

Spherically Symmetric Quantum Geometry: Hamiltonian Constraint

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Abstract

Variables adapted to the quantum dynamics of spherically symmetric models are introduced, which further simplify the spherically symmetric volume operator and allow an explicit computation of all matrix elements of the Euclidean and Lorentzian Hamiltonian constraints. The construction fits completely into the general scheme available in loop quantum gravity for the quantization of the full theory as well as symmetric models. This then presents a further consistency check of the whole scheme in inhomogeneous situations, lending further credence to the physical results obtained so far mainly in homogeneous models. New applications in particular of the spherically symmetric model in the context of black hole physics are discussed.

1 Introduction

Loop quantum gravity [1, 2, 3] provides a candidate for a non-perturbative, background independent quantization of general relativity which has already led to several results concerning the quantum structure of space and time. There are, however, also open issues mainly in the context of understanding the dynamics and the semiclassical limit. While dealing with these problems in full generality is complicated, one can isolate particular aspects by looking at reduced situations where only a select class of degrees of freedom is considered. This class of degrees of freedom needs to be adapted to the physical situation of interest, which is most commonly done by employing symmetry reduction. In the case of loop quantum gravity, or any diffeomorphism invariant quantum theory of connections, there is a general scheme to introduce symmetries at the level of quantum states and basic operators [4].

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Homogeneous models [5, 6, 7, 8, 9] in the context of loop quantum cosmology [10, 11, 12] are by now well-understood both from the dynamical point of view and concerning semiclassical properties [13, 14, 15, 16]. They have led to tests of and new insights into the full theory, and resulted in many physical applications [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. However, field theory aspects which play an essential role in the full theory cannot be tested by restricting oneself to homogeneous cases such that a generalization to inhomogeneous models is needed. The simplest inhomogeneous model is the spherically symmetric one since it has only one physical degree of freedom [28, 29, 30]. States and basic operators [31], as well as the volume operator [32], have been derived along the lines of a loop quantization based on connection variables, resulting in explicit expressions in particular for all volume eigenvalues. Having an explicitly known volume spectrum, which is not available in the full theory [33, 34, 35, 36, 37], was one of the main ingredients that resulted in direct calculations in homogeneous models, and so one may expect similar applications of the spherically symmetric model. In particular the Hamiltonian constraint contains the volume operator in commutators with holonomies [38], which need to be computed in order to know the constraint equation explicitly. However, it turned out that in the quantization of [32] the spherically symmetric volume operator has eigenstates different from the triad eigenstates on which holonomies would have a simple action. This implies that even with an explicitly known volume spectrum commutators of the volume operator with holonomies are hard to compute in general. As a consequence, coefficients of the constraint equation would have a complicated form.

An additional complication for setting up the Hamiltonian constraint is presented by the non-vanishing spin connection. As discussed in [9], one often has to split off non-vanishing components of the spin connection from holonomies along homogeneous directions in order to ensure the correct classical limit. The main observation of the present paper will be the fact that one can advantageously split off the connection components already at the level of states, not only when constructing the Hamiltonian constraint operator. We will show that this can be done in a way consistent with both the full theory and the treatment in homogeneous models, and even leads to a considerable simplification of the volume operator. That this is possible depends sensitively on non-trivial properties of the spin connection and extrinsic curvature for spherically symmetric configurations, which also hold true in polarized cylindrical wave models. Thus, similar constructions can be done in other models, where the method of [32] to quantize the volume operator and to find its spectrum would not work or where calculations would be more complicated [39, 40].

We will first discuss the spherically symmetric classical phase space and constraints in connection variables in Sec. 2. After computing the spin connection we will start Sec. 3 with preliminary aspects of the classical limit, which motivates the introduction of a canonical transformation to variables better suited to a loop quantization and its investigation. The loop representation is then briefly done in Sec. 4, before Sec. 5 presents the Hamiltonian constraint operator as the main part of this paper. This allows several conclusions also on general aspects of loop quantum gravity and physical properties to be discussed in Sec. 6

2 Classical phase space

A loop quantization of gravitational systems is based on real Ashtekar variables [41, 42] which are given by the $\text{su}(2)$ -connection $A_a^i = \Gamma_a^i + \gamma K_a^i$ and its momentum, the densitized triad E_i^a . In the definition of A , Γ_a^i are spin connection components compatible with the triad, K_a^i are extrinsic curvature components and $\gamma \in \mathbb{R}^+$ is the Barbero–Immirzi parameter [42, 43]. In spherical symmetry, one only considers connections and triads which are invariant under rotations up to gauge transformations, which implies the general form

$$A = A_x(x)\Lambda_3 dx + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)d\vartheta + (A_1(x)\Lambda_2 - A_2(x)\Lambda_1) \sin \vartheta d\varphi + \Lambda_3 \cos \vartheta d\varphi \quad (1)$$

and

$$E = E^x(x)\Lambda_3 \sin \vartheta \frac{\partial}{\partial x} + (E^1(x)\Lambda_1 + E^2(x)\Lambda_2) \sin \vartheta \frac{\partial}{\partial \vartheta} + (E^1(x)\Lambda_2 - E^2(x)\Lambda_1) \frac{\partial}{\partial \varphi} \quad (2)$$

with real functions A_x, A_1, A_2, E^x, E^1 and E^2 on the radial manifold B coordinatized by x (see, e.g., [4, 12]). The $\text{su}(2)$ -matrices Λ_I are constant and are identical to $\tau_I = -\frac{i}{2}\sigma_I$ or a rigid rotation thereof, which can be eliminated by partially fixing the Gauss constraint. The functions E^x, E^1 and E^2 on B are canonically conjugate to A_x, A_1 and A_2 :

$$\Omega_B = \frac{1}{2\gamma G} \int_B dx (dA_x \wedge dE^x + 2dA_1 \wedge dE^1 + 2dA_2 \wedge dE^2) \quad (3)$$

with the gravitational constant G .

These variables are subject to constraints which are obtained by inserting the invariant forms into the full expressions. We have the Gauss constraint

$$G[\lambda] = \int_B dx \lambda (E^{x'} + 2A_1 E^2 - 2A_2 E^1) \approx 0 \quad (4)$$

generating $U(1)$ -gauge transformations, the diffeomorphism constraint

$$D[N_x] = \int_B dx N_x (2A_1' E^1 + 2A_2' E^2 - A_x E^{x'}) \quad (5)$$

and the Hamiltonian constraint

$$\begin{aligned} H[N] &= (2G)^{-1} \int_B dx N (|E^x|((E^1)^2 + (E^2)^2))^{-1/2} \\ &\quad \times (2E^x(E^1 A_2' - E^2 A_1') + 2A_x E^x (A_1 E^1 + A_2 E^2) + (A_1^2 + A_2^2 - 1)((E^1)^2 + (E^2)^2) \\ &\quad - (1 + \gamma^2) (2K_x E^x (K_1 E^1 + K_2 E^2) + (K_1^2 + K_2^2)((E^1)^2 + (E^2)^2))) \\ &=: -H^E[N] + P[N] \end{aligned} \quad (6)$$

where H^E is the first (so-called Euclidean) part depending explicitly on connection components and P the second part depending on extrinsic curvature components (which are themselves functions of A_a^i and E_i^a).

In [31] variables

$$A_\varphi(x) := \sqrt{A_1(x)^2 + A_2(x)^2}, \quad (8)$$

$$E^\varphi(x) := \sqrt{E^1(x)^2 + E^2(x)^2} \quad (9)$$

and $\alpha(x), \beta(x)$ defined by

$$\Lambda_\varphi^A(x) =: \Lambda_1 \cos \beta(x) + \Lambda_2 \sin \beta(x), \quad (10)$$

$$\Lambda_E^\varphi(x) =: \Lambda_1 \cos(\alpha(x) + \beta(x)) + \Lambda_2 \sin(\alpha(x) + \beta(x)). \quad (11)$$

for the internal directions

$$\Lambda_\varphi^A(x) := (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)/A_\varphi(x), \quad (12)$$

$$\Lambda_E^\varphi(x) := (E^1(x)\Lambda_2 - E^2(x)\Lambda_1)/E^\varphi(x) \quad (13)$$

were introduced. Similarly, we have $\Lambda_\vartheta^A(x) = -\Lambda_1 \sin \beta(x) + \Lambda_2 \cos \beta(x)$ and the analogous expression for Λ_E^ϑ . These variables are adapted to a loop quantization in that holonomies along integral curves of generators of the symmetry group are of the form $\exp A_\varphi \Lambda_\varphi^A$.

However, E^φ is *not* the momentum conjugate to A_φ , which instead is given by

$$P^\varphi(x) := 2E^\varphi(x) \cos \alpha(x). \quad (14)$$

Canonical coordinates are thus the conjugate pairs $A_x, E^x; A_\varphi, P^\varphi; \beta, P^\beta$ with

$$P^\beta(x) := 2A_\varphi(x)E^\varphi(x) \sin \alpha(x) = A_\varphi(x)P^\varphi(x) \tan \alpha(x) \quad (15)$$

The momenta as basic variables will directly be quantized, resulting in flux operators with equidistant discrete spectra. But unlike in the full theory and homogeneous models, the resulting quantum representation has a volume operator which does not commute with flux operators. This follows since volume is determined by triad components, in particular E^φ which is related to P^φ in a rather complicated way *involving the connection component* A_φ . Thus, the volume operator has eigenstates different from flux eigenstates, which makes the computation of commutators with holonomies more complicated [32]. An alternative way to derive volume eigenvalues in a quantization based on the variables A_x, A_1 and A_2 is being pursued in [39, 40].

In this paper we will considerably simplify the formalism by applying a canonical transformation such that now E^φ plays the role of a basic momentum variable. This will, of course, change the configuration variables which will no longer be purely connection components. At first sight, this seems to render the procedure unsuitable for a loop quantization where basic operators make use of holonomies of the connection. After a transformation of the canonical variables, holonomies in general will be complicated functions of the new variables such that the new quantum representation would not be suitable for a loop quantization. It turns out, however, that the special form of a spherically symmetric spin connection and extrinsic curvature for a given triad leads to new variables which are ideally suited to a loop representation even from the dynamical point of view.

3 The role of the spin connection

The co-triad corresponding to a densitized triad (2) is given by

$$e = e_x \Lambda_3 dx + e_\varphi \Lambda_E^\vartheta d\vartheta + e_\varphi \Lambda_E^\varphi \sin \vartheta d\varphi \quad (16)$$

with

$$e_\varphi = \sqrt{|E^x|} \quad \text{and} \quad e_x = \text{sgn}(E^x) \frac{E^\varphi}{\sqrt{|E^x|}}. \quad (17)$$

From this form one can compute the spin connection

$$\Gamma = -(\alpha + \beta)' \Lambda^3 dx + \frac{e'_\varphi}{e_x} \Lambda_E^\varphi d\vartheta - \frac{e'_\varphi}{e_x} \Lambda_E^\vartheta \sin \vartheta d\varphi + \Lambda_3 \cos \vartheta d\varphi \quad (18)$$

and the extrinsic curvature for lapse function N and shift N^x ,

$$K = N^{-1}(\dot{e}_x - (N^x e_x)') \Lambda_3 dx + N^{-1}(\dot{e}_\varphi - N^x e'_\varphi) \Lambda_E^\vartheta d\vartheta + N^{-1}(\dot{e}_\varphi - N^x e'_\varphi) \Lambda_E^\varphi \sin \vartheta d\varphi. \quad (19)$$

We define the φ -components of Γ and K as

$$\Gamma_\varphi := -\frac{e'_\varphi}{e_x} = -\frac{E^{x'}}{2E^\varphi} \quad , \quad K_\varphi := N^{-1}(\dot{e}_\varphi - N^x e'_\varphi) \quad (20)$$

which combines to $A_\varphi = \sqrt{\Gamma_\varphi^2 + \gamma^2 K_\varphi^2}$ since the two internal φ -directions are orthogonal.

3.1 Classical limit

The form of spin connection and extrinsic curvature is important to understand classical regimes in which extrinsic curvature is small. Since the spin connection in general is not a tensor, a similar statement about the smallness of Γ would not have invariant meaning since one can always choose local coordinates such that the components Γ_a^i are arbitrarily small. However, in a symmetric model this may not be true for all components because within the model only coordinate transformations preserving the invariant form (1), (2) are allowed. Then, some components of the spin connection can become covariant such that statements about their magnitude become meaningful. In the above form, for instance, it is clear that the x -component of the spin connection is not of definite magnitude since locally one can simply gauge the angle $\alpha + \beta$ to be constant. The last term in (18), on the other hand, does not depend on the fields at all and is thus always of the order one. In general, components of the spin connection in inhomogeneous directions, just as all components in the full theory, do not have invariant meaning, while components along symmetry orbits are gauge invariant. (This follows easily from the transformation properties of a connection where the inhomogeneous part $g^{-1}dg$ only contributes to inhomogeneous directions such as x when the gauge transformation g is required to preserve the symmetry and is thus constant along symmetry orbits.) Moreover, Γ_φ is covariant and transforms as a scalar because both $E^{x'}$ and E^φ are densities.

Covariant components of the spin connection carry information about the intrinsic curvature of symmetry orbits. In contrast to extrinsic curvature, this is not necessarily small in classical regimes as can be seen for the spin connection (18) specialized to the Schwarzschild solution. Inserting $e_\varphi = x$ and $e_x = (1 - 2M/x)^{-1/2}$ leads to $\Gamma_\varphi = 1 + O(x^{-1})$ which is not small at large radii where classical properties are to be expected. Thus, also the Ashtekar connection component $A_\varphi = \sqrt{\Gamma_\varphi^2 + \gamma^2 K_\varphi^2}$ is not small in this regime but of the order one. Consequently, angular holonomies of the form $\exp A_\varphi \Lambda_\varphi^A$ cannot be expanded in A_φ for semiclassical considerations. It is then more complicated to quantize classical expressions which are polynomial in A_φ , a prominent example being the Hamiltonian constraint, because they first need to be expressed through holonomies such that the classical expression is reproduced in semiclassical regimes.

One can arrive at expressions expandable in connection components if one explicitly subtracts the spin connection for homogeneous directions. Instead of working with holonomies of A_φ one would then construct operators from holonomies of, in the above case, $A_\varphi - 1$ which would be small. This procedure can in fact be described as a general scheme which works in all homogenous models and gives a satisfactory evolution there (i.e. stable in the sense of [44] and with the correct classical limit).

The same procedure can be followed in the spherically symmetric model for holonomies along symmetry orbits. However, in the variables described so far the same complication as with the volume operator arises: In order to subtract the angular spin connection components at the operator level we need to quantize the spin connection which, since it is related to triad components, results in a complicated expression in terms of canonical variables A_φ and P^φ .

3.2 Canonical triad variables

The preceding remarks have shown that many steps in the usual constructions of a loop quantization become much more complicated when triad components are not among the basic canonical variables. On the other hand, applying a canonical transformation such that P^φ is replaced by E^φ may lead to more complicated configuration variables which are no longer related to holonomies in a simple way. We will now see that the explicit form (18) of the spin connection and (19) of extrinsic curvature in spherical symmetry allows one to perform a suitable canonical transformation and at the same time facilitate the subtraction of spin connection components already at the state level. The subtraction in the constraint operator can then be done easily. There is a difference to the treatment of homogeneous models in [9] where the spin connection was used explicitly only at the operator level. However, there the subtraction could also have been done at the state level such that the procedure here is consistent with [9]. Moreover, we will see that inhomogeneous directions are not affected by the subtraction and we are also consistent with the full theory. We thus have crucial tests available by models which are in between homogeneous ones and the full theory, as discussed in more detail below after explicit constructions will be available.

Since the momentum of A_x is already given by a triad component E^x , it will be unchanged by our canonical transformation and we can focus on the variables $A_\varphi, P^\varphi; \beta, P^\beta$.

Using the definitions (14) and (15) of P^φ and P^β in the canonical Liouville form and trading in E^φ for P^φ results in

$$\begin{aligned}
P^\varphi dA_\varphi + P^\beta d\beta &= 2E^\varphi \cos \alpha dA_\varphi + P^\beta d\beta \\
&= E^\varphi d(2A_\varphi \cos \alpha) - 2E^\varphi A_\varphi d \cos \alpha + P^\beta d\beta \\
&= E^\varphi d(2A_\varphi \cos \alpha) + P^\eta d\eta.
\end{aligned} \tag{21}$$

In the last line we now have E^φ as momentum of the configuration variable $2A_\varphi \cos \alpha$, and the old $P^\eta = P^\beta$ as momentum of the angle $\eta := \alpha + \beta$ determining the internal triad direction.

As a function of the original variables, $A_\varphi \cos \alpha$ looks complicated and does not seem to be related to holonomies. In fact, α is a function of both A_φ and the momenta P^φ and P^β such that it cannot be expressed as a function of holonomies in the original variables alone. However, the structure of (18) and (19) shows that there is a simple geometrical meaning to the new configuration variable conjugate to E^φ . Here it is important to notice that the internal directions along a given angular direction of a spherically symmetric Γ in (18) are always perpendicular to those of E (note that Λ_E^ϑ and Λ_E^φ are exchanged in (18) compared to (2)), while the corresponding extrinsic curvature components are parallel to those of E . Since A is obtained by summing Γ and K , we can write

$$A_\varphi \Lambda_\varphi^A = \Gamma_\varphi \bar{\Lambda} + \gamma K_\varphi \Lambda$$

with $\Lambda := \Lambda_E^\varphi$ and $\bar{\Lambda} := \Lambda_E^\vartheta$. This implies

$$A_\varphi \cos \alpha = A_\varphi \Lambda_\varphi^A \cdot \Lambda = \gamma K_\varphi \tag{22}$$

where the left equality is the definition of α . Thus, the new configuration variable is simply proportional to the extrinsic curvature component K_φ which we can view as obtained by subtracting the spin connection from the Ashtekar connection. Note that it is well known in the full theory that extrinsic curvature components are conjugate to triad components. But as we have seen for Ashtekar variables, this does not imply that the φ -components as defined here are conjugate (while E^1, E^2 would obviously be conjugate to A_1, A_2 as well as K_1, K_2). The non-trivial fact is thus that in contrast to the angular Ashtekar components, the angular extrinsic curvature component as configuration variable allows one to use triad components as momenta. As the derivation shows, this depends crucially on properties of the spherically symmetric spin connection and extrinsic curvature. That E^φ is conjugate to K_φ follows from the fact that E and K have the same internal directions, while the orthogonality of internal directions in Γ to those of E is relevant for details of the canonical transformation.

3.3 Hamiltonian constraint

In the original spherically symmetric variables the Hamiltonian constraint has Euclidean part

$$H^E[N] = -(2G)^{-1} \int_B dx N(x) |E^x|^{-1/2} \left((A_\varphi^2 - 1) E^\varphi + 2 \cos \alpha A_\varphi E^x (A_x + \beta') - 2 \sin \alpha E^x A'_\varphi \right).$$

With the new canonical triad variables, using

$$A'_\varphi \sin \alpha = (A_\varphi \sin \alpha)' - A_\varphi \alpha' \cos \alpha = \Gamma'_\varphi - \gamma K_\varphi \alpha'$$

we have

$$H^E[N] = -(2G)^{-1} \int_B dx N(x) |E^x|^{-1/2} \left((\Gamma_\varphi^2 + \gamma^2 K_\varphi^2 - 1) E^\varphi + 2\gamma K_\varphi E^x (A_x + \eta') - 2E^x \Gamma'_\varphi \right). \quad (23)$$

Moreover, the Lorentzian part is simply of the form

$$P[N] = -(2G)^{-1} (1 + \gamma^2) \int_B dx N(x) |E^x|^{-1/2} (K_\varphi^2 E^\varphi + 2K_\varphi K_x E^x)$$

which, using $\gamma K_x = A_x - \Gamma_x = A_x + \eta'$, combines easily with the terms already present in (23):

$$H[N] = -(2G)^{-1} \int_B dx N(x) |E^x|^{-1/2} \left((1 - \Gamma_\varphi^2 + K_\varphi^2) E^\varphi + 2\gamma^{-1} K_\varphi E^x (A_x + \eta') + 2E^x \Gamma'_\varphi \right). \quad (24)$$

4 Loop representation

With the new spherically symmetric configuration variables $A_x, \gamma K_\varphi, \eta$ the construction of the quantum representation proceeds identically to that in [31], just by replacing everywhere A_φ with γK_φ and β with η . The resulting Hilbert space is then spanned by an orthonormal basis in terms of spin network states

$$T_{g,k,\mu}(A) = \prod_{e \in g} \exp\left(\frac{1}{2} i k_e \int_e A_x(x) dx\right) \prod_{v \in V(g)} \exp(i \mu_v \gamma K_\varphi(v)) \exp(i k_v \eta(v)) \quad (25)$$

with edge labels $k_e \in \mathbb{Z}$ and vertex labels $\mu_v \in \mathbb{R}$ and $k_v \in \mathbb{Z}$ for graphs g in the 1-dimensional radial manifold B .

By definition in (9), A_φ is always non-negative which is sufficient because a sign change in both components A_1 and A_2 can always be compensated by a gauge rotation. The extrinsic curvature component K_φ , on the other hand, is measured relatively to the internal direction Λ_E^φ in (19) and thus both signs are possible: $K_\varphi \in \mathbb{R}$. This makes the representation technically easier to deal with, but could be done similarly with connection components upon defining $\tilde{A}_\varphi := A_\varphi \operatorname{sgn} K_\varphi$. (See also [45] for a discussion in a homogeneous model.)

Similarly, we immediately obtain the action of flux operators quantizing the momenta E^x, E^φ and P^η (using the Planck length $\ell_P = \sqrt{G\hbar}$):

$$\hat{E}^x(x) T_{g,k,\mu} = \gamma \ell_P^2 \frac{k_{e^+(x)} + k_{e^-(x)}}{2} T_{g,k,\mu} \quad (26)$$

$$\int_{\mathcal{I}} \hat{E}^\varphi T_{g,k,\mu} = \gamma \ell_P^2 \sum_{v \in \mathcal{I}} \mu_v T_{g,k,\mu} \quad (27)$$

$$\int_{\mathcal{I}} \hat{P}^\eta T_{g,k,\mu} = 2\gamma \ell_P^2 \sum_{v \in \mathcal{I}} k_v T_{g,k,\mu} \quad (28)$$

where $e^\pm(x)$ are the two edges (or two parts of a single edge) meeting in x . (Compared to the expression for \hat{P}^φ in [31] there is a factor of $\frac{1}{2}$ missing in (27) since E^φ is conjugate to $\gamma K_\varphi/G$, while P^φ was defined to be conjugate to $A_\varphi/2G$; see (21). Note also that in some previous papers $\ell_{\text{P}}^2 = 8\pi G\hbar$ was used.) Spin networks, as expected, are thus eigenstates of the flux operators with the crucial difference to [31] being that now \hat{E}^φ is among the fluxes.

This allows us to quantize the volume $V(\mathcal{I}) = 4\pi \int_{\mathcal{I}} dx \sqrt{|E^x|} E^\varphi$ of a region $\mathcal{I} \subset B$ immediately after regularizing as in [32], resulting in the volume operator

$$\hat{V}(\mathcal{I}) = 4\pi \int_{\mathcal{I}} dx |\hat{E}^\varphi(x)| \sqrt{|\hat{E}^x(x)|} \quad (29)$$

where $\hat{E}^\varphi(x)$ is the distribution valued operator

$$\hat{E}^\varphi(x) T_{g,k,\mu} = \gamma \ell_{\text{P}}^2 \sum_{v \in B} \delta(v, x) \mu_v T_{g,k,\mu}.$$

Note that, just as A_φ in (9), also E^φ is defined to be non-negative in (9). Thus, only labels $\mu_v \geq 0$ would be allowed. Again, it is technically easier to allow first all values $\mu_v \in \mathbb{R}$ and in the end require physical states to be symmetric under $\mu_v \mapsto -\mu_v$ corresponding to solving a residual gauge transformation. We thus write explicitly absolute values around $\hat{E}^\varphi(x)$ and μ_v . The volume operator then has eigenstates (25) with eigenvalues

$$V_{k,\mu} = 4\pi \gamma^{3/2} \ell_{\text{P}}^3 \sum_v |\mu_v| \sqrt{\frac{1}{2} |k_{e^+(v)} + k_{e^-(v)}|}. \quad (30)$$

In contrast to [32], the eigenvalues follow immediately and the eigenstates are identical to flux eigenstates.

The Gauss constraint and the fluxes it depends on are unmodified as compared to [31] (except for the $-$ sign in front of P^β which was written by mistake in [31]) such that classically it still is given by

$$G[\lambda] = \int_B dx \lambda (E^{x'} + P^n) \approx 0 \quad (31)$$

and quantized to

$$\hat{G}[\lambda] T_{g,k,\mu} = \gamma \ell_{\text{P}}^2 \sum_v \lambda(v) (k_{e^+(v)} - k_{e^-(v)} + 2k_v) T_{g,k,\mu} = 0. \quad (32)$$

As before, this is solved directly and imposes the condition

$$k_v = -\frac{1}{2} (k_{e^+(v)} - k_{e^-(v)}) \quad (33)$$

on gauge invariant states. Inserting this in (25) we eliminate the integer valued vertex labels k_v and obtain the general form of gauge invariant spherically symmetric spin networks

$$T_{g,k,\mu} = \prod_e \exp\left(\frac{1}{2} i k_e \int_e (A_x + \eta') dx\right) \prod_v \exp(i \mu_v \gamma K_\varphi(v)). \quad (34)$$

They depend only on the gauge invariant configuration variables $A_x + \eta' = \gamma K_x$ and K_φ .

5 Hamiltonian constraint

There is a general procedure to quantize objects such as the Euclidean part

$$H^E[N] = -(8\pi G)^{-1} \int_{\Sigma} d^3x N(x) \epsilon^{ijk} F_{ab}^i \frac{E_j^a E_k^b}{\sqrt{|\det E|}}$$

of the Hamiltonian constraint in the full theory, where the curvature components $F_{ab}^i(x)$ of the Ashtekar connection are expressed via a holonomy around a small loop starting at x , and the factor containing triad components is obtained from a commutator between holonomies and the volume operator [38]. This procedure can also be adapted to symmetric models, which allows detailed tests of its viability on general grounds. While the commutator part goes through almost unchanged in symmetric models (see, however, [46]), the quantization of curvature components is different. For one obtains the correct components only in an expansion of the holonomy for a closed loop, which implies that the exponent appearing in this holonomy must be small. This is easily achieved in the full theory by making the loops small enough in coordinate length, which results automatically in the continuum limit in which a regulator is removed. A similar argument can be applied to inhomogeneous directions in symmetric models such as the radial direction in the system studied here. We can then use, e.g.,

$$\exp i \int_v^{v_+} A_x dx \sim 1 + i\epsilon A_x(v) + O(\epsilon^2)$$

when $\epsilon = v_+ - v$ is the coordinate distance between two vertices v and v_+ connected by the edge.

For homogeneous directions, however, this argument cannot be applied since there is no free edge length available which could be chosen small. Homogeneous connection components transform as scalars and thus are implemented via ‘point holonomies’ [47, 48, 4, 7] such as $\exp i\gamma K_\varphi(v)$ rather than holonomies along edges. An expansion

$$\exp iA(v) \sim 1 + iA(v) + O(A^2)$$

is then possible only in regimes where the relevant connection component A is small. (Alternatively, one can employ a limit $\gamma \rightarrow 0$ as in [13], but it must still be shown to exist in an inhomogeneous construction. This may also require conditions on the basic algebra used.) Since the expansions are done in order to reproduce the classical limit, only exponents in holonomies are allowed which become small in semiclassical regimes. In general, there will be other regimes in which the components are not small, which will give rise to perturbative quantum corrections [49, 50, 51, 15, 52, 53].

5.1 Classical limit

Smallness in semiclassical regimes is not guaranteed for A_φ , as discussed before in Sec. 3.1, and so one would need to subtract off the spin connection from this component along the

lines of [9] in order to ensure the correct classical limit. This is complicated in the spherically symmetric model since the spin connection, containing E^φ , would be a complicated operator in a loop quantization based on configuration variables containing A_φ . In the quantization described here, \hat{E}^φ and thus $\hat{\Gamma}_\varphi$ are simpler, but a subtraction is not even necessary since now K_φ is a basic configuration variable. Since the extrinsic curvature component is small in semiclassical regimes, we can work directly with holonomies $\exp i\gamma K_\varphi$ which are basic and also appear in spin network states (25).

When there is non-vanishing intrinsic curvature in a model, one can proceed by splitting the Hamiltonian constraint into two parts which will be quantized separately [9]. The first one just contains extrinsic curvature components besides triad components and will be quantized by employing holonomies around suitable loops. In our case, this part is classically given by the contribution

$$H_K[N] = -(2G)^{-1} \int_B dx N(x) |E^x|^{-1/2} (K_\varphi^2 E^\varphi + 2\gamma^{-1} K_\varphi (A_x + \eta') E^x) \quad (35)$$

to (24). The second part contains the curvature of the spin connection, which in our case results in

$$H_\Gamma[N] = -(2G)^{-1} \int_B dx N(x) |E^x|^{-1/2} ((1 - \Gamma_\varphi^2) E^\varphi + 2\Gamma'_\varphi E^x) . \quad (36)$$

Both parts consist of two terms containing E^φ and E^x , respectively. Via $E^\varphi/\sqrt{|E^x|} \propto \{A_x, V\}$ and $\sqrt{|E^x|} \propto \{K_\varphi, V\}$ these terms will contain commutators $h_x[h_x^{-1}, \hat{V}]$ and $h_\varphi[h_\varphi^{-1}, \hat{V}]$, respectively, when quantized. Here and in what follows we use holonomies

$$h_x(e) = \exp\left(\int_e A_x(x) dx \Lambda_3\right) \quad (37)$$

$$h_\vartheta(v, \delta) = \exp(\gamma\delta K_\varphi(v) \bar{\Lambda}) \quad (38)$$

$$h_\varphi(v, \delta) = \exp(\gamma\delta K_\varphi(v) \Lambda) \quad (39)$$

adapted to the symmetry in order to construct the operator. The edge length and the parameter δ have to be chosen for the construction; their roles will become clear later on.

The spin connection part H_Γ will be completed by using the expression (20) for Γ_φ and the triad operators. Since classically the inverse of E^φ appears, one has to use techniques as in [54, 55] in order to obtain a densely defined operator. For the extrinsic curvature part H_K we need to construct appropriate products of holonomies in order to model loops resulting in the correct curvature components. Suitable products can be read off from the general form [38]

$$\hat{H} \propto \sum_{v,i,j,k} N(v) \epsilon^{ijk} \text{tr}(h_{ij} h_k [h_k^{-1}, \hat{V}]) \quad (40)$$

for a quantization, where we sum over vertices v and triples i, j, k of edges and use the lapse function N smearing the constraint. Holonomies h_K for each edge appear in the

commutators, multiplied with a loop holonomy h_{ij} lying in a plane spanned by the two other edges of the triple. In a symmetric context, this scheme takes the form [56, 6, 8, 9]

$$\hat{H} \propto \sum_{v,I,J,K} N(v) \epsilon^{IJK} \text{tr}(h_I h_J h_I^{-1} h_J^{-1} h_K [h_K^{-1}, \hat{V}])$$

where h_{ij} is constructed from holonomies available in the symmetric model, where now I, J, K correspond to coordinates adapted to the symmetry. In general, each holonomy h_I for a homogeneous direction appears with a parameter δ (which we will use but not specify in what follows) with δ^2 being analogous to the size of the loop in h_{ij} . This parameter allows one to do formal expansions of functions of holonomies, and to check if the correct classical limit results to leading order, but this limit will also be valid for small K_φ , as expected in semiclassical regimes, even if δ is of the order one.

In spherical symmetry, $\{I, J, K\}$ is $\{x, \vartheta, \varphi\}$ such that we have two essentially different contributions for $K = x$ and $K = \vartheta, \varphi$, respectively. They lead to operators

$$\text{tr}(h_\vartheta(v) h_\varphi(v) h_\vartheta(v)^{-1} h_\varphi(v)^{-1} h_x [h_x^{-1}, \hat{V}])$$

and

$$\text{tr}(h_x h_\vartheta(v_+) h_x^{-1} h_\vartheta(v)^{-1} h_\varphi(v) [h_\varphi(v)^{-1}, \hat{V}])$$

where the x -holonomy is computed for an edge connecting the vertex v to a new one, v_+ . Using x -holonomies for small edge length ϵ and expanding in δ (or using small K_φ in a semiclassical regime) we see that both terms indeed give us the expected curvature components. We first expand the product of holonomies

$$\begin{aligned} h_\vartheta h_\varphi h_\vartheta^{-1} h_\varphi^{-1} &= \exp(\gamma \delta K_\varphi \bar{\Lambda}) \exp(\gamma \delta K_\varphi \Lambda) \exp(-\gamma \delta K_\varphi \bar{\Lambda}) \exp(-\gamma \delta K_\varphi \Lambda) \\ &= (1 + \gamma \delta K_\varphi \bar{\Lambda} + O(\delta^2))(1 + \gamma \delta K_\varphi \Lambda + O(\delta^2)) \\ &\quad \times (1 - \gamma \delta K_\varphi \bar{\Lambda} + O(\delta^2))(1 - \gamma \delta K_\varphi \Lambda + O(\delta^2)) \\ &= 1 + \frac{1}{2} \gamma^2 \delta^2 K_\varphi^2 + \gamma^2 \delta^2 K_\varphi^2 \Lambda_3 + O(\delta^3) \end{aligned}$$

and

$$\begin{aligned} h_x h_\vartheta(v_+) h_x^{-1} h_\vartheta(v)^{-1} &= e^{\int A_x \Lambda_3} \exp(\gamma \delta K_\varphi(v_+) \bar{\Lambda}(v_+)) e^{-\int A_x \Lambda_3} \exp(-\gamma \delta K_\varphi(v) \bar{\Lambda}(v)) \\ &= (1 + \int A_x \Lambda_3 + O(\epsilon^2))(1 + \gamma \delta K_\varphi(v_+) \bar{\Lambda}(v_+) + O(\delta^2)) \\ &\quad \times (1 - \int A_x \Lambda_3 + O(\epsilon^2))(1 - \gamma \delta K_\varphi(v) \bar{\Lambda}(v) + O(\delta^2)) \\ &= 1 + \gamma \delta K_\varphi(v_+) \bar{\Lambda}(v_+) - \gamma \delta K_\varphi(v) \bar{\Lambda}(v) + \frac{1}{2} \gamma \delta K_\varphi(v_+) \int A_x \Lambda(v_+) \\ &\quad + \frac{1}{2} \gamma \delta K_\varphi(v) \int A_x \Lambda(v) - \gamma^2 \delta^2 K_\varphi(v) K_\varphi(v_+) \bar{\Lambda}(v) \bar{\Lambda}(v_+) \\ &\quad + O(\delta^3) + O(\epsilon^2) \end{aligned}$$

which will be multiplied by the commutators $h_x [h_x^{-1}, \hat{V}]$ and $h_\varphi(v) [h_\varphi(v)^{-1}, \hat{V}]$, respectively. To leading order in ϵ and δ , the first commutator is proportional to Λ_3 , while the second one is proportional to $\Lambda(v)$. Thus, the relevant traces to check the classical limit are

$$-2 \text{tr}(h_\vartheta h_\varphi h_\vartheta^{-1} h_\varphi^{-1} \Lambda_3) \sim \gamma^2 \delta^2 K_\varphi^2$$

and

$$\begin{aligned}
-2 \operatorname{tr}(h_x h_\vartheta(v_+) h_x^{-1} h_\vartheta(v)^{-1} \Lambda(v)) &\sim -2\gamma\delta K_\varphi(v_+) \operatorname{tr}(\bar{\Lambda}(v_+) \Lambda(v)) \\
&\quad -\gamma\delta K_\varphi(v_+) \int A_x \operatorname{tr}(\Lambda(v_+) \Lambda(v)) - \gamma\delta K_\varphi(v) \int A_x \operatorname{tr}(\Lambda(v)^2) \\
&= \gamma\delta K_\varphi(v_+) \sin \Delta\eta \\
&\quad + \frac{1}{2}\gamma\delta K_\varphi(v_+) \int A_x \cos \Delta\eta + \frac{1}{2}\gamma\delta K_\varphi(v) \int A_x \\
&= \gamma\delta\epsilon K_\varphi(v)(A_x(v) + \eta'(v)) + O(\delta\epsilon^2)
\end{aligned}$$

with $\Delta\eta = \eta(v_+) - \eta(v)$. With these expressions we obtain the right coefficients of E^x and E^φ in H_K to ensure the correct classical limit.

5.2 Operator

The preceding discussion, together with collecting numerical factors, shows that the operator

$$\begin{aligned}
\hat{H}[N] &= \frac{i}{2\pi G\gamma^3\delta^2\ell_P^2} \sum_{v,\sigma=\pm 1} \sigma N(v) \operatorname{tr}\left((h_\vartheta h_\varphi h_\vartheta^{-1} h_\varphi^{-1} - h_\varphi h_\vartheta h_\varphi^{-1} h_\vartheta^{-1} \right. \\
&\quad \left. + 2\gamma^2\delta^2(1 - \hat{\Gamma}_\varphi^2)\Lambda_3) h_{x,\sigma}[h_{x,\sigma}^{-1}, \hat{V}] \right. \\
&\quad \left. + \left(h_{x,\sigma} h_\vartheta(v_\sigma) h_{x,\sigma}^{-1} h_\vartheta(v)^{-1} - h_\vartheta(v) h_{x,\sigma} h_\vartheta(v_\sigma)^{-1} h_{x,\sigma}^{-1} + 2\gamma^2\delta \int_\sigma \hat{\Gamma}'_\varphi \Lambda(v) \right) h_\varphi[h_\varphi^{-1}, \hat{V}] \right. \\
&\quad \left. + \left(h_\varphi(v) h_{x,\sigma} h_\varphi(v_\sigma)^{-1} h_{x,\sigma}^{-1} - h_{x,\sigma} h_\varphi(v_\sigma) h_{x,\sigma}^{-1} h_\varphi(v)^{-1} + 2\gamma^2\delta \int_\sigma \hat{\Gamma}'_\varphi \bar{\Lambda}(v) \right) h_\vartheta[h_\vartheta^{-1}, \hat{V}] \right)
\end{aligned} \tag{41}$$

corresponds to the correct classical expression where in the continuum limit coordinate differentials ϵ from h_x and $\int \Gamma'_\varphi$ become the integration measure. (The sign factor σ denotes the orientation of the radial edge running to the right (+) or left (-) of the vertex. If it is not specified, it is understood to be positive without crucial changes for the other orientation. Moreover, \int_σ in front of the derivative of the spin connection component indicates that it has to be integrated between v and v_σ as a result of the discretization.) To evaluate the action explicitly we now use $\exp(A\Lambda) = \cos \frac{1}{2}A + 2\Lambda \sin \frac{1}{2}A$ for all holonomies. This gives, for instance,

$$\begin{aligned}
h_\vartheta h_\varphi h_\vartheta^{-1} h_\varphi^{-1} &= \cos^2(\frac{1}{2}\gamma\delta K_\varphi) + \sin^2(\frac{1}{2}\gamma\delta K_\varphi) \cos(\gamma\delta K_\varphi) + \sin^2(\gamma\delta K_\varphi)\Lambda_3 \\
&\quad + 4 \cos(\frac{1}{2}\gamma\delta K_\varphi) \sin^3(\frac{1}{2}\gamma\delta K_\varphi)(\Lambda_\vartheta - \Lambda_\varphi)
\end{aligned}$$

and

$$\begin{aligned}
h_x[h_x^{-1}, \hat{V}] &= \hat{V} - \cos(\frac{1}{2} \int A_x) \hat{V} \cos(\frac{1}{2} \int A_x) - \sin(\frac{1}{2} \int A_x) \hat{V} \sin(\frac{1}{2} \int A_x) \\
&\quad + 2\Lambda_3 \left(\cos(\frac{1}{2} \int A_x) \hat{V} \sin(\frac{1}{2} \int A_x) - \sin(\frac{1}{2} \int A_x) \hat{V} \cos(\frac{1}{2} \int A_x) \right)
\end{aligned}$$

which, when combined with the contribution where ϑ and φ are exchanged and traced, yields the term

$$\begin{aligned} & -2 \operatorname{tr} \left((h_{\vartheta} h_{\varphi} h_{\vartheta}^{-1} h_{\varphi}^{-1} - h_{\varphi} h_{\vartheta} h_{\varphi}^{-1} h_{\vartheta}^{-1}) h_{x,\sigma} [h_{x,\sigma}^{-1}, \hat{V}] \right) = 2 \sin^2(\gamma \delta K_{\varphi}) \\ & \quad \times \left(\cos(\tfrac{1}{2} f A_x) \hat{V} \sin(\tfrac{1}{2} f A_x) - \sin(\tfrac{1}{2} f A_x) \hat{V} \cos(\tfrac{1}{2} f A_x) \right) \end{aligned}$$

as one part of the constraint operator.

Similarly, we obtain

$$\begin{aligned} & -2 \operatorname{tr} \left((h_x h_{\vartheta}(v_+) h_x^{-1} h_{\vartheta}(v)^{-1} - h_{\vartheta}(v) h_x h_{\vartheta}(v)^{-1} h_x^{-1}) h_{\varphi} [h_{\varphi}^{-1}, \hat{V}] \right. \\ & \quad \left. + (h_{\varphi}(v) h_x h_{\varphi}^{-1}(v_+) h_x^{-1} - h_x h_{\varphi}(v_+) h_x^{-1} h_{\varphi}^{-1}(v)) h_{\vartheta} [h_{\vartheta}^{-1}, \hat{V}] \right) \\ & = 4 \cos(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) \sin(\tfrac{1}{2} \gamma \delta K_{\varphi}(v_+)) \sin(f A_x - \eta(v) + \eta(v_+)) \\ & \quad \times \left(\cos(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) \hat{V} \sin(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) - \sin(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) \hat{V} \cos(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) \right) \end{aligned} \quad (42)$$

The spin connection component $\Gamma_{\varphi} = -\frac{E^{x'}}{2E^{\varphi}}$ and its integrated spatial derivative $\int_{\sigma} \Gamma'_{\varphi} = \Gamma_{\varphi}(v_{\sigma}) - \Gamma_{\varphi}(v)$ can be quantized using flux operators, where we need to choose a discretization for the derivatives and be careful with the inverse of E^{φ} because the flux operators are not invertible. The latter problem can be solved in a manner by now common in loop quantum gravity, expressing the classical inverse as a Poisson bracket between holonomies and only positive powers of flux operators [54]. The derivatives are also straightforward to deal with because the total expressions are scalar. We can thus write

$$\Gamma_{\varphi}(v) = -\frac{E^{x'}}{2E^{\varphi}} = -\frac{1}{4} \left(\frac{E^x(v_+) - E^x(v)}{\int_+ E^{\varphi}} - \frac{E^x(v_-) - E^x(v)}{\int_- E^{\varphi}} \right) + O(\epsilon),$$

treating the two neighboring vertices symmetrically, where now all expressions in the numerators and denominators are scalar and can be quantized. Note that in this manner $\hat{\Gamma}_{\varphi}$ becomes non-zero only in vertices. There are several choices involved in the construction, choosing a quantizable form for the inverse [57, 58] and a discretization of the derivative, but qualitative aspects to be discussed in what follows are not affected.

5.2.1 Action

To write down the action on triad eigenstates explicitly it is convenient to split the vertex contribution to the operator into three parts, $\hat{H}_v = \hat{H}_L + \hat{H}_C + \hat{H}_R$ with

$$\hat{H}_{R/L} = \frac{-i}{\pi G \gamma^3 \delta^2 \ell_P^2} \cos(\tfrac{1}{2} \gamma \delta K_{\varphi}(v)) \sin(\tfrac{1}{2} \gamma \delta K_{\varphi}(v_{\pm})) \sin \left(\int_v^{v_{\pm}} A_x - \eta(v) + \eta(v_{\pm}) \right) \Delta_{\varphi} \hat{V} \quad (43)$$

depending on K_{φ} in v and v_+ or v_- , respectively (receiving contributions from the two bottom lines in (41) for $\sigma = +$ for R and $\sigma = -$ for L) and

$$\hat{H}_C = \frac{-i}{\pi G \gamma^3 \delta^2 \ell_P^2} \left(\sin^2(\gamma \delta K_{\varphi}) \Delta_x \hat{V} + \gamma^2 \delta^2 (1 - \hat{\Gamma}_{\varphi}^2) \Delta_x \hat{V} + \gamma^2 \delta \hat{\Gamma}'_{\varphi} \Delta_{\varphi} \hat{V} \right) + \hat{H}_{\text{matter},v} \quad (44)$$

depending on K_φ only in v (with contributions from the top lines in (41) for both $\sigma = +$ and $\sigma = -$) where

$$\Delta_x \hat{V} := \cos(\tfrac{1}{2} \int A_x) \hat{V} \sin(\tfrac{1}{2} \int A_x) - \sin(\tfrac{1}{2} \int A_x) \hat{V} \cos(\tfrac{1}{2} \int A_x) \quad (45)$$

and

$$\Delta_\varphi \hat{V} := \cos(\tfrac{1}{2} \gamma \delta K_\varphi(v)) \hat{V} \sin(\tfrac{1}{2} \gamma \delta K_\varphi(v)) - \sin(\tfrac{1}{2} \gamma \delta K_\varphi(v)) \hat{V} \cos(\tfrac{1}{2} \gamma \delta K_\varphi(v)). \quad (46)$$

Since the expression for V in terms of k_+ and k_- only depends on the sum $k_+ + k_-$ and the operator $\Delta_x \hat{V}$ turns out to be diagonal on triad eigenstates, we do not need to distinguish between the versions integrating to v_+ and v_- , respectively, if v_\pm are new vertices. This is different if v_\pm already exist as vertices of the original graph which, however, does not crucially change coefficients. We will thus suppress this additional possibility in the notation.

Acting on a vertex v , only labels of that vertex, μ , and its two neighboring ones, μ_\pm as well as the connecting edge labels, k_\pm are changed. We therefore drop all other labels in the notation for states $|\mu_-, k_-, \mu, k_+, \mu_+\rangle$, symbolically

$$|\mu_-, k_-, \mu, k_+, \mu_+\rangle = \begin{array}{c} \dots \quad k_- \quad k_+ \quad \dots \\ \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \\ \mu_- \quad \mu \quad \mu_+ \end{array}, \quad (47)$$

which have connection representation

$$\begin{aligned} \langle K_\varphi, A_x | \mu_-, k_-, \mu, k_+, \mu_+ \rangle &:= \exp(i\mu_- \gamma K_\varphi(v_-)) \exp\left(\frac{1}{2} i k_- \int_{v_-}^v (A_x + \eta') dx\right) \exp(i\mu \gamma K_\varphi(v)) \\ &\quad \exp\left(\frac{1}{2} i k_+ \int_v^{v_+} (A_x + \eta') dx\right) \exp(i\mu_+ \gamma K_\varphi(v_+)). \end{aligned}$$

The action of the contributions to the Hamiltonian constraint then is, first,

$$\begin{aligned} \hat{H}_C |\vec{\mu}, \vec{k}\rangle &= \frac{\ell_P}{2\sqrt{2}G\gamma^{3/2}\delta^2} \left(|\mu| \left(\sqrt{|k_+ + k_- + 1|} - \sqrt{|k_+ + k_- - 1|} \right) \right. \\ &\quad \times (|\mu_-, k_-, \mu + 2\delta, k_+, \mu_+\rangle + |\mu_-, k_-, \mu - 2\delta, k_+, \mu_+\rangle \\ &\quad \left. - 2(1 + 2\gamma^2\delta^2(1 - \Gamma_\varphi^2(\vec{\mu}, \vec{k}))) |\mu_-, k_-, \mu, k_+, \mu_+\rangle \right) \\ &\quad - 4\gamma^2\delta^2 \operatorname{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|} \Gamma'_\varphi(\vec{\mu}, \vec{k}) |\mu_-, k_-, \mu, k_+, \mu_+\rangle \\ &\quad + \hat{H}_{\text{matter},v} |\mu_-, k_-, \mu, k_+, \mu_+\rangle, \end{aligned} \quad (48)$$

where for Γ_φ and its derivative we have to insert eigenvalues as functions of graph labels. Depending on the quantization chosen, this may require further labels beyond k_+ and k_- written here explicitly. Similarly, the expression $\sqrt{|k_+ + k_- + 1|} - \sqrt{|k_+ + k_- - 1|}$ which occurs if v_\pm were not vertices of the original graph can depend on other labels if those vertices were already present; these possibilities will be discussed below but coefficients

here do not change the main results of the present paper. In the term containing Γ'_φ we introduced the function

$$\text{sgn}_{\delta/2}(\mu) := \frac{1}{\delta}(|\mu + \delta/2| - |\mu - \delta/2|) = \begin{cases} 1 & \text{for } \mu \geq \delta/2 \\ 2\mu/\delta & \text{for } -\delta/2 < \mu < \delta/2 \\ -1 & \text{for } \mu \leq -\delta/2 \end{cases}$$

as it follows from $\Delta_\varphi \hat{V}$ and also occurs in

$$\begin{aligned} \hat{H}_R |\vec{\mu}, \vec{k}\rangle &= \frac{\ell_P}{4\sqrt{2}G\gamma^{3/2}\delta^2} \text{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|} \\ &\times (|\mu_-, k_-, \mu + \frac{1}{2}\delta, k_+ + 2, \mu_+ + \frac{1}{2}\delta\rangle - |\mu_-, k_-, \mu + \frac{1}{2}\delta, k_+ + 2, \mu_+ - \frac{1}{2}\delta\rangle \\ &+ |\mu_-, k_-, \mu - \frac{1}{2}\delta, k_+ + 2, \mu_+ + \frac{1}{2}\delta\rangle - |\mu_-, k_-, \mu - \frac{1}{2}\delta, k_+ + 2, \mu_+ - \frac{1}{2}\delta\rangle \\ &- |\mu_-, k_-, \mu + \frac{1}{2}\delta, k_+ - 2, \mu_+ + \frac{1}{2}\delta\rangle + |\mu_-, k_-, \mu + \frac{1}{2}\delta, k_+ - 2, \mu_+ - \frac{1}{2}\delta\rangle \\ &- |\mu_-, k_-, \mu - \frac{1}{2}\delta, k_+ - 2, \mu_+ + \frac{1}{2}\delta\rangle + |\mu_-, k_-, \mu - \frac{1}{2}\delta, k_+ - 2, \mu_+ - \frac{1}{2}\delta\rangle) \end{aligned} \quad (49)$$

and analogously for \hat{H}_L where k_- and μ_- change instead of k_+ and μ_+ .

It is convenient to suppress the vertex labels and write explicitly only changes in k_e with coefficients given by vertex operators changing only vertex labels but potentially depending on neighboring edge labels. To that end we introduce $\hat{C}_0(\vec{k}) := \hat{H}_C$ together with

$$\begin{aligned} \hat{C}_{R\pm}(\vec{k}) |\mu_-, \mu, \mu_+\rangle &:= \pm \frac{\ell_P}{4\sqrt{2}G\gamma^{3/2}\delta^2} \text{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|} (|\mu_-, \mu + \frac{1}{2}\delta, \mu_+ + \frac{1}{2}\delta\rangle \\ &- |\mu_-, \mu + \frac{1}{2}\delta, \mu_+ - \frac{1}{2}\delta\rangle + |\mu_-, \mu - \frac{1}{2}\delta, \mu_+ + \frac{1}{2}\delta\rangle \\ &- |\mu_-, \mu - \frac{1}{2}\delta, \mu_+ - \frac{1}{2}\delta\rangle) \end{aligned} \quad (50)$$

$$\begin{aligned} \hat{C}_{L\pm}(\vec{k}) |\mu_-, \mu, \mu_+\rangle &:= \pm \frac{\ell_P}{4\sqrt{2}G\gamma^{3/2}\delta^2} \text{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|} (|\mu_- + \frac{1}{2}\delta, \mu + \frac{1}{2}\delta, \mu_+\rangle \\ &- |\mu_- - \frac{1}{2}\delta, \mu + \frac{1}{2}\delta, \mu_+\rangle + |\mu_- + \frac{1}{2}\delta, \mu - \frac{1}{2}\delta, \mu_+\rangle \\ &- |\mu_- - \frac{1}{2}\delta, \mu - \frac{1}{2}\delta, \mu_+\rangle) \end{aligned} \quad (51)$$

and the constraint becomes schematically

$$\begin{aligned} \hat{H}[N]\psi(\vec{k}) &= \sum_v N(v) (\hat{C}_0(\vec{k})\psi(\dots, k_-, k_+, \dots) \\ &+ \hat{C}_{R+}(\vec{k})\psi(\dots, k_-, k_+ + 2, \dots) + \hat{C}_{R-}(\vec{k})\psi(\dots, k_-, k_+ - 2, \dots) \\ &+ \hat{C}_{L+}(\vec{k})\psi(\dots, k_- + 2, k_+, \dots) + \hat{C}_{L-}(\vec{k})\psi(\dots, k_- - 2, k_+, \dots)). \end{aligned} \quad (52)$$

5.2.2 Difference equation

Until now, the particular choice of vertices v_\pm only affected the form of coefficients in \hat{C}_0 which will not be very important for understanding the evolution scheme. Now, we assume that v_\pm have already been present in the graph before acting with the constraint operator.

The number of labels in a state then does not change after acting and we can write the constraint equation equivalently as a set of coupled difference equations.

Upon transforming to the triad representation by expanding $|\psi\rangle = \sum_{\vec{k}, \vec{\mu}} \psi(\vec{k}, \vec{\mu}) |\vec{k}, \vec{\mu}\rangle$, the constraint equation $\hat{H}[N]|\psi\rangle = 0$ for all N becomes equivalent to the set

$$\begin{aligned} & \hat{C}_{R+}(k_-, k_+ - 2)^\dagger \psi(\dots, k_-, k_+ - 2, \dots) + \hat{C}_{R-}(k_-, k_+ + 2)^\dagger \psi(\dots, k_-, k_+ + 2, \dots) \\ & + \hat{C}_{L+}(k_- - 2, k_+)^\dagger \psi(\dots, k_- - 2, k_+, \dots) + \hat{C}_{L-}(k_- + 2, k_+)^\dagger \psi(\dots, k_- + 2, k_+, \dots) \\ & + \hat{C}_0(k_-, k_+)^\dagger \psi(\dots, k_-, k_+, \dots) = 0 \end{aligned} \quad (53)$$

of coupled difference equations, one for each vertex. The adjoints are taken in the vertex Hilbert spaces only, i.e.

$$\begin{aligned} \hat{C}_{R\pm}(k)^\dagger |\mu_-, \mu, \mu_+\rangle & := \mp \frac{\ell_P}{4\sqrt{2}G\gamma^{3/2}\delta^2} \sqrt{|k_+ + k_-|} \left(\text{sgn}_{\delta/2}(\mu + \frac{1}{2}\delta) |\mu_-, \mu + \frac{1}{2}\delta, \mu_+ + \frac{1}{2}\delta\rangle \right. \\ & \quad - \text{sgn}_{\delta/2}(\mu + \frac{1}{2}\delta) |\mu_-, \mu + \frac{1}{2}\delta, \mu_+ - \frac{1}{2}\delta\rangle \\ & \quad + \text{sgn}_{\delta/2}(\mu - \frac{1}{2}\delta) |\mu_-, \mu - \frac{1}{2}\delta, \mu_+ + \frac{1}{2}\delta\rangle \\ & \quad \left. - \text{sgn}_{\delta/2}(\mu - \frac{1}{2}\delta) |\mu_-, \mu - \frac{1}{2}\delta, \mu_+ - \frac{1}{2}\delta\rangle \right) \end{aligned} \quad (54)$$

So far, we used the ordering with the quantization of triad components to the right, and the resulting operators are thus not symmetric. After the construction we can now also consider other orderings, most importantly the symmetric one. For this ordering, the coefficient operator $\hat{C}_{R+}(k_-, k_+ - 2)^\dagger$ in (53) is replaced by $\frac{1}{2}(\hat{C}_{R+}(k_-, k_+ - 2)^\dagger + \hat{C}_{R-}(k_-, k_+))$. From (50) and (54) one thus obtains coefficients proportional to $\text{sgn}_{\delta/2}(\mu \pm \frac{1}{2}\delta) \sqrt{|k_+ + k_- - 2|} + \text{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|}$ for the terms multiplying $\psi(\dots, k_-, k_+ - 2, \dots)$, the sign in $\mu \pm \delta/2$ depending on whether μ is raised or lowered in the term. It is easy to see that these values, unlike the coefficients $\text{sgn}_{\delta/2}(\mu) \sqrt{|k_+ + k_-|}$ for the non-symmetric ordering, are non-zero for all k_\pm if $\mu \neq \mp \frac{1}{4}\delta$. If $\mu = \mp \frac{1}{4}\delta$, the coefficients become zero for $k_+ + k_- = 1$. The coefficients in the non-symmetric ordering, on the other hand, are zero if $k_+ + k_- = 0$ irrespective of the value of μ . This has consequences for the singularity problem as discussed briefly below and in more detail in [59].

5.3 Regularization issues and anomalies

The special nature of 1-dimensional graphs relevant for spherically symmetric spin network states makes some issues in the context of the Hamiltonian constraint more complicated than in the full theory. For instance, to have a well-defined operator after regulators are removed it is necessary that the constraint operator (41) annihilates spin network states based on graphs without any vertices. Otherwise, there would remain infinitely many contributions of the same non-zero value after the continuum limit is performed.

5.3.1 Action on special vertices

For this issue it is sufficient to consider the commutators of the volume operator with holonomies appearing on the right hand side of the constraint operator. While the volume

operator annihilates states without vertices, this is not obvious for the second contribution to the commutator where we first act with a holonomy. The explicit form of the matrix elements, however, shows that the commutators appearing in the constraint always annihilate states with zero vertex label μ_v such that we obtain a well-defined operator from the non-symmetric ordering in the continuum limit. Note that the argumentation here is less trivial than in the full theory, where one was able to refer to planarity of vertices obtained after multiplying with an edge holonomy in the commutator. Here, one has to use an explicit computation of the commutator which then results in the same conclusion.

As in the full theory, however, we do not obtain a well-defined operator if we use the symmetric ordering before removing the regulator. Nevertheless, one can certainly take the continuum limit of the non-symmetric operator, which is densely defined, and symmetrize it by adding its adjoint. This does result in a well-defined symmetric operator, showing that there is still an ordering ambiguity, but it is different from the symmetric ordering considered before because the adjoint removes vertices. This remark also applies to the full theory: regularization arguments cannot fix ordering ambiguities since a reordering is always possible, and may even be forced for other reasons, after the construction of operators. To fix ordering ambiguities one needs additional arguments, such as the singularity statements provided below.

A closer look at the behavior in vertices reveals that the situation can be seen as related to planarity at least heuristically. For the conclusion that the action of the Hamiltonian constraint is zero if there is no vertex relies on the presence of $\text{sgn}(\mu_v)$ in some terms which would otherwise not vanish for $\mu_v = 0$. (Since some of the signs are replaced by $\text{sgn}_{\delta/2}(\mu_v \pm \delta/2)$ in the symmetric ordering, the continuum limit does not exist in this case.) The absolute values, as noted earlier, appear because of the residual gauge transformation $\mu_v \mapsto -\mu_v$, which itself is a consequence of isotropy in the spherical orbits. Connection components in the ϑ - and φ -directions then merge into one local kinematical degree of freedom, and in that sense vertices created by a single holonomy, as they appear in commutators, can be considered planar. This is not automatic, however, for more general vertices since they can be interpreted as incorporating φ as well as ϑ information and are thus not annihilated. In polarized cylindrical symmetry, where similar states and operators can be written, there is no such isotropy in orbits and there are two independent vertex labels. Planar vertices then only arise if one of the labels vanishes, in which case they are automatically annihilated by commutators as they always contain at least one of the two vertex labels as a factor.

Similarly, in the full theory one can use the planar nature of newly created vertices to conclude that two Hamiltonian constraint operators with different lapse functions commute [38] and then discuss the issue of anomalies. Again, we do not have this argument at our disposal since there is no notion of planarity in the reduced, 1-dimensional manifold. If the interpretation of planarity given above is used, there is no simple argument since now different holonomies, one for the vertex dependence and one for the commutator, have to be considered. This may indicate that the argument for anomaly freedom in the full theory is too simple, as it has indeed been criticized before [60, 61] on heuristic grounds of not giving local propagating degrees of freedom, and for other reasons more recently

in [62]. In this context, one-dimensional models can be helpful because they provide two treatable situations where local physical degrees may (polarized Einstein–Rosen) or may not (spherical symmetry) be present (see also [4]). Nevertheless, the Hamiltonian constraint operators appear very similar such that their properties do not obviously show how propagating degrees of freedom would be realized.

5.3.2 Quantization without creating new vertices

To understand this issue it is helpful to consider different versions of the constraint operator, and different regularization procedures. The main question is how the new vertices v_{\pm} , analogous to new edges in the full theory, are chosen. The standard way is to create a new vertex next to v not present in the original graph. In the limit removing the regulator, v_{\pm} approach v which allows the expansion of edge holonomies. This is the situation where one needs to make sure that the operator has zero action if there is no vertex present originally because the classical constraint regulated by a Riemann sum has contributions everywhere. It is also the situation where, in the full theory, the argument for anomaly freedom of [63] applies.

Alternatively, v_{\pm} can be chosen to be always the next vertices already present in the original state. This would clearly prohibit the resulting operator from being viewed as a regulated expression as before, since v_{\pm} cannot approach v at a fixed state. The viewpoint here is instead that one has an operator which reproduces the classical expression only for suitable semiclassical states, which still is to be proven, but not on arbitrary states (which in general show quantum behavior or at least corrections). The first property would be one part of the justification for using that operator, and one condition for it to be given is that expansions of holonomies and exponentials are valid in semiclassical regimes, which we have demonstrated before. In addition, there must be other consistency conditions which guarantee that the solution space to the operator is big enough for sufficiently many semiclassical states. Here, one usually encounters the anomaly issue: If the classical constraint algebra is not mimicked in a certain sense by the quantum operators, the solution space can be too strongly restricted. In what sense this is to be ensured will have to be determined by understanding the physics of models.

In the prescription for the constraint where v_{\pm} are already present as vertices originally, the technical issue of verifying the commutator algebra is more involved but in addition to the fact that the symmetric ordering is here well-defined it has the advantage that the constraint equations can more easily be written as a system of coupled difference equations. When v_{\pm} are created as new vertices, on the other hand, there will be new degrees of freedom involved after each action of the constraint which makes it more difficult to be formulated as a set of equations with a given number of independent variables. When difference equations are available, the anomaly issue is related to the question of whether or not all coupled difference equations are consistent with each other, i.e. whether or not a consistent recurrence scheme can be formulated to solve the equations from given initial and boundary conditions. This turns out to be possible [59] despite of the fact that the absence of anomalies is not clear yet. There are thus sufficiently many solutions for this

version of the operator. Still, an analysis of possible anomalies is of considerable interest as it can, for instance, reduce the ambiguities discussed, e.g., in [64], but has to rely on explicit computations which will not be pursued here.

5.3.3 Diffeomorphism constraint

In the preceding paragraphs we only discussed the Hamiltonian constraint and operator equations it implies, but the constraint algebra also has to be tested in combination with the diffeomorphism constraint. As in the full theory [65], there is no operator for the infinitesimal diffeomorphism constraint, but the exponentiated version corresponding to finite diffeomorphisms can be quantized analogously [31]. It simply moves vertices implying that their position in the background manifold B has no physical meaning. Also the coordinate position of vertices v_{\pm} , if they are created by the Hamiltonian constraint, is meaningless at the diffeomorphism invariant level, but creating a new vertex between two given ones is still meaningful at the level of diffeomorphism classes of graphs. The version of the constraint operator where no new vertices are created can thus be formulated equally at the diffeomorphism invariant or non-invariant sectors. When new vertices are created, on the other hand, we need to choose their positions, or work directly on diffeomorphism invariant classes without any such choice being necessary.

The latter procedure, i.e. formulating Hamiltonian constraints which create new vertices directly at the diffeomorphism invariant level can indeed be done despite general statements to the contrary sometimes encountered in the literature. Those statements point out the fact that the lapse function N appears in the smeared Hamiltonian constraint and also in a quantization such as (40). The factors $N(v)$ in vertex contributions of the operator are not diffeomorphism invariant which renders a simultaneous imposition of the Hamiltonian and diffeomorphism constraints impossible.

However, after quantization each vertex contribution is a well-defined operator and has to be imposed independently because the $N(v)$ are free. The smeared constraint operator should be seen as a collection of all the individual vertex constraint operators in a compact manner. In contrast to the classical case where only the smeared constraints have well-defined Poisson brackets, the vertex operators have well-defined commutators as a result of the spatial discreteness contained in a spin network state. One can thus consistently work with the vertex contributions \hat{H}_v for all $v \in B$ (which are zero whenever \hat{H}_v acts on a state which does not have v as a vertex) together with the diffeomorphism constraint. Since $N(v)$ is no longer involved, there is no problem of defining the Hamiltonian constraints on the diffeomorphism invariant sector.

6 Discussion

In this paper we have constructed the Hamiltonian constraint operator for spherically symmetric models within a loop quantization and computed its full action. Explicit calculations were facilitated by a choice of new variables which are a mixture of connection

and extrinsic curvature components. This led to several simplifications as compared to the pure connection variables used previously in [31, 32, 39, 40], all related to the fact that the volume operator simplifies. A volume operator with explicitly known spectrum is also available for the pure connection variables [32], but its eigenstates are not identical to flux eigenstates. This fact implies complicated expressions for commutators of the volume operator with holonomies which appear in the Hamiltonian constraint. In particular the commutator $[h_\varphi, \hat{V}] = [\cos(\frac{1}{2}A_\varphi), \hat{V}] + 2[\sin(\frac{1}{2}A_\varphi)\Lambda_\varphi^A(\beta), \hat{V}]$ with an angular holonomy is cumbersome since $\sin \frac{1}{2}A_\varphi$ has a complicated action on volume eigenstates. Moreover, in this framework β does not commute with the volume operator such that there are several different commutators in the constraint.

In the new variables introduced here, on the other hand, volume eigenstates are identical to flux eigenstates and the relevant angle η does commute with the volume. Thus, there are less commutators with different action, and each of them is easy to compute. The relevant calculations of matrix elements of the constraint operator are no more involved than in homogeneous cases [6, 8]; only the constraint equation itself is more complicated to solve or discuss since the system involves infinitely many kinematical degrees of freedom.

6.1 Testing the full theory

Symmetric models are introduced to test a possible full theory in simpler situations and to derive physical applications in a more direct way. For reliable results it is essential that a model is as close to a full formulation as possible in order to ensure that there are no artefacts from using simplifications of the model.¹ In this respect one may worry that the introduction of new variables here spoils the relation to the full theory since it leads to key simplifications in an otherwise barely treatable system. Yet, the quantization in new variables is in many respects *closer* to the full theory than the previous one, which we illustrate with the following examples in the context of this paper.

First, the volume operator is constructed immediately from flux operators and does not contain functions of connection components which need to be rewritten before they can be represented on the Hilbert space. Thus, the volume operator is less ambiguous than in the pure connection variables of [32], a feature shared by both the full volume operator [33, 34, 66] and that of homogeneous models [67]. In this context we can also discuss the issue of level splitting which was observed even for a single vertex contribution of the volume operator in [31]. With the simple volume operator derived here such a level splitting does not occur, and the vertex spectrum is identical to that of a homogeneous

¹This is certainly complicated in loop quantum gravity, not the least because the specific full theory is unknown so far. Nevertheless, if models are widely studied and general realizations with welcome properties, such as those concerning singularities or conceptual and phenomenological aspects, have been identified, one should hope that a full theory can be realized in such a manner that it reduces to those model situations in corresponding regimes. This viewpoint puts the burden on constructing the full theory as well as understanding its reduction. The issue is general since one is always forced to employ approximations, whether by reduction or otherwise, to understand physical applications of the full theory. Without those applications, blind constructions of possible full theories are physically empty.

model which has a single rotational axis through each point. Level splitting then occurs only when several vertices are considered, which explicitly brings in the inhomogeneous properties.

Secondly, the construction of the Hamiltonian constraint operator fits in the general scheme developed in the full theory in [38] and generalized to homogeneous models in [56, 8, 9]. In particular the presence of possibly non-vanishing covariant components of the spin connection in a symmetric model has to be dealt with in a special way compared to the full theory. This at first makes it more difficult to interpret the results since the relation to the full theory is not as close. However, the fact that it is possible to treat all homogeneous models in the same way strongly suggests that there is a general procedure and results are not caused by artefacts in a symmetric context. In this paper we have seen that this general procedure even extends to inhomogeneous symmetric models which further supports the whole construction. The new variables introduced here, even though they were motivated independently by implying a simplified volume operator, automatically implement the required subtraction of the spin connection in homogeneous components.

This is a very non-trivial test of the loop quantization procedure: by using these variables both the volume operator and the Hamiltonian constraint become closer to what is known from homogeneous models and the full theory. Moreover, this works only when the special form of spherically symmetric spin connections and extrinsic curvatures (or those of the Einstein–Rosen model for which the same procedure works) is taken into account. Otherwise, it would be far from clear that the canonical transformation employed here to make triad components into momenta amounts to a subtraction of homogeneous spin connection components. The spherically symmetric model is in between homogeneous models and the full theory, and indeed one can observe both homogeneous and inhomogeneous aspects. The subtraction is done only in the angular components which correspond to homogeneous directions, while for inhomogeneous directions we still have to use the connection component A_x as would be the case in the full theory. This automatically arises from the canonical transformation, and is essential in obtaining a constraint operator with the correct classical limit.

One may think that by way of our canonical transformation we go back from Ashtekar variables to ADM like variables with extrinsic curvature as configuration variable. This is, however, only partially true since we do not use the x -component of extrinsic curvature, but the Ashtekar connection component instead. Thus, only homogeneous directions are affected by the transformation, while the inhomogeneous components are left unchanged. In fact, using extrinsic curvature components throughout, which would be possible from the point of view of the symplectic structure, would result in a different Hamiltonian constraint operator. For instance, the dependence on η in (42) results from insertions of Λ in (41), not from η in holonomies as it would happen if A_x was replaced by K_x . Retaining A_x as configuration variable and as an argument of basic holonomies is crucial for an operator constructed along the lines of the full theory. It is certainly possible to construct constraint operators also with ADM like variables, possibly after employing a Bohr representation as suggested recently [68]. But such a quantization, in contrast to one following a general scheme for the full theory and symmetric models as employed here, does not have contact

with full quantum gravity which makes reliable conclusions more problematic.

Moreover, in ADM variables exponentials are unmotivated in contrast to using holonomies in theories of connections. As demonstrated by different examples in [31, 53, 12], a representation with discontinuous exponentiated connection or extrinsic curvature components is induced in models of loop quantum gravity by the full representation. In [31] it is also shown that in one-dimensional models one could construct many different representations where also fluxes could be quantized by discontinuous exponentials while the conjugate connection components have direct actions without exponentiation. The representation is thus far from unique if only the model is considered, and physical properties can crucially depend on it. Only by relating models to the full theory with its unique representation [69, 70] can a reliable basis for using one representation be given. From this perspective, it may be worthwhile to view the representation constructed in [71], which is related to that of [68] by a gauge fixing, also as a gauge fixing done in a loop formulation removing the connection components A_x which from our point of view cannot be replaced by extrinsic curvature components. This could provide an embedding of that representation in loop quantum gravity.

6.2 Key simplifications in symmetric models

While the general scheme discussed in the preceding subsection shows that results can be trusted as general loop properties, rather than properties belonging to a given model which may or may not be shared by other models or a full theory, it also allows one to see how key simplifications arise. They have already been exploited in physical applications of homogeneous models, and are now also realized in diagonal inhomogeneous models such as the spherically symmetric one or Einstein–Rosen waves. With these models a large class of physical situations, which already has been intensively studied classically and with diverse quantum methods, is now accessible to explicit computations in the loop framework. Applications span all areas of gravitational physics from cosmology to black holes and gravitational waves.

A technical feature shared by all these models is that the relevant form of invariant connections and triads is diagonal, i.e. Lie algebra valued components corresponding to independent directions such as x, ϑ, φ here are perpendicular in the internal direction. How this leads to simplifications has been explained in detail in [31]. This diagonalization essentially amounts to a reduction from $SU(2)$ to $U(1)$ which is the reason for a simpler volume operator. However, as seen by comparing this paper with [32], this reduction in the gauge group is necessary but not sufficient for simplifying the calculations. Also an appropriate form of canonical variables is essential, which connects simplifications in the volume operator with the construction of the Hamiltonian constraint.

The required computations then are quite similar in homogeneous and inhomogeneous models. Only the discussion and solution of the constraint equation is, of course, more complicated in inhomogeneous situations since more degrees of freedom are involved. But in all these models the constraint operator takes an analogous form, and in all models a triad representation is available. The latter feature is the main difference to the full

theory, besides the complication in explicit calculations. Writing the constraint equation in the triad representation is much more intuitive than in the connection representation and has led to many results concerning the quantum structure of classical singularities [17, 18, 72, 73, 74, 75, 16, 8, 9, 59]. Again, one may worry that this difference to the full theory implies special features of models which may be misleading. However, this is not the case since we just transform an equation which we have in the models as well as in the full theory into a different form more suitable to applications. We are, however, not changing the equation itself or its solutions.

In this triad representation we then obtain difference equations, which are of different complexity: ordinary in the vacuum isotropic case, partial in homogeneous models or even partial in infinitely many variables in inhomogeneous models. In all cases, the nature of being a difference equation is derived from holonomy operators appearing in the constraint, with coefficients determined by commutators with the volume operator. Computing those coefficients can be done with equal ease and no new ingredients in all models, and is crucial for the discussion of singularities (see also [46] for a recent summary).

Another shared feature of the models is the relation between the Euclidean and Lorentzian constraints, which is always simple and only involves a rescaling of the holonomy part of the constraint. This is in contrast to the full theory where the relation is much more complicated and the Lorentzian constraint, so far, can only be quantized by introducing additional commutators between the Euclidean constraint and the volume operator [38]. At this point it is not clear if the simplification in symmetric models is general enough to lead to reliable results. This can, however, be tested directly since the procedure of the full theory is certainly available in the models, too. Usually, one first studies the simpler version in order to understand the physical implications, before more complicated choices can be compared (see, e.g., [74] compared to [75]). So far, the more complicated version only implies higher order equations and more involved coefficients, but no crucial differences.

6.3 Physical applications

The key simplifications discussed before indicate that physical applications can be obtained in an explicit form even at the dynamical level. There are several features common with the formalism of homogeneous models, such as the explicitly known matrix elements of the Hamiltonian constraint in the Euclidean and Lorentzian form, the fact that they are sufficiently simple functions of the spin network labels and, crucially, the availability of a triad representation. The latter facilitates in particular the analysis of a neighborhood of classical singularities in order to see if they still present a boundary to the quantum evolution. This is crucial for an interpretation, not the formulation of models; the lack of a triad representation in the full theory is thus not problematic for trusting the model.

However, there are certainly also complications as compared to homogeneous models with only finitely many kinematical degrees of freedom. The spherically symmetric quantum constraint equation in the triad representation is a partial difference equation in potentially infinitely many variables and, depending on the form of the quantization, the number of degrees of freedom involved at each time step may not even be constant (if the

constraint operator creates new vertices). Moreover, there are many more ways to approach a classical singularity on midisuperspace, depending also on gauge choices, and there are different versions of singularities. This makes general statements about their removal or persistence more complicated since for that classical singularities first have to be located in midisuperspace before the possibility of unique extensions of physical wave functions is discussed. Yet, in one-dimensional models this is possible and shows many crucial new features [59]. The ordering, for instance is more restricted than in homogeneous models and a symmetric one is preferred. While homogeneous models are non-singular for the non-symmetric and symmetric ordering because the evolution can be continued even in cases where coefficients of the difference equation can become zero [17], in inhomogeneous models this is not possible at least for the direct operator following from a non-symmetric version. As noted before, the symmetric ordering, on the other hand, has generically non-zero coefficients in such a manner that the evolution does not break down. Requiring a non-singular evolution thus restricts the ordering choices in inhomogeneous models.

For other physical applications it is helpful to have approximation schemes available which allow one to isolate the essential quantum modifications to classical equations. In homogeneous models, effective classical equations of the form introduced in [19, 73, 76] have been essential in many recent physical applications in the context of cosmology [20, 77, 23, 24, 78, 79, 80, 25, 81, 22, 82, 49, 83, 84, 21, 85]. These are classical equations in that they are ordinary differential equations in coordinate time which are much easier to handle than difference equations [18, 73, 74, 75, 86, 87, 88, 89, 90, 16]. Quantum effects are then imported by comparing the expectation value of the quantum constraint with the classical constraint equation, which can also be analyzed and justified by comparing the motion of quantum wave packets with solutions to the effective equations [14, 15]. (This is part of a general procedure which also contains the usual effective action techniques [91, 92, 52, 93].) Similarly, effective classical equations for inhomogeneous models, which then would be partial differential equations in space-time coordinates, would be much easier to deal with than a difference equation in infinitely many variables. Moreover, mathematical expertise from geometrical analysis would be available to arrive at general conclusions.

In cosmological models, several quantum effects have been observed which give rise to different terms in effective classical equations. The first and most direct one was the modification of matter Hamiltonians at small volume [54, 55], which makes classically diverging energy densities finite along effective trajectories and also played a role for the removal of cosmological singularities. The same modification is available in a spherically symmetric model with matter; but since here even a vacuum model would have a classical (Schwarzschild) singularity, it cannot be expected to be sufficient for a non-singular effective formulation.

Other modifications come from the gravitational part of the constraint and are thus available even in the vacuum case. There are perturbative corrections which can be interpreted as being analogous to higher curvature terms, and non-perturbative ones in the case of a non-vanishing spin connection. (The latter can be viewed as providing a natural cut-off on intrinsic curvature, analogously to the extrinsic curvature cut-off from a quantum matter Hamiltonian in a homogeneous situation.) The non-perturbative modifications

have mainly been used so far in the Bianchi IX model where they remove the classical chaos [22, 82]. They are derived by quantizing the spin connection potential term of the Hamiltonian constraint in accordance with the general scheme described above. Since the spin connection contains inverse powers of the triad components, it will obtain modifications from the quantization of inverse powers [54, 55] similarly to a matter Hamiltonian.

Here, the same procedure is available where we also have an intrinsic curvature potential with the spin connection depending on inverse powers of E^φ . Using eigenvalues of the quantized potential in the classical constraint equation then provides effective classical equations showing one main quantum effect from the cut-off of intrinsic curvature. Comparison with the Schwarzschild solution shows that this indeed provides modifications at the right place, namely close to the classical singularity where E^φ is small. In asymptotic regimes or around the horizon of massive black holes, on the other hand, E^φ is large and so the behavior there remains classical. This would be different had we used other metric variables such as the co-triad or metric instead of the densitized triad as, e.g., in [68]. Then the component whose inverse appears in the spin connection (20) would be e_x , which for Schwarzschild is large at the horizon but also at the classical singularity and so there would be no quantum corrections there. In asymptotic regimes, on the other hand, e_x would be of the order one: modifications would be noticeable but completely unwanted. Thus, similarly to homogeneous models [8] we see that the issue of singularity removal will crucially depend on the canonical variables used. Densitized triad components, which we have to use anyway since they are part of the basis of the full background independent quantization, are well-suited for this aim. These statements are only preliminary as of now because no consistent set of effective equations for inhomogeneous cases has been derived so far.

Besides singularities, horizons would be the second interesting aspect to be studied. As just noted, we do not expect strong quantum modifications there at least for massive black holes. But there are other interesting aspects which are usually related to quantum models, such as the issue of black hole entropy, horizon degrees of freedom and fluctuations, and Hawking radiation. All these issues, for instance in the context of the scenario of [27], require solving the Hamiltonian constraint which can provide important feedback on the viability of a given quantization scheme, and thus test the full theory and restrict quantization ambiguities. The isolated or dynamical horizon framework [94, 95, 96, 97, 98, 99] provides an ideal setting for an analysis at the classical and quantum levels. If a horizon is isolated or slowly evolving [100], certain terms in the quantum constraint become negligible such that the constraint simplifies [101]. This provides an approximation scheme to understand the physics at the horizon or, in a perturbative form, around the horizon. Also dynamical processes can then be studied in a controlled manner because not just isolated but also slowly evolving horizons are allowed. In this way one can derive physical information about black hole or other horizons, but also study general issues of the Hamiltonian constraint such as observables.

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References

- [1] C. Rovelli, Loop Quantum Gravity, *Living Rev. Rel.* 1 (1998) 1, [gr-qc/9710008], <http://www.livingreviews.org/Articles/Volume1/1998-1rovelli>; *Quantum Gravity* (Cambridge University Press, Cambridge, 2004)
- [2] T. Thiemann, Introduction to Modern Canonical Quantum General Relativity, [gr-qc/0110034]
- [3] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, *Class. Quantum Grav.* 21 (2004) R53–R152, [gr-qc/0404018]
- [4] M. Bojowald and H. A. Kastrup, Symmetry Reduction for Quantized Diffeomorphism Invariant Theories of Connections, *Class. Quantum Grav.* 17 (2000) 3009–3043, [hep-th/9907042]
- [5] M. Bojowald, Loop Quantum Cosmology: I. Kinematics, *Class. Quantum Grav.* 17 (2000) 1489–1508, [gr-qc/9910103]
- [6] M. Bojowald, Isotropic Loop Quantum Cosmology, *Class. Quantum Grav.* 19 (2002) 2717–2741, [gr-qc/0202077]
- [7] A. Ashtekar, M. Bojowald, and J. Lewandowski, Mathematical structure of loop quantum cosmology, *Adv. Theor. Math. Phys.* 7 (2003) 233–268, [gr-qc/0304074]
- [8] M. Bojowald, Homogeneous loop quantum cosmology, *Class. Quantum Grav.* 20 (2003) 2595–2615, [gr-qc/0303073]
- [9] M. Bojowald, G. Date, and K. Vandersloot, Homogeneous loop quantum cosmology: The role of the spin connection, *Class. Quantum Grav.* 21 (2004) 1253–1278, [gr-qc/0311004]
- [10] M. Bojowald and H. A. Morales-Técotl, Cosmological applications of loop quantum gravity, In: Proceedings of the Fifth Mexican School (DGFM): The Early Universe and Observational Cosmology, *Lect. Notes Phys.* 646 (2004) 421–462 (Springer-Verlag), [gr-qc/0306008]
- [11] M. Bojowald, Loop quantum cosmology, In: *100 Years of Relativity – Space-Time Structure: Einstein and Beyond*, Ed.: A. Ashtekar, World Scientific, Singapore, 2005, [gr-qc/0505057]
- [12] M. Bojowald, Loop Quantum Cosmology, *Living Rev. Relativity*, to appear

- [13] M. Bojowald, The Semiclassical Limit of Loop Quantum Cosmology, *Class. Quantum Grav.* 18 (2001) L109–L116, [gr-qc/0105113]
- [14] M. Bojowald, P. Singh, and A. Skirzewski, Coordinate time dependence in quantum gravity, *Phys. Rev. D* 70 (2004) 124022, [gr-qc/0408094]
- [15] P. Singh and K. Vandersloot, Semi-classical States, Effective Dynamics and Classical Emergence in Loop Quantum Cosmology, *Phys. Rev. D* 72 (2005) 084004, [gr-qc/0507029]
- [16] A. Ashtekar, T. Pawłowski, and P. Singh, in preparation
- [17] M. Bojowald, Absence of a Singularity in Loop Quantum Cosmology, *Phys. Rev. Lett.* 86 (2001) 5227–5230, [gr-qc/0102069]
- [18] M. Bojowald, Dynamical Initial Conditions in Quantum Cosmology, *Phys. Rev. Lett.* 87 (2001) 121301, [gr-qc/0104072]
- [19] M. Bojowald, Inflation from quantum geometry, *Phys. Rev. Lett.* 89 (2002) 261301, [gr-qc/0206054]
- [20] S. Tsujikawa, P. Singh, and R. Maartens, Loop quantum gravity effects on inflation and the CMB, *Class. Quantum Grav.* 21 (2004) 5767–5775, [astro-ph/0311015]
- [21] G. M. Hossain, Primordial Density Perturbation in Effective Loop Quantum Cosmology, *Class. Quantum Grav.* 22 (2005) 2511–2532, [gr-qc/0411012]
- [22] M. Bojowald and G. Date, Quantum suppression of the generic chaotic behavior close to cosmological singularities, *Phys. Rev. Lett.* 92 (2004) 071302, [gr-qc/0311003]
- [23] P. Singh and A. Toporensky, Big Crunch Avoidance in $k = 1$ Semi-Classical Loop Quantum Cosmology, *Phys. Rev. D* 69 (2004) 104008, [gr-qc/0312110]
- [24] J. E. Lidsey, D. J. Mulryne, N. J. Nunes, and R. Tavakol, Oscillatory Universes in Loop Quantum Cosmology and Initial Conditions for Inflation, *Phys. Rev. D* 70 (2004) 063521, [gr-qc/0406042]
- [25] D. J. Mulryne, R. Tavakol, J. E. Lidsey, and G. F. R. Ellis, An emergent universe from a loop, *Phys. Rev. D* 71 (2005) 123512, [astro-ph/0502589]
- [26] M. Bojowald, R. Goswami, R. Maartens, and P. Singh, A black hole mass threshold from non-singular quantum gravitational collapse, *Phys. Rev. Lett.* 95 (2005) 091302, [gr-qc/0503041]
- [27] A. Ashtekar and M. Bojowald, Black hole evaporation: A paradigm, *Class. Quantum Grav.* 22 (2005) 3349–3362, [gr-qc/0504029]

- [28] T. Thiemann and H. A. Kastrup, Canonical Quantization of Spherically Symmetric Gravity in Ashtekar’s Self-Dual Representation, *Nucl. Phys. B* 399 (1993) 211–258, [gr-qc/9310012]
- [29] H. A. Kastrup and T. Thiemann, Spherically Symmetric Gravity as a Completely Integrable System, *Nucl. Phys. B* 425 (1994) 665–686, [gr-qc/9401032]
- [30] K. V. Kuchař, Geometrodynamics of Schwarzschild Black Holes, *Phys. Rev. D* 50 (1994) 3961–3981
- [31] M. Bojowald, Spherically Symmetric Quantum Geometry: States and Basic Operators, *Class. Quantum Grav.* 21 (2004) 3733–3753, [gr-qc/0407017]
- [32] M. Bojowald and R. Swiderski, The Volume Operator in Spherically Symmetric Quantum Geometry, *Class. Quantum Grav.* 21 (2004) 4881–4900, [gr-qc/0407018]
- [33] C. Rovelli and L. Smolin, Discreteness of Area and Volume in Quantum Gravity, *Nucl. Phys. B* 442 (1995) 593–619, [gr-qc/9411005], Erratum: *Nucl. Phys. B* 456 (1995) 753
- [34] A. Ashtekar and J. Lewandowski, Quantum Theory of Geometry II: Volume Operators, *Adv. Theor. Math. Phys.* 1 (1997) 388–429, [gr-qc/9711031]
- [35] R. De Pietri and C. Rovelli, Geometry Eigenvalues and the Scalar Product from Recoupling Theory in Loop Quantum Gravity, *Phys. Rev. D* 54 (1996) 2664–2690
- [36] T. Thiemann, Closed Formula for the Matrix Elements of the Volume Operator in Canonical Quantum Gravity, *J. Math. Phys.* 39 (1998) 3347–3371, [gr-qc/9606091]
- [37] J. Brunnemann and T. Thiemann, Simplification of the Spectral Analysis of the Volume Operator in Loop Quantum Gravity, [gr-qc/0405060]
- [38] T. Thiemann, Quantum Spin Dynamics (QSD), *Class. Quantum Grav.* 15 (1998) 839–873, [gr-qc/9606089]
- [39] D. E. Neville, The volume operator for spin networks with planar or cylindrical symmetry, [gr-qc/0511005]
- [40] D. E. Neville, The volume operator for singly polarized gravity waves with planar or cylindrical symmetry, [gr-qc/0511006]
- [41] A. Ashtekar, New Hamiltonian Formulation of General Relativity, *Phys. Rev. D* 36 (1987) 1587–1602
- [42] J. F. Barbero G., Real Ashtekar Variables for Lorentzian Signature Space-Times, *Phys. Rev. D* 51 (1995) 5507–5510, [gr-qc/9410014]

- [43] G. Immirzi, Real and Complex Connections for Canonical Gravity, *Class. Quantum Grav.* 14 (1997) L177–L181
- [44] M. Bojowald and G. Date, Consistency conditions for fundamentally discrete theories, *Class. Quantum Grav.* 21 (2004) 121–143, [gr-qc/0307083]
- [45] A. Ashtekar and M. Bojowald, Quantum Geometry and the Schwarzschild Singularity, [gr-qc/0509075]
- [46] M. Bojowald, Degenerate Configurations, Singularities and the Non-Abelian Nature of Loop Quantum Gravity, [gr-qc/0508118]
- [47] T. Thiemann, Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories, *Class. Quantum Grav.* 15 (1998) 1487–1512, [gr-qc/9705021]
- [48] A. Ashtekar, J. Lewandowski, and H. Sahlmann, Polymer and Fock representations for a Scalar field, *Class. Quantum Grav.* 20 (2003) L11–L21, [gr-qc/0211012]
- [49] G. Date and G. M. Hossain, Effective Hamiltonian for Isotropic Loop Quantum Cosmology, *Class. Quantum Grav.* 21 (2004) 4941–4953, [gr-qc/0407073]
- [50] K. Banerjee and G. Date, Discreteness Corrections to the Effective Hamiltonian of Isotropic Loop Quantum Cosmology, *Class. Quant. Grav.* 22 (2005) 2017–2033, [gr-qc/0501102]
- [51] K. Vandersloot, On the Hamiltonian Constraint of Loop Quantum Cosmology, *Phys. Rev. D* 71 (2005) 103506, [gr-qc/0502082]
- [52] A. Ashtekar, M. Bojowald, and J. Willis, in preparation
- [53] M. Bojowald, H. H. Hernández, and H. A. Morales-Técolt, Perturbative degrees of freedom in loop quantum gravity: Anisotropies, [gr-qc/0511058]
- [54] T. Thiemann, QSD V: Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories, *Class. Quantum Grav.* 15 (1998) 1281–1314, [gr-qc/9705019]
- [55] M. Bojowald, Inverse Scale Factor in Isotropic Quantum Geometry, *Phys. Rev. D* 64 (2001) 084018, [gr-qc/0105067]
- [56] M. Bojowald, Loop Quantum Cosmology III: Wheeler-DeWitt Operators, *Class. Quantum Grav.* 18 (2001) 1055–1070, [gr-qc/0008052]
- [57] M. Bojowald, Quantization ambiguities in isotropic quantum geometry, *Class. Quantum Grav.* 19 (2002) 5113–5130, [gr-qc/0206053]
- [58] M. Bojowald, Loop Quantum Cosmology: Recent Progress, In: Proceedings of the International Conference on Gravitation and Cosmology (ICGC 2004), Cochin, India, *Pramana* 63 (2004) 765–776, [gr-qc/0402053]

- [59] M. Bojowald, Non-singular black holes and degrees of freedom in quantum gravity, *Phys. Rev. Lett.* 95 (2005) 061301, [gr-qc/0506128]
- [60] L. Smolin, The Classical Limit and the Form of the Hamiltonian Constraint in Non-Perturbative Quantum General Relativity, [gr-qc/9609034]
- [61] D. E. Neville, Long range correlations in quantum gravity, *Phys. Rev. D* 59 (1999) 044032, [gr-qc/9803066]
- [62] H. Nicolai, K. Peeters, and M. Zamaklar, Loop quantum gravity: an outside view, *Class. Quantum Grav.* 22 (2005) R193–R247, [hep-th/0501114]
- [63] T. Thiemann, Anomaly-Free Formulation of Non-Perturbative, Four-Dimensional Lorentzian Quantum Gravity, *Phys. Lett. B* 380 (1996) 257–264, [gr-qc/9606088]
- [64] A. Perez, On the regularization ambiguities in loop quantum gravity, [gr-qc/0509118]
- [65] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann, Quantization of Diffeomorphism Invariant Theories of Connections with Local Degrees of Freedom, *J. Math. Phys.* 36 (1995) 6456–6493, [gr-qc/9504018]
- [66] K. Giesel and T. Thiemann, Consistency Check on Volume and Triad Operator Quantisation in Loop Quantum Gravity I, [gr-qc/0507036]
- [67] M. Bojowald, Loop Quantum Cosmology: II. Volume Operators, *Class. Quantum Grav.* 17 (2000) 1509–1526, [gr-qc/9910104]
- [68] V. Husain and O. Winkler, Quantum resolution of black hole singularities, *Class. Quantum Grav.* 22 (2005) L127–L133, [gr-qc/0410125]
- [69] J. Lewandowski, A. Okołów, H. Sahlmann, and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, [gr-qc/0504147]
- [70] C. Fleischhack, Representations of the Weyl Algebra in Quantum Geometry, [math-ph/0407006]
- [71] V. Husain and O. Winkler, Quantum black holes from null expansion operators, *Class. Quantum Grav.* 22 (2005) L135–L141, [gr-qc/0412039]
- [72] M. Bojowald, Initial Conditions for a Universe, *Gen. Rel. Grav.* 35 (2003) 1877–1883, [gr-qc/0305069]
- [73] M. Bojowald and K. Vandersloot, Loop quantum cosmology, boundary proposals, and inflation, *Phys. Rev. D* 67 (2003) 124023, [gr-qc/0303072]
- [74] M. Bojowald and F. Hinterleitner, Isotropic loop quantum cosmology with matter, *Phys. Rev. D* 66 (2002) 104003, [gr-qc/0207038]

- [75] F. Hinterleitner and S. Major, Isotropic Loop Quantum Cosmology with Matter II: The Lorentzian Constraint, *Phys. Rev. D* 68 (2003) 124023, [gr-qc/0309035]
- [76] P. Singh, Effective State Metamorphosis in Semi-Classical Loop Quantum Cosmology, *Class. Quantum Grav.* 22 (2005) 4203–4216, [gr-qc/0502086]
- [77] M. Bojowald, J. E. Lidsey, D. J. Mulryne, P. Singh, and R. Tavakol, Inflationary Cosmology and Quantization Ambiguities in Semi-Classical Loop Quantum Gravity, *Phys. Rev. D* 70 (2004) 043530, [gr-qc/0403106]
- [78] G. V. Vereshchagin, Qualitative Approach to Semi-Classical Loop Quantum Cosmology, *JCAP* 07 (2004) 013, [gr-qc/0406108]
- [79] N. J. Nunes, Inflation: A graceful entrance from Loop Quantum Cosmology, *Phys. Rev. D* 72 (2005) 103510, [astro-ph/0507683]
- [80] M. Bojowald, R. Maartens, and P. Singh, Loop Quantum Gravity and the Cyclic Universe, *Phys. Rev. D* 70 (2004) 083517, [hep-th/0407115]
- [81] M. Bojowald, Original Questions, *Nature* 436 (2005) 920–921
- [82] M. Bojowald, G. Date, and G. M. Hossain, The Bianchi IX model in loop quantum cosmology, *Class. Quantum Grav.* 21 (2004) 3541–3569, [gr-qc/0404039]
- [83] G. Date and G. M. Hossain, Genericity of inflation in isotropic loop quantum cosmology, *Phys. Rev. Lett.* 94 (2005) 011301, [gr-qc/0407069]
- [84] G. Date and G. M. Hossain, Genericity of Big Bounce in isotropic loop quantum cosmology, *Phys. Rev. Lett.* 94 (2005) 011302, [gr-qc/0407074]
- [85] S. Hofmann and O. Winkler, The Spectrum of Fluctuations in Inflationary Quantum Cosmology, [astro-ph/0411124]
- [86] D. Cartin, G. Khanna, and M. Bojowald, Generating function techniques for loop quantum cosmology, *Class. Quantum Grav.* 21 (2004) 4495–4509, [gr-qc/0405126]
- [87] D. Cartin and G. Khanna, Absence of pre-classical solutions in Bianchi I loop quantum cosmology, *Phys. Rev. Lett.* 94 (2005) 111302, [gr-qc/0501016]
- [88] G. Date, Pre-classical solutions of the vacuum Bianchi I loop quantum cosmology, *Phys. Rev. D* 72 (2005) 067301, [gr-qc/0505030]
- [89] D. Green and W. Unruh, Difficulties with Closed Isotropic Loop Quantum Cosmology, *Phys. Rev. D* 70 (2004) 103502, [gr-qc/0408074]
- [90] M. Bojowald and A. Rej, Asymptotic Properties of Difference Equations for Isotropic Loop Quantum Cosmology, *Class. Quantum Grav.* 22 (2005) 3399–3420, [gr-qc/0504100]

- [91] A. Ashtekar and T. A. Schilling, Geometrical Formulation of Quantum Mechanics, In: *On Einstein's Path: Essays in Honor of Engelbert Schücking*, Ed.: A. Harvey, pages 23–65, (Springer, New York, 1999), [gr-qc/9706069]
- [92] M. Bojowald and A. Skirzewski, Effective Equations of Motion for Quantum Systems, [math-ph/0511043]
- [93] J. Willis, *On the Low-Energy Ramifications and a Mathematical Extension of Loop Quantum Gravity*, PhD thesis, The Pennsylvania State University, 2004
- [94] A. Ashtekar, C. Beetle, and S. Fairhurst, Isolated Horizons: A Generalization of Black Hole Mechanics, *Class. Quantum Grav.* 16 (1999) L1–L7, [gr-qc/9812065]
- [95] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski, and J. Wiśniewski, Isolated Horizons and their Applications, *Phys. Rev. Lett.* 85 (2000) 3564–3567, [gr-qc/0006006]
- [96] A. Ashtekar, A. Corichi, and K. Krasnov, Isolated Horizons: the Classical Phase Space, *Adv. Theor. Math. Phys.* 3 (2000) 419–478, [gr-qc/9905089]
- [97] A. Ashtekar and B. Krishnan, Dynamical Horizons: Energy, Angular Momentum, Fluxes and Balance Laws, *Phys. Rev. Lett.* 89 (2002) 261101, [gr-qc/0207080]
- [98] A. Ashtekar and B. Krishnan, Dynamical Horizons and their Properties, *Phys. Rev. D* 68 (2003) 104030, [gr-qc/0308033]
- [99] A. Ashtekar and B. Krishnan, Isolated and dynamical horizons and their applications, *Living Rev. Rel.* 7 (2004) 1–77, [gr-qc/0407042]
- [100] I. Booth and S. Fairhurst, The first law for slowly evolving horizons, *Phys. Rev. Lett.* 92 (2004) 011102, [gr-qc/0307087]
- [101] M. Bojowald and R. Swiderski, Spherically Symmetric Quantum Horizons, *Phys. Rev. D* 71 (2005) 081501(R), [gr-qc/0410147]