P_{a_1...a_6}^{(2)}S_{(2)}^{a_1...a_6}$$
$$+ \frac{1}{9!}P_{a_0|a_1...a_8}^{(3)}S_{(3)}^{a_0|a_1...a_8} + \cdots \tag{4.4}$$

and

$$\mathcal{Q} = \frac{1}{2}Q_{ab}^{(0)}J_{(0)}^{ab} + \frac{1}{3!}Q_{abc}^{(1)}J_{(1)}^{abc} + \frac{1}{6!}Q_{a_1...a_6}^{(2)}J_{(2)}^{a_1...a_6}$$
$$+ \frac{1}{9!}Q_{a_0|a_1...a_8}^{(3)}J_{(3)}^{a_0|a_1...a_8} + \cdots . \tag{4.5}$$

Although $\mathfrak{k}$ is not a graded Lie algebra in view of (2.15), we will nevertheless exploit its filtered structure and assign a 'level' to the various terms in the expansion of $\mathcal{Q}$ as above. We also define for later convenience the partially covariantised derivative

$$\mathcal{D}^{(0)} P^{(\ell)} = \partial P^{(\ell)} - \left[Q^{(0)}, P^{(\ell)} \right] - \left[Q^{(\ell)}, P^{(0)} \right]. \tag{4.6}$$

This is the derivative also appearing in table 1. Unless stated otherwise, we will work in the following in (almost) *triangular gauge* for $\mathcal{V}(t) \in E_{10}$; this implies [20]

$$P^{(\ell)} = Q^{(\ell)} \qquad \text{for } \ell > 0. \tag{4.7}$$

Of course, in a general gauge, this relation will no longer hold.

In order to obtain an explicit expression for $\mathcal{Q}$ and $\mathcal{P}$ in terms of coset manifold coordinates, and to write the bosonic equations of motion in the standard second order form, one must, of course, choose an explicit parametrisation $\mathcal{V}(t) = \mathcal{V}\big(h(t), A^{(3)}(t), A^{(6)}(t), \dots\big)$ as was done in [19, 25]. This choice is naturally subject to 'general coordinate transformations' on the coset space, that is, to non-linear field redefinitions of the basic fields (which maintain the triangular gauge). For this reason, the relation between the coset fields appearing in the exponential parametrisation of $\mathcal{V}$ and the ones appearing in supergravity depends on coordinate choices in field space. It is therefore convenient (and entirely sufficient for our purposes) to work only with the $K(E_{10})$ objects $\mathcal{Q}$ and $\mathcal{P}$, and with 'flat' indices, where this coordinate dependence is not visible.

The Lagrangian of the one-dimensional model is assumed to be of the form [25, 27]

$$L = \frac{1}{4n} \langle \mathcal{P} | \mathcal{P} \rangle - \frac{i}{2} (\Psi | \mathcal{D} \Psi)_{\text{vs}} + i n^{-1} (\Psi_t | \mathcal{S})_{\text{s}} \tag{4.8}$$

where $\langle \cdot | \cdot \rangle$ is the invariant bilinear form on the $\mathfrak{e}_{10}$ Lie algebra and $(\cdot | \cdot)$ are the invariant forms on the $K(E_{10})$ spinor representations of section 2.3. The expression $\mathcal{S}$ denotes the (rescaled) supersymmetry constraint and is proportional to $\mathcal{P} \odot \Psi$ which is a certain projection from the tensor product $\mathcal{P} \otimes \Psi$ to a Dirac-spinor representation of $K(E_{10})$, and Ψ_t is a Dirac-spinor Lagrange multiplier. Starting from supergravity we will be more explicit below as to what we can say about this projection and how the supersymmetry constraint $\mathcal{P} \odot \Psi$ can be expressed in terms of $E_{10}/K(E_{10})$ coset variables.

We will also investigate the invariance of the Lagrangian (4.8) under local supersymmetry transformations with transformation parameter ϵ in a Dirac-spinor representation of $K(E_{10})$. Schematically, these will be of the form

$$\begin{aligned} \delta_\epsilon \mathcal{P} &= \mathcal{D} \Sigma + [\Lambda, \mathcal{P}], & \delta_\epsilon n &= i \epsilon^T \Psi_t, \\ \delta_\epsilon \Psi &= \epsilon \odot \mathcal{P}, & \delta_\epsilon \Psi_t &= \mathcal{D} \epsilon, \end{aligned} \tag{4.9}$$

where Σ and Λ are fermion bilinears constructed out of Ψ and ϵ. Note that the 'supersymmetry gauge-fixed' action obtained by imposing $\Psi_t = 0$ is then expected to be invariant under residual 'quasi-rigid' supersymmetry transformations constrained to satisfy $\mathcal{D} \epsilon = 0$.

Since we do not have the expressions for (4.9) to arbitrary levels we cannot present a complete analysis of how local supersymmetry is realised in (4.8) — as will be argued

shortly the action (4.8) for unfaithful fermions will fail to simultaneously possess $K(E_{10})$ symmetry and local supersymmetry.

The equations of motion are obtained by varying (4.8) with respect to $\mathcal{P}$ and Ψ. In the gauge $\Psi_t = 0$ they are to lowest order in fermions

$$\mathcal{D}(n^{-1}\mathcal{P}) = 0 \qquad \Leftrightarrow \qquad n\partial_t(n^{-1}\mathcal{P}) - [\mathcal{Q}, \mathcal{P}] = 0 \qquad (4.10)$$

$$\mathcal{D}\Psi = 0 \qquad \Leftrightarrow \qquad \partial_t\Psi - \mathcal{Q} \cdot \Psi = 0. \qquad (4.11)$$

In the last equation, the $K(E_{10})$ gauge connection $\mathcal{Q}$ acts in the appropriate (here: vector-spinor) representation. For a general representation R we write the $K(E_{10})$ covariant derivative $\mathcal{D}$ as

$$\overset{R}{\mathcal{D}} = \partial_t - \left(\frac{1}{2} Q^{(0)}_{ab} \overset{R}{J}{}^{ab}_{(0)} + \frac{1}{3!} Q^{(1)}_{abc} \overset{R}{J}{}^{abc}_{(1)} + \frac{1}{6!} Q^{(2)}_{a_1 \ldots a_6} \overset{R}{J}{}^{a_1 \ldots a_6}_{(2)} \right.$$
$$\left. + \frac{1}{9!} Q^{(3)}_{a_0|a_1 \ldots a_8} \overset{R}{J}{}^{a_0|a_1 \ldots a_8}_{(3)} + \cdots \right), \qquad (4.12)$$

where $\overset{R}{J}$ is the form a $K(E_{10})$ generator takes in the representation R.

Employing the unfaithful vector-spinor of section 2 in (4.11) we note the (potentially pathological) feature that the components of $\mathcal{Q}$ in the ideal $\mathfrak{i}_{\text{vs}}$ of (2.45) do not couple to the fermionic field Ψ. Hence the unfaithful spinor Ψ couples only to a very restricted subset of the bosonic σ-model degrees of freedom. Similar features can be anticipated when writing down $K(E_{10})$-covariant supersymmetry transformation rules with the unfaithful fields, as will be discussed below.

Varying with respect to the Lagrange multipliers n (ensuring invariance under time reparametrisation) and Ψ_t (hopefully linked to local supersymmetry) gives the constraints of (4.8)

$$\langle \mathcal{P} | \mathcal{P} \rangle = 0 \quad , \qquad \mathcal{P} \odot \Psi = 0. \qquad (4.13)$$

5. Supergravity and the $E_{10}/K(E_{10})$ σ-model

Now we turn to the comparison of the supergravity expressions of section 3 and the σ-model expressions of the preceding section. The method adopted here differs from the one used in previous work in that we shall start from postulating a correspondence *between the fermionic variables*, and deduce from it the correspondence between the bosonic variables (previously, we started from the bosonic equations of motion to derive the supergravity-coset dictionary). Accordingly, we *stipulate* as starting point the following correspondence between the supergravity fermions (3.4) and the unfaithful $K(E_{10})$ spinor representations of section 2 by identifying (in addition to the bosonic identification $(Ng^{-1/2})(t, \mathbf{x}_0) \equiv n_{\text{coset}}(t)$ of eq. (3.2))

$$\psi_a(t, \mathbf{x}_0) = \Psi_a(t) ,$$
$$\psi_t(t, \mathbf{x}_0) = \Psi_t(t)$$
$$\varepsilon(t, \mathbf{x}_0) = \epsilon(t) , \qquad (5.1)$$

with the supergravity objects on the left hand side evaluated at a fixed but arbitrary spatial point $\mathbf{x}_0$, and the $K(E_{10})$ objects on the right hand side. We will truncate systematically spatial frame gradients of the fermionic fields in the supergravity expressions. Proceeding from (5.1), we then infer the bosonic correspondence from an analysis of the supersymmetry variations, employing techniques that had already been successfully used in [31–33].

5.1 Re-derivation of the bosonic 'dictionary'

We first re-derive the bosonic 'dictionary' which accompanies (5.1) by using as input the supersymmetry variation of the eleven-dimensional gravitino ψ_t in (3.7) and comparing it with the expected supersymmetry variation (4.9) of the coset Lagrange multiplier Ψ_t, in which $\mathcal{D}$ denotes the $K(E_{10})$ covariant derivative (4.12) in the unfaithful Dirac-spinor representation (2.23). The basic equation which will allow us to extend the dictionary to the bosonic sector is

$$\delta_\varepsilon \psi_t \overset{!}{=} \delta_\epsilon \Psi_t = \overset{\text{s}}{\mathcal{D}}\epsilon = (\partial_t - \overset{\text{s}}{\mathcal{Q}})\epsilon. \tag{5.2}$$

We can expand the right hand side of (5.2) from (4.12) and (2.23) as

$$\mathcal{D}\epsilon = \partial_t \epsilon - \frac{1}{4} Q^{(0)}_{ab} \Gamma^{ab} \epsilon - \frac{1}{12} Q^{(1)}_{a_1 a_2 a_3} \Gamma^{a_1 a_2 a_3} \epsilon - \frac{1}{2 \cdot 6!} Q^{(2)}_{a_1 \dots a_6} \Gamma^{a_1 \dots a_6} \epsilon$$
$$- \frac{1}{6 \cdot 7!} Q^{(3)}_{b|b a_1 \dots a_7} \Gamma^{a_1 \dots a_7} \epsilon + \cdots \tag{5.3}$$

where the dots stand for higher level contributions. Comparing this expression with (3.7) we can read off the identification ('dictionary')

$$Q^{(0)}_{ab}(t) = -N\omega_{0\,ab}(t, \mathbf{x}_0) = -N\Omega_{0[a\,b]}(t, \mathbf{x}_0),$$
$$Q^{(1)}_{abc}(t) = NF_{0abc}(t, \mathbf{x}_0),$$
$$Q^{(2)}_{a_1 \dots a_6}(t) = -\frac{1}{4!} N\epsilon_{a_1 \dots a_6 b_1 \dots b_4} F_{b_1 \dots b_4}(t, \mathbf{x}_0),$$
$$Q^{(3)}_{b|b a_1 \dots a_7}(t) = -\frac{3}{4} N\epsilon_{a_1 \dots a_7 b_1 b_2 b_3} \Omega_{b_1 b_2\,b_3}(t, \mathbf{x}_0). \tag{5.4}$$

This fixes only the trace part $Q^{(3)}_{b|b a_1 \dots a_7}$ of the level three gauge connection — as was to be expected since $\bar{Q}^{(3)}_{a_0|a_1 \dots a_8}$ is contracted with a generator contained in the ideal of the Dirac representation. However, demanding that the expression for the full $Q^{(3)}$ does not involve this trace separately, leads to[12]

$$Q^{(3)}_{a_0|a_1 \dots a_8}(t) = \frac{3}{4} N\epsilon_{a_1 \dots a_8 b_1 b_2} \Omega_{b_1 b_2\,a_0}(t, \mathbf{x}_0). \tag{5.5}$$

In triangular gauge (4.7) we can now directly infer from this the corresponding dictionary for the $\mathcal{P}$ components

$$P^{(1)}_{abc}(t) = NF_{0abc}(t, \mathbf{x}_0)$$

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[12]Alternatively, the complete result for $Q^{(3)}$ can be deduced by matching the Rarita-Schwinger equation (3.15) with its $K(E_{10})$ covariant form (5.10), see following section.

$$P_{a_1 \ldots a_6}^{(2)}(t) = -\frac{1}{4!} N \epsilon_{a_1 \ldots a_6 b_1 \ldots b_4} F_{b_1 \ldots b_4}(t, \mathbf{x}_0)$$

$$P_{a_0 | a_1 \ldots a_8}^{(3)}(t) = \frac{3}{4} N \epsilon_{a_1 \ldots a_8 b_1 b_2} \Omega_{b_1 b_2 \, a_0}(t, \mathbf{x}_0) \qquad (5.6)$$

The only undetermined piece at this point is $P_{ab}^{(0)}$. Its explicit expression follows either from inspection of (5.8) below and comparison with (3.9), or alternatively from splitting the Cartan form associated with the GL(10) submatrix of the 'unendlichbein' $\mathcal{V}$ into its symmetric and anti-symmetric parts; either way the result is

$$P_{ab}^{(0)}(t) = -N \omega_{(a\,b)0}(t, \mathbf{x}_0) = -e_{(a}{}^m \partial_t e_{mb)}(t, \mathbf{x}_0) \qquad (5.7)$$

The above list is identical to the dictionary derived in [19] (and also [25]) if one follows through all changes in convention.[13] We emphasize again that the present analysis did not involve the bosonic equations of motion, but only the supersymmetry variations and the unfaithful $K(E_{10})$ spinor representations. The correspondence for $n = N g^{-1/2}$ was already motivated in (3.2) for the simple form of the variation (3.6).

It remains to rewrite the variation (3.9) of the 320 gravitino components ψ_a in coset quantities. For this we use (i) the dictionary (5.6), (ii) the unfaithful Dirac-spinor ϵ and (iii) the identification $\psi_a = \Psi_a$ as the correspondence for the 320 components. Putting everything together leads to the following expression for $\delta_\epsilon \Psi_a$:

$$\delta_\epsilon \Psi_a = n^{-1} \Gamma^0 \Bigg[\frac{1}{2} P_{ac}^{(0)} \Gamma^c - \frac{1}{36} P_{c_1 c_2 c_3}^{(1)} \left(\Gamma_a{}^{c_1 c_2 c_3} - 6 \delta_a^{c_1} \Gamma^{c_2 c_3} \right)$$

$$- \frac{1}{3 \cdot 6!} P_{c_1 \ldots c_6}^{(2)} \left(\Gamma_a{}^{c_1 \ldots c_6} - 3 \delta_a^{c_1} \Gamma^{c_2 \ldots c_6} \right) + \frac{3}{9!} P_{a | c_1 \ldots c_8}^{(3)} \Gamma^{c_1 \ldots c_8}$$

$$- \frac{12}{9!} P_{b | b c_1 \ldots c_7}^{(3)} \Gamma_a{}^{c_1 \ldots c_7} \Bigg] \epsilon + g^{1/2} \left(\partial_a \varepsilon + \frac{1}{4} (g^{-1} \partial_a g) \epsilon \right). \qquad (5.8)$$

This expression can be shown to be $K(E_{10})$ covariant for all terms involving $P^{(0)}$ and $P^{(1)}$. $K(E_{10})$ covariance here means that transforming on the r.h.s. ϵ in the representation (2.23) and $\mathcal{P}$ as a coset element results in a vector-spinor transformation (2.26) for $\delta_\epsilon \Psi_a$. Anticipating a fully supersymmetric and $K(E_{10})$ covariant formulation, the above formula has already been written out in (4.9) in the somewhat symbolic form $\delta_\epsilon \Psi = \epsilon \odot \mathcal{P}$.

5.2 $K(E_{10})$ covariant form of RS equation

Like the supersymmetry variations, the fermionic equations of motion can be cast into a $K(E_{10})$ covariant form. More specifically, we would like to rewrite the Rarita-Schwinger equation (3.15) as a $K(E_{10})$ covariant 'Dirac equation' involving the unfaithful fermion representation Ψ_a, according to

$$\mathcal{E}_a \stackrel{!}{=} (\mathcal{D}\Psi)_a \equiv \big((\partial_t - \mathcal{Q}) \Psi \big)_a. \qquad (5.9)$$

---

[13]In comparison with [19] and [25], the relative normalisation is given by $P_{abc}^{(1)\,\mathrm{here}} = \frac{1}{2} D A_{abc}^{\mathrm{coset}} = \frac{1}{2} N F_{0abc}^{\mathrm{DHN}} = N F_{0abc}^{\mathrm{DKN}} = N F_{0abc}^{\mathrm{here}}$.

This ansatz is, of course, motivated by the $K(E_{10})$ covariant spinor equation (4.11); more explicitly, we now proceed from

$$(\overset{\text{vs}}{\mathcal{D}}\Psi)_a = \partial_t\Psi_a - \left(\frac{1}{2}Q_{ab}^{(0)}\overset{\text{vs}}{J}_{(0)}^{ab}\cdot\Psi + \frac{1}{3!}Q_{abc}^{(1)}\overset{\text{vs}}{J}_{(1)}^{abc}\cdot\Psi \right.\tag{5.10}$$
$$\left. +\frac{1}{6!}Q_{a_1...a_6}^{(2)}\overset{\text{vs}}{J}_{(2)}^{a_1...a_6}\cdot\Psi + \frac{1}{9!}Q_{a_0|a_1...a_8}^{(3)}\overset{\text{vs}}{J}_{(3)}^{a_0|a_1...a_8}\cdot\Psi + \cdots\right)_a.$$

where the explicit expressions for the $K(E_{10})$ generators $\overset{\text{vs}}{J}_{(\ell)}$ are to be substituted from eqs. (2.26) and (2.30). The resulting expression must then be compared with (3.15) in order to re-obtain the bosonic dictionary. At this point, (5.9) provides a consistency check on the results we have obtained so far since all bosonic quantities in (3.15) have found corresponding coset partners in the dictionary (5.6), and the form of the unfaithful representation is known from (2.26). Indeed, performing all the required substitutions on the r.h.s. of (5.10), we find complete agreement between $\mathcal{E}_a$ and (5.10), *except* for the terms in the last line of (3.15) which involve spatial gradients of the fermions, the lapse and the trace of the spin connection.

The agreements established at this stage provide a very non-trivial consistency check. To underline this point, let us have a closer look at how this agreement works for the level three terms, as this is the most intricate part of the computation. According to the dictionary (5.4) the level three terms involve the spatial part of the spin connection ω_{abc} in (3.15) which needs to be re-expressed in terms of the anholonomy Ω_{abc}. In order to establish (5.9), we therefore need to match explicitly

$$\Omega_{abc}\Gamma^0\Gamma^b\Psi^c + \frac{1}{2}\Omega_{abc}\Gamma^0\Gamma^{bcd}\Psi_d - \frac{1}{4}\Omega_{bca}\Gamma^0\Gamma^{bcd}\Psi_d - \frac{1}{8}\Omega_{bcd}\Gamma^0\Gamma^{bcd}\Psi_a$$
$$\overset{!}{=} -\left(\frac{1}{9!}Q_{a_0|a_1...a_8}^{(3)}J_{(3)}^{a_0|a_1...a_8}\cdot\Psi\right)_a.\tag{5.11}$$

Using the duality

$$\Omega_{abc} = \frac{2}{3\cdot 8!}N^{-1}\epsilon_{abd_1...d_8}Q_{c|d_1...d_8}^{(3)}\tag{5.12}$$

from the dictionary (5.4) we find that the left hand side of (5.11) becomes

$$= -\frac{2}{3\cdot 8!}\left(Q_{b_0|b_1...b_8}^{(3)}\Gamma_a{}^{b_1...b_8}\Psi^{b_0} + 2Q_{c|cb_1...b_7}^{(3)}\Gamma^{b_1...b_7}\Psi_a\right.$$
$$\left. -28Q_{c|cb_1...b_7}^{(3)}\Gamma_a{}^{b_1...b_6}\Psi^{b_7} + 8Q_{a|b_1...b_8}^{(3)}\Gamma^{b_1...b_7}\Psi^{b_8}\right).\tag{5.13}$$

It is gratifying that equation (5.13) indeed agrees completely with the transformation property (2.31) deduced abstractly from purely algebraic considerations in the unfaithful $K(E_{10})$ vector-spinor representation (with transformation parameter $\Lambda^{(3)}$ replaced by $Q^{(3)}$).

5.3 Supersymmetry constraint

We also rewrite the supersymmetry constraint (3.12) in coset quantities. Defining $\mathcal{S} = Ng^{1/4}\tilde{\mathcal{S}}$ we write

$$\mathcal{S} = \frac{1}{2}\left(P_{ab}^{(0)}\Gamma^0\Gamma^a - P_{cc}^{(0)}\Gamma^0\Gamma_b\right)\Psi^b + \frac{1}{4}P_{c_1c_2c_3}^{(1)}\Gamma^0\Gamma^{c_1c_2}\Psi^{c_3}$$

$$-\frac{1}{2\cdot 5!}P^{(2)}_{c_1\ldots c_6}\Gamma^0\Gamma^{c_1\ldots c_5}\Psi^{c_6} + \frac{1}{6\cdot 6!}P^{(3)}_{b|bc_1\ldots c_7}\Gamma^0\Gamma^{c_1\ldots c_6}\Psi^{c_7}$$

$$-\frac{1}{3\cdot 8!}P^{(3)}_{b|c_1\ldots c_8}\Gamma^0\Gamma^{c_1\ldots c_8}\Psi^b + N\Gamma^{ab}\left(\partial_a - \frac{1}{4}g^{-1}\partial_a g\right)\Psi_b. \tag{5.14}$$

As before, and in analogy with (4.9), we introduce the symbolic notation

$$\mathcal{S} = \mathcal{P}\odot\Psi \tag{5.15}$$

for the supersymmetry constraint expressed in $E_{10}/K(E_{10})$ coset variables.

From the way $\mathcal{S}$ appears in the action (4.8) we would like this expression to transform under $K(E_{10})$ in the same manner as an unfaithful Dirac-spinor. For an infinitesimal level one transformation $J^{a_1a_2a_3}_{(1)}$, we need to compare $\frac{1}{2}\Gamma^{a_1a_2a_3}\mathcal{S}$ with the expression obtained by transforming the $\mathcal{P}$ and Ψ_a symbols in (5.14). Though the terms involving $P^{(0)}$ and $P^{(1)}$ pass this check, we find that (5.14) is *not* fully covariant like a Dirac-spinor. More precisely, in the transformed expression all terms which receive contributions from $P^{(3)}$ do not give the correct result. This happens for the first time when comparing the resulting terms involving $P^{(2)}$ since $P^{(3)}$ transforms into $P^{(2)}$ under $J^{a_1a_2a_3}_{(1)}$. This deficiency is likely to be linked to the projection from the tensor product $\Psi\otimes\mathcal{P}$ of the unfaithful $K(E_{10})$ spinor with the faithful infinite-dimensional coset representation onto an unfaithful Dirac-spinor representation $\Psi\odot\mathcal{P}$, as in (5.14) where not all components of $\mathcal{P}$ appear.

5.4 Supersymmetry variation in the coset

Finally, we study the compatibility of the supersymmetry variation of the bosonic fields (via the dictionary) with the general variational structure of the coset. For a general supersymmetry coset variation we write

$$\delta_\epsilon \mathcal{V}\mathcal{V}^{-1} = \Lambda + \Sigma, \quad \Lambda\in\mathfrak{k}, \ \ \Sigma\in\mathfrak{e}_{10}\ominus\mathfrak{k} \tag{5.16}$$

as in (4.1). The advantage of writing the variation in this way is again its $K(E_{10})$ covariance: if we were to express the variations in terms of explicit 'coordinates' on $E_{10}/K(E_{10})$, these variations would be subject to possible field redefinitions just as the coordinate fields themselves. Moreover, this simple expression encapsulates all the variations of the bosonic fields (including dual magnetic potentials) in a single formula. However, the shortcomings of the unfaithful spinor representations of $K(E_{10})$ are again apparent: because Λ and Σ are bilinear expressions in some fermionic fields ϵ and Ψ of $K(E_{10})$, it is not possible to construct out of only finitely many spinor components ϵ and Ψ_a the most general Λ and Σ (which both have infinitely many independent components), and hence objects which transform in the right way under $K(E_{10})$. Lastly, the 'compensating' $K(E_{10})$ transformation with parameter Λ in (5.16) is needed to preserve the triangular gauge.[14]

Proceeding with the general analysis we deduce by combining (4.1) with (5.16) that

$$\delta_\epsilon \mathcal{P} = \mathcal{D}\Sigma + [\Lambda, \mathcal{P}],$$

$$\delta_\epsilon \mathcal{Q} = \mathcal{D}\Lambda + [\Sigma, \mathcal{P}]. \tag{5.17}$$

In triangular gauge the first equation yields schematically

$$\delta_\epsilon P^{(\ell)} = \mathcal{D}^{(0)}\Sigma^{(\ell)} - \sum_{m=1}^{\ell-1} P^{(m)}\Sigma^{(\ell-m)}, \tag{5.18}$$

where in particular only a finite number of terms contribute on the right hand side. ($\mathcal{D}^{(0)}$ is the partially covariantised derivative of eq. (4.6).)

We can compute the variation of the fields $P^{(\ell)}$ for $\ell = 0, 1$ by using the dictionary (5.6) and the supergravity variations (A.15). We find that the general coset structure (5.18) matches the supergravity result with

$$\begin{aligned}
\Sigma^{(0)}_{ab} &= -i\bar{\epsilon}\Gamma_{(a}\Psi_{b)}, \\
\Sigma^{(1)}_{a_1 a_2 a_3} &= -\frac{3}{2}i\bar{\epsilon}\Gamma_{[a_1 a_2}\Psi_{a_3]}, \\
\Lambda^{(0)}_{ab} &= i\bar{\epsilon}\Gamma_{[a}\Psi_{b]}, \\
\Lambda^{(1)}_{a_1 a_2 a_3} &= -\frac{3}{2}i\bar{\epsilon}\Gamma_{[a_1 a_2}\Psi_{a_3]}.
\end{aligned} \tag{5.19}$$

Here, we have used the correspondence with the unfaithful spinors and $\bar{\epsilon} \equiv \epsilon^T\Gamma^0$. The structure up to here is also compatible with the $K(E_{10})$ transformation on the coset: A level one transformation of the unfaithful spinors in $\Sigma^{(0)}$ yields the same combination of $\Sigma^{(1)}$ terms as transforming $\Sigma^{(1)}$ as a coset element.[15] The choice $\Sigma^{(1)} = \Lambda^{(1)}$ ensures that the supersymmetry tansformations (5.19) preserve the triangular gauge.

Insisting on the correct $K(E_{10})$ transformation properties we can also compute from the unfaithful fermions that $\Sigma^{(2)}$ has to be

$$\Sigma^{(2)}_{a_1 \ldots a_6} = 3i\bar{\epsilon}\Gamma_{[a_1 \ldots a_5}\Psi_{a_6]}. \tag{5.20}$$

This result is identical with the one that one obtains from supergravity when introducing a dual potential $A_{a_1 \ldots a_6}$, see appendix A.2.

However, transforming ϵ and Ψ in $\Sigma^{(2)}$ under level one again does *not* give the right tensor structure $\Sigma^{(3)}_{a_0|a_1 \ldots a_8}$ in agreement with the coset on level three. Rather one finds also a totally anti-symmetric piece to $\Sigma^{(3)}$. Such a breakdown is not unexpected since we knew from the start that ϵ and Ψ will not suffice to construct Σ to all levels. Again, we interpret this as the need to find the correct faithful spinor representation ϵ and Ψ in order to construct a supersymmetric *and* E_{10} invariant model.

6. Canonical structure and constraints

In this section, we consider the (Dirac) algebra of supersymmetry constraints and show that it properly closes into the bosonic constraints.[16] We give the expressions for all constraints

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[15]For this computation it is crucial that $\Sigma^{(0)}$ contains $\bar{\epsilon}$ and not simply ϵ^T.

[16]See [48, 49] and [50] for analyses of the supersymmetry constraint algebras for canonical supergravity in four and three space-time dimensions, respectively.

in terms of the $E_{10}/K(E_{10})$ coset variables, but will leave a detailed investigation of their transformation properties under $K(E_{10})$ to future work.

6.1 Canonical Dirac brackets

The momentum conjugate to the original supergravity gravitino $\psi_a^{(11)}$ is given by (suppressing spinor indices)

$$\Pi^a = \frac{\partial \mathcal{L}}{\partial \partial_t \psi_a^{(11)}} = \frac{i}{2} E N^{-1} (\psi_b^{(11)})^T \Gamma^{ab}. \tag{6.1}$$

Because of the linear dependence of the momentum on $\psi^{(11)}$ (6.1) is tantamount to a *second class constraint*, hence we must replace Poisson by Dirac brackets in the standard fashion [51]. As a result, we can replace the momentum by $\psi^{(11)}$, thus explicitly solving the constraint (6.1). This yields

$$(\psi_a^{(11)})^T = -\frac{2ig^{-1/2}}{9} \Pi^b \left(8\delta_{ab} + \Gamma_{ab} \right), \tag{6.2}$$

and the canonical (Dirac) brackets

$$\left\{ \psi_a^{(11)}, (\psi_b^{(11)})^T \right\} = -2ig^{-1/2} \left(\delta_{ab} - \frac{1}{9} \Gamma_a \Gamma_b \right). \tag{6.3}$$

Substituting the redefinition (3.4) and making use of the fermionic correspondence (5.1), we finally obtain (now with all the indices written out)

$$\frac{i}{2} \left\{ \Psi_{a\alpha}, \Psi_{b\beta} \right\} = \delta_{ab} \delta_{\alpha\beta} - \frac{1}{9} \left(\Gamma_a \Gamma_b \right)_{\alpha\beta}. \tag{6.4}$$

The canonical bracket (6.4) is the same one would have derived from the E_{10} model (4.8) since the kinetic term for ψ_a is the same.

For the explicit computation of the canonical brackets $\{\mathcal{S}, \mathcal{S}^T\}$ we note that the second entry in this bracket corresponds to the (matrix) transpose of $\mathcal{S}$. Hence, the antisymmetric Γ-matrices appearing in $\mathcal{S}^T$ change sign relative to $\mathcal{S}$, that is, for $\Gamma^{(p)}$ with $p = 2, 3, 6, 7, 10$.

6.2 Constraint algebra

From (6.4) and (5.14) one can now compute $\{\mathcal{S}, \mathcal{S}\}$. We restrict attention here to the purely bosonic terms (originating from $\mathcal{P}\mathcal{P}\{\Psi, \Psi\}$), and will thus not consider fermionic bilinears (coming from $\Psi\Psi\{\mathcal{P}, \mathcal{P}\}$). Since $\mathcal{S}$ is an SO(10) Dirac spinor, we can decompose this symmetric tensor product into its irreducible SO(10) pieces, i.e. in a Γ-basis, using all the symmetric SO(10) Γ-matrices, see appendix A. The result of this computation is

$$\frac{i}{2} \left\{ \mathcal{S}_\alpha, \mathcal{S}_\beta \right\} = \mathcal{C}^{(0)} \delta_{\alpha\beta} + \mathcal{C}^{(3)}_{c_1 \ldots c_9} \Gamma^{c_1 \ldots c_9}_{\alpha\beta} + \mathcal{C}^{(4)}_{c_1 \ldots c_8} \Gamma^{c_1 \ldots c_8}_{\alpha\beta}$$
$$+ \mathcal{C}^{(5)}_{c_1 \ldots c_5} \Gamma^{c_1 \ldots c_5}_{\alpha\beta} + \mathcal{C}^{(6)}_{c_1 \ldots c_4} \Gamma^{c_1 \ldots c_4}_{\alpha\beta}. \tag{6.5}$$

where the constraints are labelled by the level of the corresponding contributions as in table 1 (and thus *not* by the number of Γ-matrix indices!). A further term proportional

to a single Γ-matrix $\Gamma^a_{\alpha\beta}$, which is allowed in principle, does not show up in our present calculation below. As we already mentioned in the introduction, the structure of the terms on the r.h.s. of (6.5) is reminiscent of the 'central charge representation' $L(\Lambda_1)$ of E_{11} first introduced in [36]. Let us therefore briefly relate the terms to the more familiar terms in the $D = 11$ supersymmetry algebra, which contains the following SO(1, 10) central charges (see e.g. [52] and references therein)

- P_A (translation operator): Reduced to SO(10) this yields a scalar object ($\mathcal{C}^{(0)}$ above) and an SO(10) vector (dual to $\mathcal{C}^{(3)}$ above). These are to be interpreted as the Hamiltonian and diffeomorphism constraint of the theory. (In a spatial reduction of $D = 11$ supergravity to type IIA supergravity, these are the D0 brane and the momentum charge for gravitational waves.)

- Z_{AB} (M2 brane charge): Reduced to SO(10) this yields a vector and a two-form (dual to $\mathcal{C}^{(4)}$ above). The interpretation of the latter in the present context is as the Gauss constraint of the theory. (In IIA language, these would be interpreted as the central charges to which the D2-brane and the fundamental string couple as well as their duals.)

- $Z_{A_1\ldots A_5}$ (M5 brane charge): Reduction to SO(10) gives a four-form ($\mathcal{C}^{(6)}$ above) and a five-form ($\mathcal{C}^{(5)}$ above). In the present context, they can be interpreted as part of the Bianchi identities on the four-form potential and on the gravity sector. (In IIA language, these would be interpreted as the central charges to which the D4-brane and the NS5 brane couple as well as their duals.)

Assuming that $\mathcal{S}$ transforms as a spinor representation of $K(E_{10})$, one may ask what the $K(E_{10})$ decomposition of the r.h.s. of (6.5) would be. If $\mathcal{S}$ were an unfaithful Dirac spinor **32**, the symmetric product of two **32** representations would be reducible under $K(E_{10})$ into a scalar representation ($\mathcal{C}^{(0)}$ above) and a remaining piece of dimension 527, by the $K(E_{10})$ invariance of (2.32). However, as we pointed out already, with the present dictionary, $\mathcal{S}$ does *not* transform properly, so that any match to $K(E_{10})$ representations is bound to be incomplete.

We now give the result of the computation of $\{\mathcal{S}, \mathcal{S}\}$ in this SO(10) basis. For the scalar (level zero) part we find

$$
\mathcal{C}^{(0)} = \frac{1}{4} P^{(0)}_{ab} P^{(0)}_{ab} - \frac{1}{4} P^{(0)}_{aa} P^{(0)}_{bb} + \frac{1}{12} P^{(1)}_{a_1 a_2 a_3} P^{(1)}_{a_1 a_2 a_3} + \frac{1}{12 \cdot 5!} P^{(2)}_{a_1 \ldots a_6} P^{(2)}_{a_1 \ldots a_6}
$$
$$
+ \frac{1}{9!} P^{(3)}_{a_0 | a_1 \ldots a_8} P^{(3)}_{a_0 | a_1 \ldots a_8} - \frac{4}{9!} P^{(3)}_{b | b a_1 \ldots a_7} P^{(3)}_{b | b a_1 \ldots a_7}. \tag{6.6}
$$

This result is to be compared with the scalar constraint computed from the bosonic σ-model with the standard invariant bilinear form, which reads

$$
\frac{1}{4} \langle \mathcal{P} | \mathcal{P} \rangle = \frac{1}{4} P^{(0)}_{ab} P^{(0)}_{ab} - \frac{1}{4} P^{(0)}_{aa} P^{(0)}_{bb} + \frac{1}{12} P^{(1)}_{a_1 a_2 a_3} P^{(1)}_{a_1 a_2 a_3}
$$
$$
+ \frac{1}{12 \cdot 5!} P^{(2)}_{a_1 \ldots a_6} P^{(2)}_{a_1 \ldots a_6} + \frac{1}{2 \cdot 9!} P^{(3)}_{a_0 | a_1 \ldots a_8} P^{(3)}_{a_0 | a_1 \ldots a_8} + \ldots \tag{6.7}
$$

where the dots stand for higher level ($\ell \geq 4$) contributions. One sees that the terms up to $\ell \leq 2$ match perfectly, but mismatches appear at level $\ell = 3$: (i) the coefficient of the full mixed tableau is off by a factor of 2, and (ii) the traced tableau appears explicitly, whereas it is absent from (6.6). The r.h.s. of (6.6) can be identified with the (bosonic part of) the Hamiltonian constraint of $D = 11$ supergravity. Indeed the $\ell = 3$ terms appear in exactly the right combination (neglecting the trace $\Omega_{ab\,b}$)

$$\propto \quad \Omega_{ab\,c}\Omega_{ab\,c} - 2\Omega_{ab\,c}\Omega_{bc\,a} \tag{6.8}$$

appearing in the Einstein-Hilbert action. Another indication of the mismatch is that the level $\ell \geq 1$ contributions in (6.7) are manifestly positive, whereas the level-three term in (6.6) is not.[17]

The next contributions we consider are those proportional to a $\Gamma^{(9)}$ matrix (which is dual to $\Gamma_{(1)}\Gamma_0$) in (6.5). Explicitly one finds

$$\mathcal{C}^{(3)}_{c_1...c_9} = \frac{1}{3 \cdot 8!}P^{(0)}_{ac_1}P^{(3)}_{a|c_2...c_9} + \frac{1}{6 \cdot 6!}P^{(1)}_{c_1c_2c_3}P^{(2)}_{c_4...c_9}. \tag{6.9}$$

where antisymmetrization over the indices $[c_1 \ldots c_9]$ is understood. After a little algebra this expression reduces to (after dualisation in $[c_1 \ldots c_9]$)

$$\epsilon_{ac_1...c_9}\mathcal{C}^{(3)}_{c_1...c_9} \propto \Omega_{ab\,c}\omega_{b\,c0} - \frac{1}{3}F_{abcd}F_{0bcd} \tag{6.10}$$

in terms of the original supergravity variables. Using the the tracelessness of $\Omega_{ab\,c}$ we can rewrite first term on the r.h.s. as

$$\Omega_{ab\,c}\omega_{b\,c0} = \Omega_{ab\,c}(\omega_{b\,c0} - \delta_{bc}\omega_{e\,e0}) \equiv \Omega_{ab\,c}\Pi_{bc} \tag{6.11}$$

where Π_{ab} is the gravitational canonical momentum (with flat indices). Hence, ignoring spatial gradients, this is just the diffeomorphism (momentum) constraint $\mathcal{G}_{0a} = 0$ of supergravity (with the correct relative coefficient).

Next we compute the contributions proportional to $\Gamma^{(4)}$, which read

$$\mathcal{C}^{(4)}_{c_1...c_8} = -\frac{1}{4 \cdot 4! \cdot 4!}P^{(2)}_{a_1a_2c_1...c_4}P^{(2)}_{a_1a_2c_5...c_8} - \frac{1}{3 \cdot 7!}P^{(1)}_{a_1a_2c_1}P^{(3)}_{a_1|a_2c_2...c_8}$$

$$+ \frac{1}{6 \cdot 6!}P^{(1)}_{ac_1c_2}P^{(3)}_{b|bac_3...c_8}. \tag{6.12}$$

Again, the trace of $\ell = 3$ appears separately. After dualising the relevant terms the result is proportional to

$$\epsilon_{abc_1...c_8}\mathcal{C}^{(4)}_{c_1...c_8} \propto \Omega_{cd\,[a}F_{b]0cd} + \frac{1}{576}\epsilon_{abc_1...c_4d_1...d_4}F_{c_1...c_4}F_{d_1...d_4} \tag{6.13}$$

which (again neglecting spatial gradients) coincides with the Gauss constraint $\mathcal{M}_{0ab} = 0$ of the supergravity.

---

[17]Recall that the two terms in (6.8) are not of the same order near the singularity [18]. Namely, the first term is associated with a leading gravitational wall, whose normal is a real (gravitational) root, whereas the second term is subleading, and associated to an affine null root. This distinction is not respected by the decomposition of (6.6) into SO(10) irreducible tensors, as the last (trace) term in (6.6) contributes to both terms in (6.8). By contrast, at the level of the equations of motion, the leading $\ell = 3$ terms *do match* between the supergravity and the coset dynamics, and the mismatch concerns only *subleading* $\ell = 3$ terms.

The Bianchi identity $(D_{[a}F_{bcde]} = 0)$ terms are proportional to $\Gamma^{(5)}$

$$\mathcal{C}^{(5)}_{c_1...c_5} = -\frac{1}{16 \cdot 5!}P^{(2)}_{c_1 a_1...a_5}P^{(3)}_{a_1|a_2...a_5 c_2...c_5} + \frac{1}{4 \cdot 5!}P^{(2)}_{c_1 c_2 a_1...a_4}P^{(3)}_{b|b a_1...a_4 c_3 c_4 c_5}. \quad (6.14)$$

Upon use of the dictionary (5.6) this agrees with the appropriately truncated version of the supergravity Bianchi constraint.

Finally we find contributions of the form $[P^{(3)}]^2$ which are proportional to $\Gamma^{(4)}$. They are (with anti-symmetrisation over $[c_1 \ldots c_4]$)

$$\mathcal{C}^{(6)}_{c_1...c_4} = -\frac{1}{9 \cdot 7!}P^{(3)}_{c_1|c_2 a_1...a_7}P^{(3)}_{c_3|c_4 a_1...a_7} - \frac{1}{18 \cdot 6!}P^{(3)}_{c_1|c_2 c_3 a_1...a_6}P^{(3)}_{b|b c_4 a_1...a_6}. \quad (6.15)$$

Using the dictionary (5.6), the constraint $\mathcal{C}^{(6)} = 0$ is equivalent to the Ω Bianchi identity

$$\left[\partial_{[a}, [\partial_b, \partial_{c]}]\right] = \left(\partial_{[a}\Omega_{bc]}{}^e - \Omega_{[ab}{}^d\Omega_{c]d}{}^e\right)\partial_e = 0 \quad (6.16)$$

neglecting spatial gradients (that is, dropping the first term on the r.h.s.).

7. Discussion and outlook

In this paper, we have given full account of the supersymmetry variations, equations of motion and constraints of $D = 11$ supergravity (to lowest fermion order) in the framework of the $E_{10}/K(E_{10})$ σ-model defined by the action (4.8), using the bosonic and fermionic correspondences (5.6) and (5.1). In addition, we have developed the rudiments of a structure theory for $K(E_{10})$, where, however, many important questions remain open. By studying the $K(E_{10})$ properties of various supergravity expressions in section 5 we have found strong evidence for a correspondence between supergravity and the fermionic $E_{10}/K(E_{10})$ σ-model, with complete agreement up to and including A_9 level $\ell = 2$, but also a number of discrepancies starting at level $\ell = 3$. Most of these can be traced back to our use of unfaithful $K(E_{10})$ spinor representation for the fermionic fields. This makes the need for the construction of faithful spinor representations more urgent. The task is made harder by the fact that standard tools of representation theory are unavailable here; in particular, we do not expect that the required representations of $K(E_{10})$ are of highest or lowest weight type. We have exposed some unusual (and, *a priori* unexpected) features of $\mathfrak{k}$ related to the existence of unfaithful fermionic representations, especially the existence of non-trivial ideals in $\mathfrak{k}$, and pointed out that these ideals may furnish new types of representations (and thus may also shed a new light on the 'gradient conjecture' of [19]). One possibility for constructing faithful spinor representations of $K(E_{10})$ was already mentioned in [30, 25], namely to consider tensor products of unfaithful spinor representations (e.g. the Dirac-spinor ϵ) with faithful bosonic representations (e.g. the coset $\mathcal{P}$). We have not explored this possibility in much detail, but note that a similar construction was recently proposed in the context of maximal ($N = 16$) supergravity in two space-time dimensions and for the involutory algebra $K(E_9)$ [53].

We leave to future work a better understanding of the extension of the multiplet of bosonic constraints studied above, that is the Hamiltonian constraint $\mathcal{C}^{(0)}$ and the remaining constraints $\mathcal{C}^{(3)}, \mathcal{C}^{(4)}, \mathcal{C}^{(5)}, \mathcal{C}^{(6)}$, to a *bona fide* (and presumably infinite-dimensional)

multiplet of constraints carrying a (faithful) representation of $K(E_{10})$. An interesting speculation is that this infinite set of constraints might constrain the 'velocity' $\mathcal{P}$ of the coset particle to lie in a 'mass-shell', which might be small enough to zoom on the very restricted affine representations entering the 'gradient conjecture' of [19].

A. $D = 11$ supergravity

We give an explicit transcription of the conventions of Cremmer, Julia and Scherk (CJS) [1]. As a warning to the reader we note that in eq. (3.4) redefined fermions are introduced which have the same letters as the ones used in this appendix but are different.

A.1 General conventions

Unlike [1] we work with the 'mostly plus' signature for the eleven-dimensional Lorentz metric

$$\eta_{\text{here}}^{AB} = \text{diag}(- + \cdots +) = -\eta_{\text{CJS}}^{AB}, \qquad A, B = 0, \ldots, 10. \tag{A.1}$$

In order to maintain the SO(1, 10) Clifford algebra[18] $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$ the Γ-matrices change accordingly to

$$\Gamma_{\text{here}}^M = -i\Gamma_{\text{CJS}}^M. \tag{A.2}$$

Furthermore, we set the $D = 11$ anti-symmetric tensor with upper indices to

$$\epsilon_{\text{here}}^{0\ldots10} = \epsilon_{\text{CJS}}^{0\ldots10} = +1 \tag{A.3}$$

and our Γ-matrices satisfy

$$\Gamma^0 \cdots \Gamma^{11} = +\epsilon^{0\ldots10} \mathbf{1}_{32} = +\mathbf{1}_{32}. \tag{A.4}$$

An explicit representation in a Majorana basis is given by (cf. appendices of [21])

$$\Gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} = \mathcal{C} \ , \quad \Gamma^{10} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ , \quad \Gamma^a = \begin{pmatrix} 0 & \tilde{\gamma}^a \\ \tilde{\gamma}^a & 0 \end{pmatrix} \tag{A.5}$$

---

[18]If no subscript appears on an object it is in the 'here' conventions.

with the real symmetric 16×16 SO(9) $\tilde{\gamma}^a$ matrices for $a = 1, \ldots, 9$; $\mathcal{C}$ is the charge conjugation matrix. Consequently, our (32×32) SO(10) matrices Γ^a (now $a = 1, \ldots, 10$) are real and symmetric, and furthermore obey

$$\Gamma^{a_1 \ldots a_{10}} = \epsilon^{a_1 \ldots a_{10}} \Gamma_0 \qquad (= -\epsilon^{a_1 \ldots a_{10}} \Gamma^0), \tag{A.6}$$

where the ten-dimensional SO(10) invariant epsilon symbol is $\epsilon^{1 \ldots 10} = +1$. From this we deduce

$$\Gamma^{a_1 \ldots a_k} = \frac{1}{(10-k)!} (-1)^{\frac{(k+1)(k+2)}{2}} \epsilon^{a_1 \ldots a_k a_{k+1} \ldots a_{10}} \Gamma_{a_{k+1} \ldots a_{10}} \Gamma_0. \tag{A.7}$$

Among the anti-symmetric products $\Gamma^{(p)}$ of p SO(10) Γ-matrices the symmetric ones occur for $p = 0, 1, 4, 5, 8, 9$ and the anti-symmetric ones occur for $p = 2, 3, 6, 7, 10$.

As a rule (with the exception of $\epsilon_{\text{here}}^{M_1 \ldots M_{11}} = \epsilon_{\text{CJS}}^{M_1 \ldots M_{11}}$) we identify *covariant* tensors with the corresponding objects in CJS [1]: $E_{N\,\text{here}}{}^A = E_{N\,\text{CJS}}{}^A$, $\omega_{M\,A\,\text{here}}{}^B = \omega_{M\,A\,\text{CJS}}{}^B$, $A_{MNP}^{\text{here}} = A_{MNP}^{\text{CJS}}$, $\psi_M^{\text{here}} = \psi_M^{\text{CJS}}$. One must then be careful about changes in derived objects, such as $G_{MN} \equiv \eta_{AB} E_M{}^A E_N{}^B$ (for which $G_{MN}^{\text{here}} = -G_{MN}^{\text{CJS}}$), or $\psi_{\text{here}}^M = -\psi_{\text{CJS}}^M$. Similarly, the conjugate fermions are related by

$$(\bar{\psi}_M)_{\text{here}} \equiv (\psi_M^{\text{here}})^T \Gamma_{\text{here}}^0 = -i(\psi_M)^T \Gamma_{\text{CJS}}^0 = -i(\bar{\psi}_M)_{\text{CJS}}. \tag{A.8}$$

This implies $\bar{\psi}_{\text{here}}^M = +i\bar{\psi}_{\text{CJS}}^M$. There is no complex conjugation since ψ_M is in a real (Majorana) representation of the SO(1, 10) group. A Majorana spinor consists of 32 *real* (anti-commuting) components. Note also that conjugation reverses the order of the (anti-commuting) fermions. In the main body of the paper, the fermions of this appendix will be written with an additional superscript [11] in order to distinguish them from certain redefined fermions which are more useful for studying the relation to E_{10} (cf. (3.4)).

The Lorentz covariant derivative on the vielbein $E_M{}^A$ is given by

$$D_M(\omega) E_N{}^A := \partial_M E_N{}^A + \omega_M{}^{AB} E_{NB} = \Gamma_{MN}{}^P E_P{}^A \tag{A.9}$$

with the standard Christoffel symbol $\Gamma_{MN}{}^P$ and spin connection ω_{ABC} (in flat indices) given in terms of the anholonomy coefficients Ω_{ABC} by

$$\omega_{ABC} = \frac{1}{2}(\Omega_{ABC} + \Omega_{CAB} - \Omega_{BCA}),$$
$$\Omega_{ABC} \equiv E_A{}^M E_B{}^N (\partial_M E_{NC} - \partial_N E_{NC}). \tag{A.10}$$

The Riemann tensor is defined via the commutator of two Lorentz covariant derivatives, viz.

$$D_M = \partial_M + \frac{1}{4}\omega_{M\,AB}\Gamma^{AB} \quad \Rightarrow \quad [D_M, D_N] = \frac{1}{4}R_{MNAB}\Gamma^{AB}. \tag{A.11}$$

which gives

$$R_{MNA}{}^B = \partial_M \omega_{N\,A}{}^B - \partial_N \omega_{M\,A}{}^B + \omega_{M\,A}{}^E \omega_{N\,E}{}^B - \omega_{N\,A}{}^E \omega_{M\,E}{}^B, \tag{A.12}$$

or, in flat indices,

$$R_{ABCD} = \partial_A \omega_{BCD} - \partial_B \omega_{ACD} + \Omega_{AB}{}^E \omega_{ECD}$$
$$+ \omega_{AC}{}^E \omega_{B\,ED} - \omega_{BC}{}^E \omega_{A\,ED}. \tag{A.13}$$

A.2 Action and supersymmetry variations

Modulo higher order fermionic terms, the Lagrangian of $D = 11$ supergravity [1] in our conventions reads[19]

$$E^{-1}\mathcal{L} = \frac{1}{4}R - \frac{i}{2}\bar{\psi}_M\Gamma^{MNP}D_N\psi_P - \frac{1}{48}F^{MNPQ}F_{MNPQ}$$
$$-\frac{i}{96}\left(\bar{\psi}_M\Gamma^{MNPQRS}\psi_S + 12\bar{\psi}^N\Gamma^{PQ}\psi^R\right)F_{NPQR}$$
$$+\frac{2E^{-1}}{(144)^2}\epsilon^{M_1...M_{11}}F_{M_1...M_4}F_{M_5...M_8}A_{M_9...M_{11}}. \qquad (A.14)$$

The field strength is $F_{MNPQ} \equiv 4\partial_{[M}A_{NPQ]}$. The supersymmetry variations are, in our conventions,

$$\delta E_M{}^A = i\bar{\varepsilon}\Gamma^A\psi_M$$
$$\delta\psi_M = D_M\varepsilon + \frac{1}{144}\left(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR}\right)\varepsilon F_{NPQR}$$
$$\delta A_{MNP} = -\frac{3}{2}i\bar{\varepsilon}\Gamma_{[MN}\psi_{P]} \qquad (A.15)$$

The parameter ε is a **32**-component spinor of SO(1, 10), related to the one of [1] by $\varepsilon_{\text{here}} = \varepsilon_{\text{CJS}}$, and $\bar{\varepsilon}_{\text{here}} = -i\bar{\varepsilon}_{\text{CJS}}$.

For completeness we also give the variation of the dual 'magnetic' potential $\tilde{A}_{M_1...M_6}$. This can be derived by adding to the action a term [54] (ignoring the FFA term for the moment)

$$\mathcal{L}' = \frac{1}{4! \cdot 7!}\epsilon^{MNPQRS_1...S_6}F_{MNPQ}\partial_R\tilde{A}_{S_1...S_6} \qquad (A.16)$$

such that the variation with respect to $\tilde{A}_{M_1...M_6}$ enforces the Bianchi identity for F_{MNPQ}. Requiring the action with the addition (A.16) to be supersymmetric, a little algebra shows that

$$\delta\tilde{A}_{M_1...M_6} = 3i\bar{\varepsilon}\Gamma_{[M_1...M_5}\psi_{M_6]} \qquad (A.17)$$

which is thus the 'magnetic' analog of the last variation in (A.15). Extensions of the supersymmetry transformation rules including dual and ten-form potentials were recently studied in [55, 56].

A.3 Equations of motion and constraints

Neglecting terms quadratic in the fermions, the bosonic equations of motion with $\psi_M = 0$ are (always *flat* indices)

$$R_{AB} = \frac{1}{3}F_{ACDE}F_B{}^{CDE} - \frac{1}{36}\eta_{AB}F_{CDEF}F^{CDEF} \qquad (A.18)$$

$$D_A F^{ABCD} = -\frac{1}{576}\epsilon^{BCDE_1...E_4F_1...F_4}F_{E_1...E_4}F_{F_1...F_4} \qquad (A.19)$$

---

[19]We set the constant $\kappa_{11} = 1$ [1], which in terms of the Newton constant is $4\pi G_{11} = 1$.

with the Lorentz covariant derivative in flat indices $D_A \equiv E_A{}^M D_M(\omega)$. In the text (and in table 1) we use the notation

$$\mathcal{G}_{AB} \equiv R_{AB} - \frac{1}{2}\eta_{AB}R - \frac{1}{3}F_A{}^{CDE}F_{BCDE} + \frac{1}{24}\eta_{AB}F_{CDEF}F^{CDEF},$$

$$\mathcal{M}^{BCD} \equiv D_A F^{ABCD} + \frac{1}{576}\epsilon^{BCDE_1\ldots E_4 F_1\ldots F_4}F_{E_1\ldots E_4}F_{F_1\ldots F_4}. \tag{A.20}$$

Furthermore, we have the Bianchi identities

$$D_{[A_1}F_{A_2 A_3 A_4 A_5]} = 0, \tag{A.21}$$

$$D_{[A_1}\Omega_{A_2 A_3]}{}^B = 0. \tag{A.22}$$

where the Lorentz covariant derivative in (A.22) does not act on the B index.

The Rarita-Schwinger equation for the gravitino is[20]

$$\mathcal{E}^A \equiv \Gamma^{ABC}D_B(\omega)\psi_C + \frac{1}{48}\Big(\Gamma^{ABCDEF}\psi_F + 12\eta^{AB}\Gamma^{CD}\psi^E\Big)F_{BCDE} = 0 \tag{A.23}$$

A more convenient form, used in the text, is

$$\Gamma^B\big(\hat{D}_A\psi_B - \hat{D}_B\psi_A\big) = 0 \quad \text{with} \quad \hat{D}_A := D_A(\omega) + \mathcal{F}_A \tag{A.24}$$

where

$$\mathcal{F}_A := +\frac{1}{144}\Big(\Gamma_A{}^{BCDE} - 8\delta_A^B\Gamma^{CDE}\Big)F_{BCDE} \tag{A.25}$$

Again these equations have been given in terms of *flat* indices, whose use is a crucial ingredient in our construction.

In a Hamiltonian (canonical) formulation, the above equations split into *equations of motion* (describing the evolution in time) and *constraints* (which must be imposed on the initial data), and whose relation to the σ-model quantities is displayed in table 1. The equations of motion consist of the components $\mathcal{G}_{ab}$, $\mathcal{M}_{abc}$ and $\mathcal{E}_a$, while the constraints are:

$$\begin{aligned}
\mathcal{G}_{00} &\approx 0 &\leftrightarrow& \quad \text{Hamiltonian (scalar) constraint} \\
\mathcal{G}_{0a} &\approx 0 &\leftrightarrow& \quad \text{diffeomorphism constraint} \\
\mathcal{M}_{0ab} &\approx 0 &\leftrightarrow& \quad \text{Gauss constraint} \\
\mathcal{E}_0 &\approx 0 &\leftrightarrow& \quad \text{supersymmetry constraint}
\end{aligned} \tag{A.26}$$

B. Consistency conditions for representations

Proposition. *Let $\mathfrak{k}_1$ be a (finite-dimensional) vector space with basis (x_i) and $\tilde{\mathfrak{k}}$ be the free Lie algebra (over $\mathbb{R}$) generated by the x_i. Let $\mathfrak{r} \subset \tilde{\mathfrak{k}}$ be an ideal in $\tilde{\mathfrak{k}}$. Denote the quotient Lie algebra $\tilde{\mathfrak{k}}/\mathfrak{r}$ by $\mathfrak{k}$. Finally, let V be a module of $\mathfrak{k}_1$, i.e. we have a map $\rho_1 : \mathfrak{k}_1 \to End(V)$.*[21]

---

[20]Note that $D_A(\omega)\psi_B = \partial_A\psi_B + \omega_{ABC}\psi^C + \frac{1}{4}\omega_{ACD}\Gamma^{CD}\psi_B$.

[21]In this set-up, where $\mathfrak{k}_1$ is thought of to contain the *simple* generators x_i, ρ_1 can be any map, there are no consistency conditions imposed on it.

(i) V can be made a module of $\tilde{\mathfrak{k}}$. Denote the representation homomorphism by $\tilde{\rho} : \tilde{\mathfrak{k}} \to End(V)$.

(ii) If $\tilde{\rho}(\mathfrak{r}) = 0$, then V is also a module of $\mathfrak{k} = \tilde{\mathfrak{k}}/\mathfrak{r}$.

Proof. (i) $\tilde{\rho}$ is defined recursively on $\tilde{\mathfrak{k}} = \bigoplus_{n \geq 1} \tilde{\mathfrak{k}}_n$, where the degree of an element is the number of $\mathfrak{k}_1$ elements in the multiple commutator in the free Lie algebra. For $y_1 \in \mathfrak{k}_1$ one defines $\tilde{\rho}(y_1) := \rho_1(y_1)$. For $x_2 \in \tilde{\mathfrak{k}}_2$ represented by $x_2 = [y_1, y_1']$ in terms of two elements $y_1, y_1' \in \mathfrak{k}_1$ one *defines* $\tilde{\rho}(x_2) := \tilde{\rho}(y_1)\tilde{\rho}(y_1') - \tilde{\rho}(y_1')\tilde{\rho}(y_1)$. Similarly for the higher degrees. Consistency (and independence from the way one parametrises the next degree) is guaranteed generally, see e.g. chapter 17.5 of [57].

(ii) Define the representation homomorphism $\rho : \mathfrak{k} \to End(V)$ by $\rho(x) = \tilde{\rho}(x + \mathfrak{r})$. Independence of the representative follows generally from $\tilde{\rho}(\mathfrak{r}) = 0$ (the kernel of $\tilde{\rho}$ factors through). For this representation homomorphism one easily checks the representation property. $\qquad\square$

Corollary. *If $\mathfrak{r}$ is generated as an ideal of $\tilde{\mathfrak{k}}$ by some number of relations r_A (A in some index set), it suffices to check $\tilde{\rho}(r_A) = 0$ for the assumptions of (ii) in theorem above.*

Proof. This follows from the construction of $\tilde{\rho}$ in the free algebra via successive commutators. $\qquad\square$

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