Complexifier Coherent States for Quantum General Relativity

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Abstract

Recently, substantial amount of activity in Quantum General Relativity (QGR) has focussed on the semiclassical analysis of the theory. In this paper we want to comment on two such developments: 1) Polymer-like states for Maxwell theory and linearized gravity constructed by Varadarajan which use much of the Hilbert space machinery that has proved useful in QGR and 2) coherent states for QGR, based on the general complexifier method, with built – in semiclassical properties. We show the following:

A) Varadarajan's states *are* complexifier coherent states. This unifies all states constructed so far under the general complexifier principle.

B) Ashtekar and Lewandowski suggested a non-Abelean generalization of Varadarajan's states to QGR which, however, are no longer of the complexifier type. We construct a new class of non-Abelean complexifiers which come close to the one underlying Varadarajan's construction.

C) Non-Abelean complexifiers close to Varadarajan's induce new types of Hilbert spaces which do not support the operator algebra of QGR. The analysis suggests that if one sticks to the present kinematical framework of QGR and if kinematical coherent states are at all useful, then normalizable, graph dependent states must be used which are produced by the complexifier method as well.

D) Present proposals for states with mildened graph dependence, obtained by performing a graph average, do not approximate well coordinate dependent observables. However, graph dependent states, whether averaged or not, seem to be well suited for the semiclassical analysis of QGR with respect to coordinate independent operators.

1 Introduction

A mathematically well-defined candidate Wheeler-DeWitt (or Hamiltonian constraint) operator for canonical Quantum General Relativity (QGR) has been proposed in [1, 2] on the kinematical Hilbert space $\mathcal{H}_0$ defined by Ashtekar, Isham and Lewandowski [3]. This Hilbert space presently underlies literally all the constructions within QGR. (See [4] for an up-dated, detailed introduction to QGR and [5, 6] for non-technical overviews). Thus, there is modest hope that $\mathcal{H}_0$ indeed supports a (dual) representation of all constraint operators and Dirac observables of the theory and that there exists a well-defined quantum field theory for Lorentzian metrics (plus, possibly supersymmetric, matter) in four spacetime dimensions. It is possible to explicitly state (and to some extent even solve) the,

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necessarily discrete, quantum time evolution. Open problems are the construction of an inner product on the space of solutions and of a suitable set of gauge invariant observables.

However, before tackling these problems, the more crucial question is, whether that quantum field theory can possibly have general relativity as its classical limit. The answer to that question is far from obvious because the theory is background independent, that is, non-perturbatively defined: Usual quantization procedures, of which we know that they guarantee the correct classical limit such as for Maxwell theory, always make crucially use of the (Minkowski) background metric. However, general relativity does not distinguish any background, it is a backgound independent theory, in other words, the background metric becomes dynamical, a quantum field operator. Therefore these usual quantization techniques, usually based on Fock spaces (notice that the Minkowski metric explicitly slips into the definition of a Fock space through, e.g. the d'Alembert operator, the positive and negative frequency one-particle wave functions etc.) cannot be used. This fact manifests itself in the regularization and renormalization of the operator in [1] on the background independent Hilbert space $\mathcal{H}_0$ which employs completely new mathematical techniques which we have little experience with and it is therefore indeed not manifest whether the quantum Einstein equations proposed reduce to the classical ones in the classical limit.

To settle this question one has several possibilities. As a minimum test one should verify whether the new quantization technique works in situations where the theory can be solved explicitly, say in dimensionally or Killing reduced models of general relativity. This has been done in [2] for 2+1 gravity and in [7] for Bianchi cosmologies, in both cases with success. The analysis in [7] is based on a quantum symmetry reduction, that is, one works on the Hilbert space of the full theory and imposes the Killing constraints there, in contrast to the usual procedure of imposing the symmetry before quantization. In that sense the calculations performed in [7] are almost fully 3+1 dimensional, just that one neglects the excitations of all but a finite number of degrees of freedom, which is why this provides a really firm test of the proposal made in [1]. The agreement of the quantum theory with the classical theory demonstrated in [7] is indeed spectacularly sharp, the discrete spectrum of quantum metric operators constructed in [1] approaches the classical curve much faster (for very low quantum numbers) than one could have hoped for (correspondence principle). What is more, the operator proposed in [1] certainly suffers from quantization ambiguities but in [7] these were shown to be irrelevant for the classical limit. Finally, for the same reason that the quantization technique in in [1] made the Wheeler-DeWitt operator finite, the deeper reason being background independence, it avoids certain classical curvature singularities which could be checked explicitly in [7] where it was shown that the classical big bang singularity is avoided in the quantum theory. These discrepancies between classical and quantum predictions actually are large only in the deep Planckian regime.

These results are very promising but they must be generalized to the full theory (all excitations). To that end, a general construction principle, the so-called complexifier method, for coherent states (in principle for any canonical theory based on a cotangent bundle) has been formulated in [8] which guarantees that many semiclassical properties are built in. These states are *kinematical*, that is, they are annihilated neither by the spatial diffeomorphism constraint nor by the Hamiltonian constraint for well motivated reasons:

1) In order to construct physical coherent states we would need to spend a lot of work in order to find the entire space of solutions to all constraints which we do not want to do unless we are sure that the Hamiltonian constraint operator has the correct classical limit.

2) Whether or not the Hamiltonian conatraint operator has the correct classical limit cannot be checked on its kernel where it vanishes by definition.

3) We cannot make our coherent states at least diffeomorphism invariant because the Hamiltonian constraint does not preserve the space of diffeomorphism invariant states. We will report on the

classical limit of the Hamiltonian constraint with respect to these states elsewehere [13] and focus on another issue in the present paper.

When applying the complexifier method to QGR at the kinematical level, then it does not give rise to a normalizable (with respect to the kinematical inner product of QGR) state but rather a complex probability measure ψ which also can be considered as a *distribution* over a dense subset of the kinematical Hilbert space. The underlying reason is that the kinematical Hilbert space is not seperable, a situation that might change once we go to the spatially diffeomorphism invariant level. For instance, we showed that if the volume operator is used as the complexifier then one obtains a distributional, gauge invariant and diffeomorphism covariant coherent state. Thus, in order to obtain normalizable states one must restrict these distributions to (countable) graphs, resulting in "cut-off" states ψ_γ. This we did in [9, 10] with respect to a natural (modification of a) Laplacian operator as the complexifier considered already in [11] and obtained a complete family of gauge invariant and spatially diffeomorphism covariant coherent states for quantum general relativity These are based on coherent states that have been constructed for the phase space of a cotangent bundle over a compact gauge group (in this case SU(2)) in [12] (see also references therein) and whose semiclassical properties (such as overcompleteness, peakedness, Ehrenfest properties, minimal uncertainty, annihilation operator eigenstate property, small fluctuations) have been explicitly proved in the just mentioned works.

To shed light on the issue of the classical analysis from a seemingly completely independent perspective, Varadarajan has constructed a polymer like representation of Maxwell theory and linearized general relativity in [14] by using GNS technology in an ingenious way. More in detail, by making explicit use of the Minkowski background metric, he was able to construct an image of the usual Fock states on *a distributional extension of* the type of background independent Hilbert space $\mathcal{H}_0$ on which quantum general relativity currently is based. Varadarajan's work is ground breaking for several reasons: First of all, this is the first time that a clear link between the usual Fock Hilbert spaces of free Maxwell theory or linearized general relativity on the one hand and the polymer-like Hilbert spaces of the type in use for QGR on the other hand has been achieved in a mathematically rigorous way. Secondly, his work shows that the space of distributional connections $\overline{\mathcal{A}}$ on which the background *independent* measure underlying $\mathcal{H}_0$ is supported is also a natural support for the background *dependend* measure [15] which underlies Varadarajan's GNS Hilbert space. Finally, at least on the level of linearized general relativity we now have, for the first time, a very good idea of how to think about gravitons in the framework of QGR and a first glimpse of how to extend his results to full non-perturbative QGR.

There are many more applications of Varadarajan's technique, in particular it goes much beyond the mere purpose of constructing semiclassical or, more specifically, coherent states. In this paper we confine ourselves to a limited set of questions, namely which implications his work has for full Quantum General Relativity. More specifically, we are interested in whether the coherent states that are implicit in his construction, suitably generalized to non-Abelean gauge theories, are good candidate semiclassical states for QGR. Recalling that his states, from the point of view of $\mathcal{H}_0$, are distributions, Varadarajan's analysis suggests two conclusions:
i) There is an appropriate substitute for usual Fock states (photons, gravitons) and the usual coherent states (realized, e.g. in lasers) in full QGR.
ii) However, these states are not normalizable with respect to the kinematical inner product used in quantum general relativity, rather, Fock states and semiclassical states define a completely new representation of the canonical commutation and adjointness relations based on a new type of background dependent measure [15] on the quantum configuration space $\overline{\mathcal{A}}$ of QGR.

These speculations rest on the assumption that the answers to the following two questions are affirmative:
A) Can these results, obtained for the gauge groups $U(1)$ for Maxwell theory and $U(1)^3$ for linearized gravity, be transferred to $SU(2)$, the gauge group of fully non-linear general relativity ?
B) Do the basic operators of quantum general relativity, the one-dimensionally smeared holonomy operators and the two-dimensionally smeared electric field operators, continue to be well-defined in the new representation ?

To answer A), in [16] Ashtekar and Lewandowski already proposed such an extension to $SU(2)$ using the *key* observation that the formulas derived by Varadarajan allow for a natural generalization if one replaces the operators associated with the Lie algebra $u(1)$ by their analogs for $su(2)$. However, issue B) is more difficult to settle because Varadarajan's construction explicitly employs *three-dimensionally smeared electric fields* (while holonomies are by construction well-defined). This is quite critical since all the operators so far constructed in quantum general relativity are based on electric fields smeared in two dimensions only. In particular, in the non-Abelean case the holonomy and 3d smeared electric field operators no longer form a closed subalgebra of the Poisson algebra. Notice that we do not insist that electric field and holonomy operators are physically interesting operators *per se*, however, they serve to build more complicated, physically interesting, composite operators such as constraint operators (in the limit of small loops or surfaces).

Thus, if B), as we will show, does not hold, then one can have the following attitudes:
I) We want to keep the present, background independent and non-perturbative kinematical set-up of QGR which has proved so successful in supporting constraint operators, area operators, volume operators etc. Then the distributional states constructed by Varadarajan or those which are produced by the complexifier machinery are not good semiclassical states for QGR because the corresponding representations that they define does not support the operator algebra of QGR.
II) We do not insist on using the current formulation of QGR but wish to use the new representations of the Varadarajan type. Then the regularization and quantization of all operators of interest has to be repeated for the new Varadarajan type of representation, probably in a background dependent way.
III) It does not make sense to discuss the semiclassical analysis at the kinematical level at all. That is, one should first solve all the constraints, leading to new Hilbert spaces on the space of solutions which from the point of view of $\mathcal{H}_0$ are expected to be distributional as well, before diving into the semiclassical analysis.

Option II) would not only make all current results within quantum general relativity obsolete, it would also raise serious conceptual difficulties because the representations constructed by Varadarajan make explicit use of a background metric while we wanted to build a manifestly background independent quantum gravity theory and thus our working principle would be lost ! What would be our fundamental theory if for each background we have to construct a new representation ? Also option III) would be desastrous because we would have no possibility to check on the correctness of the Hamiltonian constraint before actually solving it.

Being optimistic that the current formulation of QGR has at least some bearing for the final theory, we thus adopt viewpont I) and the question arises what one can do with the distributional states of the Varadarajan type or the complexifier coherent states within the present stage of affairs anyway. The strategy used in [9, 10] was to cut off these distributional coherent states at given (countable) graphs resulting in families of normalizable coherent states indexed by graphs. A similar procedure has also been applied to the Varadarajan states in [16] where the analog of cut-off states were coined "shadows". (The difference between a cut-off state and a shadow is that the former form a cylindrically consistent family of complex probability measures while shadows are their

corresponding spin-network projections). Since such states are embedded into the full kinematical Hilbert space, there is hope is that they display suitable semiclassical properties. This has been demonstrated to some extent for the states [10] as already mentioned. Notice that this strategy *does not make use of the new representation of the Varadarajan type* but that either the complexifier method or Varadarajan's method just serve to produce those cut-off states or shadows respectively, the kinematical framework of QGR remains untouched.

Cut-off states have been criticized due to their graph dependence which leads to "staircase problems" with the semiclassical properties of certain types of operators such as the area operator, see the last reference in [10]. In order to remove this graph dependence, in [16] graph – averaging techniques have been advocated which are based on the Dirichlet-Voronoi construction [17].

The aim of this paper is to review these developments, to show how they are interrelated and to communicate some observations which hopefully add some conceptual and technical clarity to the subject. The architecture of this article is as follows:

In order to make this paper close to being self-contained we review the complexifier construction and the Varadarajan construction in section two, using a position space language which will help us to compare them later on. The main results then are contained in the subsequent sections:

i) *Complexifier Coherent States Versus Varadarajan States*
In section three we begin by observing that the usual coherent states on Fock spaces are also obtained via the complexifier method where for Maxwell theory the complexifier is related to the electric field energy. Using an operator correspondence $\mathcal{I}$ between pairs of 3d smeared holonomy and non-smeared elecric field operators on Fock space on the one hand and pairs of non-smeared holonomy and 3d smeared elecric fields on the QGR Hilbert space $\mathcal{H}_0$ due to Ashtekar, Isham and and Lewandowski [3] on the other hand, on which Varadarajan's beautiful construction rests, we obtain as a simple corollary that also the Varadarajan states are obtained via the complexifier method from the QGR Hilbert space. This explains the miracolous identities found in [14] which are responsible for the fact that the coefficients of the formal linear combinations of charge network states, defining the distributional image of the Fock vacuum states, can be obtained by asking that the Varadarajan states are generalized eigenstates of the image, under the just mentioned operator correspondence, of the exponentiated Fock space annihilation operator.

Once we know that the Varadarajan states come from a complexifier we can employ the techniques developed in [8] in order to examine their semiclassical properties. For instance, we find the point in phase space at which these states are peaked which implies corresponding Ehrenfest properties.

Finally we show that the Varadarajan representation can indeed be obtained just by using the QGR inner product, without recourse to the Fock representation and the isomorphism $\mathcal{I}$, although these states are not normalizable with respect to the QGR inner product. The mechanism responsible for this is very similar to the way that one obtains rigorously defined Gaussian measures $d\mu$ from formal measures of the kind $[DA]e^{-S}/Z$. This last point is conceptually interesting because it shows that despite the appearance of distributional states one can still use the original inner product in order to obtain the new representation so that it is still the "fundamental representation" $\mathcal{H}_0$ from which all others arise through usual limiting procedures of constructive quantum field theory. Thus, also the complexifier method induces new L_2 Hilbert spaces on $\overline{\mathcal{A}}$.

ii) *Electric Flux Operator*

Since the isomorphism $\mathcal{I}$ explicitly uses 3d smeared electric fields it is not surprising that the 2d smeared electric field operator is not a well-defined operator in Varadarajan's representation We then ask whether it is possible to construct a different complexifier, for the $U(1)$ theory say, which *does* give rise to a well-defined electric flux operator. We find that for complexifiers that are bilinear in the electric field (so that on the Fock side the measure is Gaussian and we can make practical computations) it is not possible to have the fluctuations of both the holonomy and electric flux operator finite, no matter how one chooses the smearing of the complexifier. This indicates an obstruction to using this type of new representations produced by Varadarajan's or the complexifier method, for the holonomy and electric flux operator algebra crucial for QGR, at least for those that come from Gaussian measures.

iii) *Generalization to $SU(2)$*

In section five we examine whether we can use the complexifier method in order to arrive at Varadarajan type of states directly for $SU(2)$ as anticipated in [16]. We find that the obvious gauge invariant $SU(2)$ generalization of the Varadarajan complexifier underlying the states considered in [14] for $U(1), U(1)^3$ is not a well-defined operator any longer due to the fact that $SU(2)$ is not Abelean, roughly speaking the corresponding operator is no longer densely (cylindrically) defined.

Next we consider the "shadows" of [16] and find that these states do not come from a complexifier, the corresponding would-be complexifier is not cylindrically consistent. This means that the methods of [8] cannot be used in order to spell out their semiclassical properties and that more work is required in order to determine them.

To improve on this, we construct a new type of well-defined complexifier on the QGR Hilbert space with the following properties:

1.

It is almost bilinear in the electric fields, more precisely it is bilinear in certain area operators which by themselves are square roots of quadratic polynomials in the electric field operators.

2.

There are several variants of this operator. One variant does not use background metric but requires three, everywhere linearly independent, families of foliations of the spatial manifold as a background structure together with a *parquet* (partition into disjoint 2d surfaces) of each leaf of these foliations, which is quite similar to the polyhedronal decompositions that have been used in [9]. There is also a family of background dependent variants of this operator where the freedom consists in choosing a covariance kernel (propagator) of a Gaussian measure. Notice that since we want to approximate a given classical metric we may use it naturally as a background mewtric in order to construct our coherent state.

3.

The resulting distributional state is gauge invariant and diffeomorphism covariant. It can be considered as the closest analog to the Abelean case. One can also define these new complexifier states for the Abelean theory and if one would use the isomorphism $\mathcal{I}$ to map the resulting representations of this type to the "Fock side", then the resulting measure would no longer be free (Gaussian) but interacting.

4.

The cut-off states resulting from this complexifier are very similar to those of [9, 10] (and also to the Abelean ones corresponding to Varadarajan's complexifier) but in contrast to those the complexifier now does have a classical limit.

Despite of these positive features, the electric flux still does not have finite fluctuations in the associated new representation induced by these distributional states. This indicates an obstruction to using these representations for the semiclassical analysis of QGR and suggests that we are forced to work with cut-off states (or maybe with a completely different type of complexifier). This is unfortunate because it would be much more natural to work with graph independent objects.

iv) *Dirichlet-Voronoi Construction*
An idea of how to milden the graph dependence of cut-off states is to perform an average over an ensemble of graphs. An extremely elegant way of how to do that is based on the Dirichlet-Voronoi [16, 17] construction which we describe in some detail. As observed by many, the resulting states or density matrices are strictly speaking zero states and zero operators respectively on the QGR Hilbert space, however, if one interchanges two limits then the density matrix turns from a trace class operator on the QGR Hilbert space into an expectation value functional which defines a new representation through the GNS construction.

We show that in this representation the holonomy operator necessarily has zero expectation value (far off the classically allowed range of being a unitary matrix) whether or not one thickens the path on which the holonomy depends. The only way of how to obtain non-zero expectation values is by averaging coherent states over a countable (necessarily finite, if the spatial manifold is compact and our graphs are piecewise analytic) number of graphs in which case we may as well work with the coherent state on the countable graph that results by taking the union of all sample graphs in the ensemble. If one does not do that but rather works with a countable averaging then expectation values for holonomy operators are non-zero but still way off their classical values.

This seems to indicate that the Dirichlet-Voronoi procedure produces nice weaves (states that approximate the intrinsic three metric, defined in terms of area operators, for instance) but definitely not semiclassical states (states approximating also the extrinsic curvature) since the holonomy expectation values come out grossly wrong. The underlying reason is the discrete structure of the QGR Hilbert space which is related to background independence. Thus, it seems that better averaging procedures have to be designed.

v) *A Resolution: Diffeomorphism Invariant Observables*
The upshot of the analysis presented in sections three trough six is as follows:
1) If the fundamental theory of quantum general relativity, based on the holonomy – electric flux operator algebra, is not to be changed then the best coherent states available at this point must be states in the kinematical QGR Hilbert space, that is, normalizable and therefore graph dependent.
2) If we do not average the graph dependent states, then they suffer from the staircase problem and electric flux and area operators do not have the correct expectation values. If we average them, then holonomy operators have the wrong expectation values.

This seems to be a desastrous situation. In section seven we propose the following way out of the dilemma:

We must remember that a given set of states may not be adapted to all possible operators of the theory. In particular, it is an *additional physical input* what the basic observables of the theory should be with respect to which a given set of states is to behave semiclassically. Our first guess was that these should be the electric flux and holonomy operators since all others

can be constructed from them. However, the above analysis shows that the coherent states constructed so far do not approximate them well. Are there better suited quantum observables ?

The analysis in [26] has revealed that while gauge invariant but coordinate dependent operators such as Wilson loop and area operators are simply not well approximated, the present coherent states are very well geared to approximating *spatially diffeomorphism invariant operators* (and more general ones which arise from volume integrals of spatial scalar densities of weight one), specifically those that use matter (or those of the type of the Hamiltonian constraint). Intuitively this happens because, physically **matter can only be where geometry is excited**. In other words, while a coordinate surface or loop can be chosen quite independently of a given graph underlying a coherent state, if the surface or loop is defined by matter then they **automatically lie within the graph**. This is, in a nutshell, the simple mechanism which avoids both the staircase problem and the vanishing of holonomy expectation values.

Notice that since the classical counterparts of such operators suffice to separate the points of the (spatially diffeomorphism invariant) classical phase space, that only those are well approximated is quite sufficient for our purposes in view of the fact that they are more physical anyway.

Thus, it might be that graph dependent coherent states, especially those based on *random graphs* [26] and those that are averaged over a countable ensemble, are maybe not so bad for the modest main application that we have in mind, namely to test whether the quantum dynamics of the theory as presently proposed has the correct classical limit (including matter).

The paper ends with a summary in section eight and in the appendices we prove some technical results.

2 Review of Complexifier Coherent States and Varadarajan States

This section reviews the complexifier coherent states construction and the Vardarajan construction in order to prepare the stage for the next section where we compare them. Readers familiar with these concepts can skip this section, which is fairly self-contained. There are two new twists in our presentation however: 1) We use entirely a position space language since we are interested later on in complexifiers which do not use a flat background metric so that Fourier transformation is not available. 2) We perform a small change of viewpoint in that we consider the distributions that come out of both constructions formally as (non-normalizable) states of the original Hilbert space at the aid of which one can pass to a new representation through a limiting procedure familiar from constructive quantum field theory.

Appendix A contains a review of the kinematical structure of diffeomorphism invariant theories of connections and an explanation why kinematical representations of the type $\mathcal{H}_0$ are a natural starting point.

2.1 The Complexifier Coherent State Concept

The purpose of the present subsection is to introduce the concept of a complexifier which can be used to *generate* coherent states. As it will become obvious, this method is quite powerful and proviedes

a clean *construction mechanism* for coherent states. The method has been introduced for the first time in [18] and is by now also appreciated by mathematicians (see the second and third reference of [12]).

Let $(\mathcal{M},\Omega)$ be a symplectic manifold with strong symplectic structure Ω (notice that $\mathcal{M}$ is allowed to be infinite dimensional). We will assume that $\mathcal{M}=T^*\mathcal{C}$ is a cotangent bundle. Let us then choose a real polarization of $\mathcal{M}$, that is, a real Lagrangean submanifold $\mathcal{C}$ which will play the role of our configuration space. Then a lose definition of a complexifier is as follows:

Definition 2.1 *A complexifier is a positive definite function C on $\mathcal{M}$ with the dimension of an action, which is smooth a.e. (with respect to the Liouville measure induced from Ω) and whose Hamiltonian vector field is everywhere non-vanishing on $\mathcal{C}$. Moreover, for each point $q\in\mathcal{C}$ the function $p\mapsto C_q(p)=C(q,p)$ grows stronger than linearly with $||p||_q$ where p is a local momentum coordinate and $||.||_q$ is a suitable norm on $T^*_q(\mathcal{C})$.*

In the course of our discussion we will motivate all of these requirements.

The reason for the name *complexifier* is that C enables us to generate a *complex polarization* of $\mathcal{M}$ from $\mathcal{C}$ as follows: If we denote by q local coordinates of $\mathcal{C}$ (we do not display any discrete or continuous labels but we assume that local fields have been properly smeared with test functions) then

$$z(m):=\sum_{n=0}^{\infty}\frac{i^n}{n!}\{q,C\}_{(n)}(m) \tag{2.1}$$

define local complex coordinates of $\mathcal{M}$ provided we can invert $z,\bar{z}$ for $m:=(q,p)$ where p are the fibre (momentum) coordinates of $\mathcal{M}$. This is granted at least locally by definition (2.1). Here the multiple Poisson bracket is inductively defined by $\{C,q\}_{(0)}=q,\ \{C,q\}_{(n+1)}=\{C,\{C,q\}_{(n)}\}$ and makes sense due to the required smoothness. What is interesting about (2.1) is that it implies the following bracket structure

$$\{z,z\}=\{\bar{z},\bar{z}\}=0 \tag{2.2}$$

while $\{z,\bar{z}\}$ is necessarily non-vanishing. The reason for this is that (2.1) may be written in the more compact form

$$z=e^{-i\mathcal{L}_{\chi_C}}q=([\varphi^t_{\chi_C}]^*q)_{t=-i} \tag{2.3}$$

where χ_C denotes the Hamiltonian vector field of C, unamibiguously defined by $i_{\chi_C}\Omega+dC=0$, $\mathcal{L}$ denotes the Lie derivative and $\varphi^t_{\chi_C}$ is the one – parameter family of symplectomorphisms generated by χ_C. Formula (2.3) displays the transformation (2.1) as the analytic extension to imaginary values of the one parameter family of diffeomorphisms generated by χ_C and since the flow generated by Hamiltonian vector fields leaves Poisson brackets invariant, (2.2) follows from the definition of a Lagrangean submanifold. The fact that we have to continued to the negative imaginary axis rather than the positive one is important in what follows and has to do with the required positivity of C.

The importance of this observation is that either of $z,\bar{z}$ are coordinates of a Lagrangean submanifold of the complexification $\mathcal{M}^{\mathbb{C}}$, i.e. a complex polarization and thus may serve to define a Bargmann-Segal representation of the quantum theory (wave functions are holomorphic functions of z). The diffeomorphism $\mathcal{M}\to\mathcal{C}^{\mathbb{C}};\ m\mapsto z(m)$ shows that we may think of $\mathcal{M}$ either as a symplectic manifold or as a complex manifold (complexification of the configuration space). Indeed, the polarization is usually a positive Kähler polarization with respect to the natural Ω-compatible complex structure on a cotangent bundle defined by local Darboux coordinates, if we choose the complexifier to be a function of p only. These facts make the associated Segal-Bargmann representation especially

attractive.

We now apply the rules of canonical quantization: a suitable Poisson algebra $\mathcal{O}$ of functions O on $\mathcal{M}$ is promoted to an algebra $\hat{\mathcal{O}}$ of operators $\hat{O}$ on a Hilbert space $\mathcal{H}$ subject to the condition that Poisson brackets turn into commutators divided by $i\hbar$ and that reality conditions are reflected as adjointness relations, that is,

$$[\hat{O},\hat{O}'] = i\hbar\widehat{\{O,O'\}} + o(\hbar),\ \ \hat{O}^\dagger = \hat{\bar{O}} + o(\hbar) \tag{2.4}$$

where quantum corrections are allowed (and in principle unavoidable except if we restrict $\mathcal{O}$, say to functions linear in momenta). We will assume that the Hilbert space can be represented as a space of square integrable functions on (a distributional extension $\overline{\mathcal{C}}$ of) $\mathcal{C}$ with respect a positive, faithful probability measure μ, that is, $\mathcal{H} = L_2(\overline{\mathcal{C}}, d\mu)$ as it is motivated by the real polarization.

The fact that C is positive motivates to quantize it in such a way that it becomes a self-adjoint, positive definite operator. We will assume this to be the case in what follows. Applying then the quantization rules to the functions z in (2.1) we arrive at

$$\hat{z} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \frac{[\hat{q},\hat{C}]_{(n)}}{(i\hbar)^n} = e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar} \tag{2.5}$$

The appearance of $1/\hbar$ in (2.5) justifies the requirement for $C/\hbar$ to be dimensionless in definition (2.1). We will call $\hat{z}$ *annihilation operator* for reasons that will become obvious in a moment.

Let now $q \mapsto \delta_{q'}(q)$ be the δ-distribution with respect to μ with support at $q = q'$. (More in mathematical terms, consider the complex probability measure, denoted as $\delta_{q'}d\mu$, which is defined by $\int \delta_{q'}d\mu f = f(q')$ for measurable f). Notice that since C is non-negative and necessarily depends non-trivially on momenta (which will turn into (functional) derivative operators in the quantum theory), the operator $e^{-\hat{C}/\hbar}$ is a *smoothening operator*. Therefore, although $\delta_{q'}$ is certainly not square integrable, the complex measure (which is probability if $\hat{C}\cdot 1 = 0$)

$$\psi_{q'} := e^{-\hat{C}/\hbar}\delta_{q'} \tag{2.6}$$

has a chance to be an element of $\mathcal{H}$. Whether or not it does depends on the details of $\mathcal{M}, \Omega, C$. For instance, if C as a function of p at fixed q has flat directions, then the smoothening effect of $e^{-\hat{C}/\hbar}$ may be insufficient, so in order to avoid this we required that C is positive definite and not merely non-negative. If C would be indefinite, then (2.6) has no chance to make sense as an L_2 function.

We will see in a moment that (2.6) qualifies as a candidate *coherent state* if we are able to analytically extend (2.6) to complex values z of q' where the label z in ψ_z will play the role of the point in $\mathcal{M}$ at which the coherent state is peaked. In order that this is possible (and in order that the extended function is still square integrable), (2.6) should be entire analytic. Now $\delta_{q'}(q)$ roughly has an integral kernel of the form $e^{i(k,(q-q'))}$ (with some pairing $<.,.>$ between tangential and cotangential vectors) which is analytic in q' but the integral over k, after applying $e^{-\hat{C}/\hbar}$, will produce an entire analytic function only if there is a damping factor which decreases faster than exponentially. This provides the intuitive explanation for the growth requirement in definition 2.1. Notice that the ψ_z are not necessarily normalized.

Let us then assume that

$$q \mapsto \psi_m(q) := [\psi_{q'}(q)]_{q'\to z(m)} = [e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \tag{2.7}$$

is an entire L_2 function. Then ψ_m is automatically an *eigenfunction of the annihilation operator* $\hat{z}$ *with eigenvalue* z since

$$\hat{z}\psi_m = [e^{-\hat{C}/\hbar}\hat{q}\delta_{q'}]_{q'\to z(m)} = [q'e^{-\hat{C}/\hbar}\delta_{q'}]_{q'\to z(m)} = z(m)\psi_m \tag{2.8}$$

where in the second step we used that the delta distribution is a generalized eigenfunction of the operator $\hat{q}$. But to be an eigenfunction of an annihilation operator *is one of the accepted definitions of coherent states* !

Next, let us verify that ψ_m indeed has a chance to be peaked at m. To see this, let us consider the self-adjoint (modulo domain questions) combinations

$$\hat{x} := \frac{\hat{z}+\hat{z}^\dagger}{2}, \;\; \hat{y} := \frac{\hat{z}-\hat{z}^\dagger}{2i} \tag{2.9}$$

whose classical analogs provide real coordinates for $\mathcal{M}$. Then we have automatically from (2.8)

$$< \hat{x} >_m := \frac{< \psi_m, \hat{x}\psi_m >}{||\psi_m||^2} = \frac{z(m)+\bar{z}(m)}{2} =: x(m) \tag{2.10}$$

and similar for y. Equation (2.10) tells us that the operator $\hat{z}$ should really correspond to the function $m \mapsto z(m),\ m \in \mathcal{M}$.

Now we compute by similar methods that

$$< [\delta\hat{x}]^2 >_m := \frac{< \psi_m, [\hat{x} - < \hat{x} >_m]^2 \psi_m >}{||\psi_m||^2} = < [\delta\hat{y}]^2 >_m = \frac{1}{2}| < [\hat{x},\hat{y}] >_m | \tag{2.11}$$

so that the ψ_m are automatically *minimal uncertainty states for* $\hat{x},\hat{y}$, moreover the fluctuations are unquenched. This is the second motivation for calling the ψ_m coherent states. Certainly one should not only check that the fluctuations are minimal but also that they are small as compared to the expectation value, at least at generic points of the phase space, in order that the quantum errors are small.

The *infinitesimal Ehrenfest* property

$$\frac{< [\hat{x},\hat{y}] >_z}{i\hbar} = \{x,y\}(m) + O(\hbar) \tag{2.12}$$

follows if we have properly implemented the canonical commutation relations and adjointness relations. The size of the correction, however, does not follow from these general considerations but the minimal uncertainty property makes small corrections plausible. Condition (2.12) supplies information about how well the symplectic structure is reproduced in the quantum theory.

For the same reason one expects that the peakedness property

$$|\frac{< \psi_m, \psi_{m'} >|^2}{||\psi_m||^2\ ||\psi_{m'}||^2} \approx \chi_{K_m}(m') \tag{2.13}$$

holds, where K_m is a phase cell with center m and Liouville volume $\approx \sqrt{< [\delta\hat{x}]^2 >_m < [\delta\hat{y}]^2 >_m}$ and χ denotes the characteristic function of a set.

Finally one wants coherent states to be overcomplete in order that every state in $\mathcal{H}$ can be expanded in terms of them. This has to be checked on a case by case analysis but the fact that our complexifier coherent states are for real z nothing else than regularized δ distributions which in turn provide a (generalized) basis makes this property plausible to hold.

After these abstract considerations, the reader should work out a known example for himself in order to fill in the necessary intuition:
Phase space: $\mathcal{M} = T^*\mathbb{R} = \mathbb{R}^2$
Configuration space: $\mathcal{C} = \mathbb{R}$
Symplectic structure: $\{q, q\} = \{p, p\} = 0,\ \{p, q\} = 1$
Complexifier: $C = p^2/2$
This complexifier certainly meets all our requirements and the ambitious reader will find out that the construction displayed above results in the usual coherent states for the harmonic oscillator as defined via energy eigenstates with Hamiltonian $H = (p^2 + q^2)/2$ up to a phase (we take $q, p, \hbar$ dimensionless for the sake of the example).

Important Remark:
1.
It is very crucial to know the map $m \mapsto z(m)$. If we are just given some states ψ_z with $z \in \mathcal{C}^{\mathbb{C}}$ then we have no way to find the point $m \in \mathcal{M}$ to which z corresponds (there are certainly infinitely many diffeomorphisms between $\mathcal{M}, \mathcal{C}^{\mathbb{C}}$) and the connection with the classical phase space is lost. Without this knowledge we cannot check, for instance, whether the infinitesimal Ehrenfest property holds. This is one of the nice things that the complexifier method automatically does for us. In order to know the function $z(m)$ we must know what the classical limit of $\hat{C}$ is, if we are just given some abstract operator without classical interpretation, then again we do not know $z(m)$. Of course, if we are given just some set of states ψ_z we could try to construct an appropriate map $m \mapsto z(m)$ as follows: Find a (complete) set of basic operators $\hat{O}$ whose fluctuations are (close to) minimal and define a map $z \mapsto O'(z) :=< \psi_z, \hat{O}\psi_z > /||\psi_z||^2$. Also define $\{O', \bar{O}'\}'(z) := \lim_{\hbar\to 0} < \psi_z, \frac{[\hat{O},\hat{O}^\dagger]}{i\hbar}\psi_z > /||\psi_z||^2$. Now construct $m \mapsto z(m)$ by asking that the pull-back functions $O(m) := O'(z(m))$ satisfy

$$\{O, \bar{O}\}(m) = \{O', \bar{O}'\}'(z(m)) \tag{2.14}$$

in other words, that the symplectic structure defined by $\{.,.\}'$ is the symplectomorphic image of the original symplectic structure $\{.,.\}$ under the canonical transformation $m \mapsto z(m)$. The reader will agree that this procedure is rather indirect and especially in field theory will be hard to carry out. Notice that by far not all symplectic structures are equivalent so that even to find appropriate operators for given ψ_z such that at least one map $m \mapsto z(m)$ exists will be a non-trivial task. The complexifier method guarantees all of this to be the case from the outset since the transformation (2.1) is a canonical transformation by construction.
2.
Usually one calls a *complete* system of states $\psi_m,\ m \in \mathcal{M}$ semiclassical for an algebra $\hat{\mathcal{O}}$ if the expectation values of the operators and their commutators in the states ψ_m reproduce the values of the corresponding functions and their Poisson brackets at the point m and if the relative errors (fluctuations) are small. Coherent states have the additional properties of being annihilation operator eigenstates, to be minimal uncertainty states and to be peaked in phase space. The complexifier method thus provides a general construction guideline, but no algorithm, to arrive at a satisfactory candidate system of semiclassical, even coherent, states for a wide class of theories. It should be kept in mind, however, that while coherent states are a natural choice for semiclassical states, they do not comprise the full set of semiclassical states and in some cases it maybe needed to employ more general classes of semiclassical states.

2.2 Complexifier Coherent States for Diffeomorphism Invariant Theories of Connections

We assume the reader to be familiar with the basics of QGR. A short introduction can be found in appendix A where we mainly introduce notation and explain why the Hilbert space $\mathcal{H}_0$ currently used in QGR is extremely natural from a representation theoretic point of view. It therefore seems to be a suitable kinematical background independent starting point for further analysis, in particular, it should be considered as the "fundamental representation" from which all others can be derived.

After having chosen a Hilbert space $\mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0)$, the only input required in the complexifier construction is the choice of a complexifier itself. We will restrict our class of choices to functions $C = C(E)$ which are gauge invariant but not necessarily diffeomorphism invariant (since we can use the $(D-)$metric to be approximated as a naturally available background metric) and only depend on the electric field to make life simple. We suppose that the associated operator $\hat{C}$ is a densely defined positive definite operator on $\mathcal{H}_0$ whose spectrum is pure point (discrete). The latter assumption is not really a restriction because operators which are constructed from (limits of) electric flux operators quite generically have this sort of spectrum, see e.g. [4]. Let $T_s,\ s \in \mathcal{S}$ be the associated uncountably infinite orthonormal basis of eigenvectors. The labels $s = (\gamma, \vec{\pi}, \vec{I})$ are triples consisting of a piecewise analytic graph γ, an array of equivalence classes of non-trivial irreducible representations π_e, one for each edge e of of γ and an array of intertwiners I_v, one for each vertex v of γ. The intertwiners are chosen in such a way that the T_s are not only gauge invariant but also eigenfunctions of $\hat{C}$. The space of possible $\vec{I}$ at given $\vec{\pi}$ is always finite dimensional and the operators of the form $\hat{C}$ can never change $\vec{\pi}, \gamma$. Thus, the T_s are just suitable linear combinations of the usual spin network functions [24].

Let λ_s be the corresponding eigenvalues. Then

$$\delta_{A'} = \sum_{s \in \mathcal{S}} T_s(A') \overline{T_s} \tag{2.15}$$

is a suitable representation of the δ distribution with respect to μ_0, i.e.

$$\int_{\overline{\mathcal{A}}} d\mu_0(A) \delta_{A'}(A) f(A) = \sum_s T_s(A') < T_s, f >= f(A') \tag{2.16}$$

and our complexifier coherent states become explicitly

$$\psi_m = \sum_{s \in \mathcal{S}} e^{-\lambda_s/\hbar} T_s(Z(m)) \overline{T_s} \tag{2.17}$$

where we have made use of the fact that the expression for $\hat{C}$ is real, $m = (A, E)$ is the point in $\mathcal{M}$ to be approximated and

$$[z_a^j(m)](x) := [{}^{(\mathbb{C})}A_a^j(m)](x) := A_a^j(x) - i\kappa \frac{\delta C}{E_j^a(x)} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{A_a^j(x), C\}_{(n)} \tag{2.18}$$

is a *complex valued G-connection* since C is supposed to be gauge invariant.

Since there are more than countably many terms different from zero in (2.17) the states ψ_m are not elements of $\mathcal{H}_0$. Rather, they define algebraic distributions in Cyl* defined by

$$\begin{aligned}
\psi_m[f] &:= < 1, \psi_m\ f > \\
&= < 1, [e^{-\hat{C}/\hbar} \delta_{A'}] f >_{A' \to Z(m)} = < \overline{f}, [e^{-\hat{C}/\hbar} \delta_{A'}] >_{A' \to Z(m)} \\
&= < e^{-\hat{C}/\hbar} \overline{f}, \delta_{A'} >_{A' \to Z(m)} = < 1, \delta_{A'} \overline{e^{-\hat{C}/\hbar} \overline{f}} >_{A' \to Z(m)} \\
&= (\delta_{A'} [\overline{e^{-\hat{C}/\hbar} \overline{f}}])_{A' \to Z(m)}
\end{aligned} \tag{2.19}$$

For $f = T_s$ the right hand side of (2.18) becomes $e^{-\lambda_s/\hbar} T_s(Z(m))$ and since C is supposed to depend on a sufficiently high power of E and since $|T_s(Z(m))|$ grows at most exponentially with $\vec{\pi}$, these numbers are actually bounded from above so that the distribution is well defined. Equivalently, we can consider ψ_m as a complex probability measure (since the δ distribution is).

Consider for each entire analytic path e the *annihilation operators*

$$\hat{g}_e := e^{-\hat{C}/\hbar} \hat{A}(e) e^{\hat{C}/\hbar} \tag{2.20}$$

which are the quantum analogs of the classical functions $Z(m)(e) = h_e({}^{(\mathbb{C})}A(m)) = g_e(m)$ where $h_e(A) = A(e)$ denotes the holonomy of A along e. Thus $g_e(m)$ is the holonomy along e of the complex connection ${}^{(\mathbb{C})}A$. The holonomy property can also be explicitly checked for the operators $\hat{g}_e$ themselves since for a composition of paths $e = e_1 \circ e_2$ we have from the holonomy property for $\hat{A}$ that

$$\hat{g}_{e_1} \hat{g}_{e_2} = e^{-\hat{C}/\hbar} \hat{A}(e_1) \hat{A}(e_2) e^{\hat{C}/\hbar} = \hat{g}_e \text{ and } \hat{g}_{e^{-1}} = (\hat{g}_e)^{-1} \tag{2.21}$$

where product and inversion is that within $G^{\mathbb{C}}$.

As one can explicitly check, ψ_m is a simultaneous generalized eigenvector of all the $\hat{g}_e$, that is,

$$\begin{aligned} (\hat{g}_e \psi_m)[f] &:= < 1, [\hat{g}_e \psi_m]\ f > \\ &= < \hat{g}_e^\dagger \overline{f}, \psi_m > = < 1, \psi_m \overline{\hat{g}_e^\dagger \overline{f}} > = \psi_m[\overline{\hat{g}_e^\dagger \overline{f}}] \\ &= h_e(Z(m)) \psi_m[f] \end{aligned} \tag{2.22}$$

The crucial point is now that although the ψ_m are not normalizable, we may be able to define a positive linear functional ω_m on our algebra of functions as expectation value functional

$$\omega_m(\hat{O}) := \frac{< \psi_m, \hat{O} \psi_m >}{||\psi_m||^2} \tag{2.23}$$

where we have *used the inner product on* $\mathcal{H}_0$ and no other additional inner product ! This is conceptually appealing because, if we can give meaning to (2.23), then we arrive at a new representation of the canonical commutation relations *which is derived from* $\mathcal{H}_0$ whence $\mathcal{H}_0$ plays the role of the fundamental representation, very much in the same way as temperature representations in ordinary quantum field theory can be derived from the Fock representation by limits of the kind performed in (2.23)

Expression (2.23) is very formal in the sense that it is the quotient of two uncountably infinite series. However, notice that we can easily give meaning to it at least for *normal ordered functions of annihilation and creation operators* as follows: For the numerator we write (the colons denote normal ordering of an operator $\hat{O} = O(\{\hat{g}_e, \hat{g}_e^\dagger\})$ which is an analytic function of the $\hat{g}_e, \hat{g}_e^\dagger$)

$$\begin{aligned} < \psi_m, : \hat{O} : \psi_m > &:= \sum_s e^{-\lambda_s/\hbar} \overline{T_s(Z(m))} < \overline{T_s}, : \hat{O} : \psi_m > \\ &= \sum_s e^{-\lambda_s/\hbar} \overline{T_s(Z(m))} [: \hat{O} : \psi_m](T_s) \\ &= O(\{g_e(m), \overline{g_e(m)}\}) \sum_s e^{-\lambda_s/\hbar} \overline{T_s(Z(m))} < \overline{T_s} \psi_m > \\ &= O(\{g_e(m), \overline{g_e(m)}\}) \sum_s e^{-2\lambda_s/\hbar} |T_s(Z(m)|^2 \\ &=: O(\{g_e(m), \overline{g_e(m)}\}) ||\psi_m||^2 \end{aligned} \tag{2.24}$$

The norm squared in the last line of (2.24) is infinite but with proper regularization understood we arrive at (after taking the regulator away)

$$\omega_m(:\hat{O}:) = O(m) = O(\{g_e(m), \overline{g_e(m)}\}) \tag{2.25}$$

which has no quantum corrections at all. Thus, if the functions $m \mapsto g_e(m)$ separate the points of $\mathcal{M}$ as e varies, then we may use them as the basic variables in the quantum theory and they, together with their adjoints, have the correct expectation values in the representation induced by ω_m via the GNS construction, moreover, that representation by construction also solves the adjointness and canonical commutation relations. Of course, (2.25) will be an interesting functional only if the normal ordering corrections of interesting operators are finite. This can only be decided in a case by case analysis.

As an illustrative example (see [8] for more details) let $Q^{ab} := E^a_j E^b_k \delta^{jk}$ and consider the diffeomorphism invariant complexifier (recall that E is a density of weight one)

$$C := \frac{1}{a\kappa} \int_\sigma d^D x (\sqrt{\det(Q)})^{1/(D-1)} \tag{2.26}$$

where a is a parameter with units of $(\hbar\kappa)^{1/(D-1)}$. Our convention is that A has dimension of cm^{-1}, thus $\frac{1}{\kappa}\int_{\mathbb{R}} dt \int_\sigma d^D x \dot{A}^j_a E^a_j$, the kinetic term in the canonical action, must have dimension of an action, therefore $E/(\hbar\kappa)$ must have dimension $\mathrm{cm}^{-(D-1)}$. Thus, in order that $C/\hbar$ be dimensionfree, a must have the said dimension. E.g. for general relativity in $D+1=4$ dimensions, $(\hbar\kappa)^{1/(D-1)} = \ell_p$ is the Planck length, (2.26) is essentially the volume functional V for σ and if we are interested in cosmological questions or scales, then $a = 1/\sqrt{\Lambda}$ would be a natural choice, where Λ is the cosmological constant. In that case the quantized complexifier would simply be given by

$$\hat{C}/\hbar = \frac{1}{a\ell_p^2}\hat{V} = \frac{\ell_p}{a}\hat{v} \tag{2.27}$$

where $\hat{v} = \hat{V}/\ell_p^3$ is the dimensionfree volume functional which has discrete spectrum with eigenvalues in multiples of ℓ_p^3. Thus $\hat{C} = t\hat{v}$ where the tiny *classicality parameter*

$$t = \frac{\ell_p}{a} = \sqrt{\hbar\kappa\Lambda} \tag{2.28}$$

has entered the stage. We easily compute the complexified connection in this case as

$${}^{(\mathbb{C})}A = A - ie/(2a) \tag{2.29}$$

where e is the dimensionfree co-triad. Thus, with the volume as the complexifier, the $g_e(m)$ indeed separate the points of $\mathcal{M}$!

However, in order to qualify as a good semi-classical state, at the very least the fluctuations of our basic operators with respect to ω_m should be small as compared to the expectation values at generic points of the phase space, in particular, they should be finite. Whether or not this is the case has to be checked for the explicit choices for C displayed in the subsequent sections.

It should be noted, however, that even if the fluctuations do not come out finite, then we can still produce *graph dependent coherent states*, which in particular *are* elements of $\mathcal{H}_0$, as follows:
Given a graph γ, consider all of its subgraphs $\gamma' \subset \gamma$ obtained by removing edges in all possible ways. Given a label s we write $s = (\gamma(s), \vec{\pi}(s), \vec{I}(s))$ and define a graph dependent $\delta-$distribution

$$\delta_{A',\gamma}(A) := \sum_{\gamma'\subset\gamma} \sum_{s;\,\gamma(s)=\gamma'} T_s(A')\overline{T_s(A)} \tag{2.30}$$

It is easy to check that (2.30) is a δ-distribution restricted to those functions on $\overline{\mathcal{A}}$ which can be written in terms of the holonomies $A(p)$ where $p \subset \gamma$. In fact, (2.30) is the *cut-off* of (2.15) with the cut-off given by the graph γ since (2.30) is the restriction of the uncountably infinite series in (2.15) to the countably infinite one in (2.30) given by restricting the sum over $s \in \mathcal{S}$ to $s \in \mathcal{S}_\gamma$ where

$$\mathcal{S}_\gamma = \{s \in \mathcal{S};\ \ \gamma(s) \subset \gamma\} \tag{2.31}$$

In fact, we can consider the Hilbert space $\mathcal{H}_{\gamma,0} = L_2(\overline{\mathcal{A}}_\gamma, d\mu_{0,\gamma})$ where $\mu_{0,\gamma}$ is the pushforward of μ_0 to the space $\overline{\mathcal{A}}_\gamma$ which is the spectrum of holonomy algebra restricted to paths within γ. Then δ_γ is in fact the $\delta-$distribution with respect to $\mu_{0\gamma}$. In other words, δ_γ is the cylindrical projection of the complex measure δ.

We now obtain normalizable graph, dependent coherent states

$$\psi_{\gamma,m}(A) = [e^{-\hat{C}/\hbar}\delta_{\gamma,A'}]_{A' \to {}^{(\mathbb{C})}A(m)} = \sum_{s \in \mathcal{S}_\gamma} e^{-\lambda_s/\hbar} T_s^{(\mathbb{C})} A(m))\overline{T_s(A)} \tag{2.32}$$

with norm

$$||\psi_{\gamma,m}||^2 = \sum_{s \in \mathcal{S}_\gamma} e^{-2\lambda_s/\hbar} |T_s({}^{(\mathbb{C})}A(m))|^2 \tag{2.33}$$

which converges due to our assumptions on the spectrum λ_s. Notice that these assumptions might not hold for the volume complexifier (the volume operator is only non-negative but not positive definite, the spectrum has flat directions and it would be crucial to know how generic these are, a problem very similar in nature (but much simpler) to the convergence proof of Euclidean Yang-Mills theory). By arguments very similar to those from above it is easy to check that the $\psi_{\gamma,m}$ are still eigenstates of the operators $\hat{g}_e$ provided that the path e lies within γ. In other words, for normal ordered functions of some set of operators $\hat{g}_e, \hat{g}_e^\dagger$ it is unimportant whether we work with the complete state ψ_m or with the cut-off state $\psi_{\gamma,m}$, as far as expectation values are concerned, as long as γ contains all the paths e under consideration. However, the fluctuations will be significantly different in general since the square of a normal ordered operator is no longer normal ordered. As we will see, it is the fluctuations which will force us to work with graph dependent coherent states.

Thus, we arrive at a *coherent state family* $\{\psi_{\gamma,m}\}_{\gamma \in \Gamma}$ for each $m \in \mathcal{M}$ where Γ denotes the set of piecewise analytic, compactly supported graphs embedded into σ. They define a complex probability measure μ_m through the consistent family of measures $d\mu_{\gamma,m} := \psi_{\gamma,m} d\mu_{0,\gamma}$.
To see that this family of measures is automatically consistent we consider for $\gamma' \subset \gamma$ the projections $p_{\gamma'\gamma} : \overline{\mathcal{A}}_\gamma \to \overline{\mathcal{A}}_{\gamma'}$ defined by restricting connections from paths within γ to paths within γ'. Now the Hilbert space $\mathcal{H}_0$ is in fact the inductive limit of the Hilbert spaces $\mathcal{H}_{\gamma,0}$, that is, there exist isometric monomorphisms

$$\hat{U}_{\gamma'\gamma} :\ \mathcal{H}_{\gamma',0} \to \mathcal{H}_{\gamma,0};\ f_{\gamma'} \mapsto p^*_{\gamma\gamma'} f_{\gamma'} \tag{2.34}$$

for all $\gamma' \subset \gamma$. These maps satisfy the consistency condition

$$\hat{U}_{\tilde{\gamma}\gamma} \hat{U}_{\gamma'\tilde{\gamma}} = \hat{U}_{\gamma'\gamma} \tag{2.35}$$

for all $\gamma' \subset \tilde{\gamma} \subset \gamma$. An operator $\hat{O}$ on $\mathcal{H}_0$ can be thought of as the inductive limit of a family of operators $\{\hat{O}_\gamma\}_{\gamma \in \Gamma}$, that is, $\hat{O}_\gamma$ is densely defined on $\mathcal{H}_\gamma$ subject to the consistency condition

$$\hat{O}_\gamma \hat{U}_{\gamma'\gamma} = \hat{U}_{\gamma\gamma'} \hat{O}_{\gamma'} \tag{2.36}$$

for all $\gamma' \subset \gamma$ (there is also a condition for the domains of definition which we skip here). Thus, in particular, the complexifier is a consistently defined operator family all of whose members are self-adjoint and positive on the respective $\mathcal{H}_{0,\gamma}$. Therefore, if $f_{\gamma'}$ depends only on connections restricted to paths within γ' we have

$$\begin{aligned}
& \int_{\mathcal{A}/\mathcal{G}_\gamma} d\mu_{\gamma,m}[p^*_{\gamma\gamma'} f_{\gamma'}] = (\int_{\mathcal{A}/\mathcal{G}_\gamma} d\mu_{\gamma,0} \delta_{A',\gamma}[e^{-\hat{C}_\gamma/\hbar} \hat{U}_{\gamma'\gamma} f_{\gamma'}])_{A\to A(\mathbb{C})} \\
= & (\int_{\mathcal{A}/\mathcal{G}_\gamma} d\mu_{\gamma,0} \delta_{A',\gamma}[\hat{U}_{\gamma'\gamma} e^{-\hat{C}_{\gamma'}/\hbar} f_{\gamma'}])_{A\to A(\mathbb{C})} = (\int_{\mathcal{A}/\mathcal{G}_{\gamma'}} d\mu_{\gamma',0} \delta_{A',\gamma'}[e^{-\hat{C}_{\gamma'}/\hbar} f_{\gamma'}])_{A\to A(\mathbb{C})} \\
= & \int_{\mathcal{A}/\mathcal{G}_{\gamma'}} d\mu_{\gamma',m} f_{\gamma'} \qquad (2.37)
\end{aligned}$$

The projective limit of these measures coincides with the measure $\psi_m d\mu_0$. The notation is abusing because it suggests that μ_m is absolutely continuous with respect to μ_0 which certainly is not the case because $\psi_m \notin L_1(\overline{\mathcal{A}}, d\mu_0)$.

Let us close this subsection by comparing with the concrete set of states constructed in [9, 10]. The states considered there do not define a complex probability measure because we wanted to keep the states background independent. However, if one gives up background independence then one can trivially repair this by making the parameter a, used there in analogy to (2.26), edge dependent. Roughly, one replaces $E_j(S_e)/a^2$ by $l_e E_j(S_e)/a^2$ where S_e is a surface, intersected by e, in a cell complex dual to the graph and the dimensionless numbers l_e satisfy $l_e + l_{e'} = l_{e\circ e'}$, $l_{e^{-1}} = l_e$, see section 4. The associated complexifier operator $\hat{C}_\gamma$ then becomes cylindrically consistent and defines an operator $\hat{C}$ and a complex probability measure if we just *define* abstract elements $g_e \in SL(2,\mathbb{C})$ subject to $g_e g_{e'} = g_{e\circ e'}$, $g_{e^{-1}} = g_e^{-1}$. The trouble, however, is that $\hat{C}$ does not have a classical limit, see [8, 9] and the remark at the end of section 2.1. Therefore we do not have a graph independent map map $m \mapsto g_e(m)$ and no classical interpretation of the distributional states a priori. This is precisely the reason why the dual cell complex has been introduced in the first place in [8, 9] because it enables us to explicitly construct a map $m \mapsto g_e^\gamma(m)$ *graph by graph* and therefore we have a semiclassical interpretation of the cut-off states.

In this paper we will propose a new set of states which *do* come from a classical complexifier whose corresponding operator is densely defined with explicitly known spectrum and which are defined for arbitrary compact gauge groups, not only Abelean. Moreover, they do not require the additional structure of the cell complex. The analysis of their semiclassical properties is beyond the scope of the present paper but since their mathematical structure is similar to those defined in [9, 10], most proofs will we be easily adaptable.

2.3 Varadarajan States

We will sketch the main ideas of Varadarajan's elegant construction [14] for the case of the gauge group $U(1)$ (Maxwell theory). The case of $U(1)^3$ (linearized gravity) is similar.

We consider classical Maxwell theory on Minkowski space, that is, $\sigma = \mathbb{R}^3$ and the spatial metric is simply δ_{ab}, we do not need to worry about positions of tensor indices. By A_a, E^a we denote the magnetic potential and electric field respectively. Due to our choice that A has dimension cm^{-1}, the Feinstruktur constant is given by $\alpha = \hbar q^2$ where $\kappa_M = q^2$ is the Maxwell coupling constant (electric charge squared). Then E has dimension cm^{-2}. Let f be an arbitrary but fixed test function of rapid decrease which is even under reflection, i.e. $f(x) = f(-x)$ and which has dimension cm^{-3}. We define

$f-$smeared connections and electric fields respectively by

$$E^a_f(x) = \int_\sigma d^3y f(x-y) E^a(y) \text{ and } A^f_a(x) = \int_\sigma d^3y f(x-y) A^a(y) \tag{2.38}$$

Let us also define the distributional and smeared form factor respectively by

$$X^a_p(x) := \int_p dy^a \delta^{(3)}(x,y) \text{ and } X^a_{p_f}(x) := \int_\sigma d^3y f(x-y) X^a_p(y) = \int_p dy^a f(x-y) \tag{2.39}$$

where p is any piecewise analytic path. Both objects have dimension cm^{-2}. Consider the following two sets of elementary variables

$$(E^a(x), A(p_f) = e^{i\int d^3x X^a_{p,f} A_a}) \text{ and } (E^a_f(x), A(p) = e^{i\int d^3x X^a_p A_a}) \tag{2.40}$$

where $A(p), A(p_f)$ denote the holonomies of the unsmeared and smeared connection respectively.

It is a remarkable feature of Abelean gauge groups that the smeared holonomy $A(p_f)$ still transforms covariantly under local gauge transformations, that is, if $A^g = -dgg^{-1}/i + A$ then with $g = e^{i\lambda}$

$$\begin{aligned} A^g(p_f) &= A(p_f) e^{-i\int d^3x \lambda_{,a} X^a_{p_f}} = A(p_f) e^{i\int d^3x \lambda \partial_a X^a_{p_f}} \\ &= A(p_f) e^{i\int d^3x \lambda(x) \int_p dy^a \partial_{x^a} f(x-y)} = A(p_f) e^{-i\int d^3x \lambda(x) \int_p dy^a \partial_{y^a} f(x-y)} \\ &= A(p_f) e^{-i\int d^3x \lambda(x) \int_p df(x-y)} = A(p_f) e^{-i\int d^3x \lambda(x)[f(x-f(p))-f(x-b(p))]} \\ &= g_f(b(p)) A(p_f) g_f(f(p))^{-1} \end{aligned} \tag{2.41}$$

where the integration by parts in the second step is justified by the temeredness of f and as before $b(p), f(p)$ denote the beginning and final point of p respectively. Here $g_f(x) = e^{i\int d^3y \lambda(y) f(x-y)}$ is the *smeared gauge transformation* which still takes values in $U(1)$. In particular, if p is a closed path then the smeared holonomies are still gauge invariant. Moreover, since already E^a is gauge invariant, so is E^a_f. *These facts no longer hold in the Non-Abelean context*, the analogs of $E_f, A(p_f)$ are no longer gauge covariant !

It is trivial to check that again, *precisely because* $U(1)$ *is Abelean*, both Poisson algebras are closed. Moreover, they are *isomorphic*

$$\begin{aligned} &\{A(p_f), A(p'_f)\} = \{E^a(x), E^b(y)\} = 0,\ \{E^a(x), A(p_f)\} = iq^2 X^a_{p_f}(x) \\ &\{A(p), A(p)\} = \{E^a_f(x), E^b_f(y)\} = 0,\ \{E^a(x), A(p)\} = iq^2 X^a_{p_f}(x) \end{aligned} \tag{2.42}$$

and the reality conditions on smeared and unsmeared objects are identical. Thus, at a purely Poisson $^*-$algebraic level we cannot distingiush between the Poisson algebras generated by these two pairs of variables although their values at a given point $m = (A_a, E^a) \in \mathcal{M}$ in the classical phase space are quite different. (To be precise, one should also prove that any algebraic identities in the subalgebra generated by the $A(p)$ is a corresponding identity in the subalgebra generated by the $A(p_f)$ for, say connections of rapid decrease. Please refer to the detailed analysis in the first reference of [14] where this proof can be found for the specific choice of f in (3.15). In what follows, we will assume this to hold for the function f under consideration.)

Consider the resulting abstract $^*-$algebras $\mathcal{A}_F$ and $\mathcal{A}_P$ respectively, to which we will refer as the *Fock algebra and polymer algebra* respectively, generated by requiring that Poisson brackets become commutators divided by $i\hbar$ and that reality conditions become $^*-$relations. That is,

$$\begin{aligned} &[\hat{A}(p_f), \hat{A}(p'_f)] = \{\hat{E}^a(x), \hat{E}^b(y)\} = 0,\ [\hat{E}^a(x), \hat{A}(p_f)\} = i\alpha X^a_{p_f}, \\ &\hat{A}(p_f)^* - \hat{A}(p_f)^{-1} = \hat{E}^a(x)^* - \hat{E}^b(x) = 0, \\ &[\hat{A}(p), \hat{A}(p')] = \{\hat{E}^a_f(x), \hat{E}^b_f(y)] = 0,\ [\hat{E}^a_f(x), \hat{A}(p)\} = i\alpha X^a_{p_f}, \\ &\hat{A}(p)^* - \hat{A}(p)^{-1} = \hat{E}^a_f(x)^* - \hat{E}^b_f(x) = 0 \end{aligned} \tag{2.43}$$

Then we have an isomorphism of *algebras

$$\mathcal{I}:\ \mathcal{A}_F \to \mathcal{A}_P;\ (\hat{A}(p_f), \hat{E}^a(x)) \mapsto (\hat{A}(p), \hat{E}^a_f(x)) \tag{2.44}$$

Let ω_F be a "Fock state", that is, a normalized, positive linear functional on the Fock algebra. Then the GNS construction provides a cyclic vector Ω_F on a Hilbert space $\mathcal{H}_F$ and a representation π_F of $\mathcal{A}_F$ such that

$$< \Omega_F, \pi_F(b)\Omega_F >_F = \omega_F(b) \tag{2.45}$$

for each $b \in \mathcal{A}_F$ and $< .,. >_F$ denotes the inner product on $\mathcal{H}_F$. Using the isomorphism (2.44) we trivially obtain a positive linear functional ω_P on $\mathcal{A}_P$ and corresponding GNS data via

$$\omega_P(\mathcal{I}(b)) := \omega_F(b),\ < \Omega_P, \pi_P(\mathcal{I}(b))\Omega_P >_P := \omega_P(\mathcal{I}(b)) \tag{2.46}$$

This works for any ω_F. Let us now consider the particular Fock representation ω_F^H induced by the Maxwell Hamiltonian on Minkowski space

$$H = \frac{1}{2q^2}\int d^3x \delta_{ab}[E^a E^b + B^a B^b] \tag{2.47}$$

where $B^a = \epsilon^{abc}\partial_b A_c$ is the magnetic field. Using the transversal projection operator

$$(P \cdot A)_a(x) = A_a(x) - \partial_a[\Delta^{-1}\partial^b A_b](x) \tag{2.48}$$

with the Laplacian $\Delta = \delta^{ab}\partial_a\partial_b$ and using the Gauss constraint $E^a_{,a} = 0$ we may rewrite H in the form

$$H = \int d^3x \delta^{ab}\bar{z}_a(x)[\hbar\sqrt{-\Delta}](P \cdot z)_b(x) \tag{2.49}$$

where we have defined the quantity ($m = (A, E) \in \mathcal{M}$ a phase space point)

$$z_a = z_a(m) = \frac{1}{\sqrt{2\alpha}}[(\sqrt[4]{-\Delta})A_a - i(\sqrt[4]{-\Delta})^{-1}E^a] \tag{2.50}$$

which has dimension $\mathrm{cm}^{-3/2}$. They obey the standard canonical brackets and reality conditions

$$\{z_a(x), z_b(y)\} = \{\bar{z}_a(x), \bar{z}_b(y)\} = 0,\ \{z_a(x), \bar{z}_b(y)\} = -iq^2/\alpha\delta_{ab}\delta(x,y),\ \overline{z_a(x)} = \bar{z}_a(x) \tag{2.51}$$

The positive linear functional ω_F^H can be formally derived by the following steps:
1) We want to represent $\hat{A}_a(x)$ and $\hat{E}_a(x)$ respectively as multiplication operator $\pi_F^H(\hat{A}_a(x)) := A_a(x)$ and functional derivative $\pi_F^H(\hat{E}^a(x)) := i\alpha\delta/\delta A_a(x)$ on a Hilbert space $\mathcal{H}_F^H$ of square integrable functions of $P \cdot A$ (they are gauge invariant and all operators should be projected by $P\cdot$).
2) The vacuum state Ω_F^H should be selected by the condition that $\pi_F^H(P \cdot \hat{z_a})$ annihilates Ω_F^H.
3)Canonical commutation relations and adjointness conditions should be implemented according to (2.52).

Condition 3) is formally satisfied if we choose $\mathcal{H}_F^H = L_2(\mathcal{A}/\mathcal{G}', [dA]/||\Omega_F^H||^2)$ where $[dA]$ is the formal infinite dimensional Lebesgue measure, $\mathcal{A}/\mathcal{G}'$ is a distributional extension of the space of smooth $U(1)$ connections modulo gauge transformations yet to be determined (it will turn out to be the space of transversal vector valued tempered distributions on $\mathbb{R}^3$) and the infinite constant $||\Omega_F^H||$ denotes the norm of the ground state.

Conditions 1) and 2) are satisfied if we define the ground state formally as

$$\Omega_F^H(A) = e^{-\frac{1}{2\alpha}\int d^3x A_a\sqrt{-\Delta}(P\cdot A)_b} \tag{2.52}$$

Notice that none of the objects $[dA], \Omega_F^H, ||\Omega_F^H||^2$ makes sense separately, however, the combination

$$d\mu_F^H := "[dA]\frac{|\Omega_F^H|^2}{||\Omega_F^H||^2}" \tag{2.53}$$

that appears in expectation values will make rigorous sense as a Gaussian measure. For instance, if p is a closed loop, then $A(p_f)$ only depends on $P \cdot A$ and we find by formally completing the square in the appearing Gaussian integral that

$$\omega_F^H(\hat{A}(p_f)) :=< \Omega_F^H, \pi_F^H(\hat{A}(p_f))\Omega_F^H >= e^{-\frac{\alpha}{4}\int d^3x \delta_{ab} X^a_{p_f} \sqrt{-\Delta}^{-1} X^b_{p_f}} \tag{2.54}$$

There is no transversal projector needed in (2.54) because for a closed path p the form factor $X^a_{p_f}$ is already transversal. Similarly we can compute expectation values of polynomials involving $\hat{E}^a(x)$ and $\hat{A}(p_f)$ by using the rules of Gaussian integrals. It should be noted, however, that $\hat{E}^a(x)$ is not an operator but rather an operator valued distribution. In order to turn it into an operator we must smear it with a test function, i.e. we must consider the operators

$$\hat{E}(l) = \int d^3x l_a(x)\hat{E}^a(x) \tag{2.55}$$

where l_a is a transversal test function of rapid decrease of dimension cm^{-1}. We compute

$$\omega_F^H(\hat{E}(l)) = 0,\ \omega_F^H(\hat{E}(l)^2) = \frac{\alpha}{2}\int d^3x (l^a\sqrt{-\Delta} l_a) \tag{2.56}$$

From this calculation we observe that if we would replace l_a by a singular expression of the form

$$Y_a^S(x) = \frac{1}{2}\int_S \epsilon_{abc} dy^b \wedge dy^b \delta^{(3)}(x,y) \tag{2.57}$$

which would be appropriate for an electric flux operator $\hat{E}(Y^S)$ then its fluctuation diverges since e.g. for $S = [-L,L]^2 \times \{0\}$

$$\begin{aligned} &\int d^3x (Y_a^S \sqrt{-\Delta} Y_a^S) = \int_{[-L,L]^2} d^2y \int_{[-L,L]^2} d^2y' [\sqrt{-\Delta}_x \delta(x,(y_1',y_2',0))]_{x=(y_1,y_2,0)} \\ = &\int \frac{d^3k}{(2\pi)^3}||k|| \int_{[-L,L]^2} d^2y \int_{[-L,L]^2} e^{i[k_1(y_1-y_1')+k_2(y_2-y_2')} \end{aligned} \tag{2.58}$$

and at least the integral over k_3 blows up. Thus, electric flux operators are not well-defined in the standard Fock representation since e.g. $||\pi_F^H(\hat{E}(Y^S)\Omega_F^H||^2 = \omega_F^H(\hat{E}(Y^S)^2) = \infty$.

Having obtained ω_F^H we now obtain ω_P^H by the above prescription. In order to compare the resulting GNS Hilbert space $\mathcal{H}_P^H$ with the standard Hilbert space $\mathcal{H}_0$ for diffeomorphism invariant theories of $U(1)$ connections we recall that a suitable orthonormal basis for $\mathcal{H}_0$ is given by the charge network states T_s with $s = (\gamma(s), \vec{n}(s))$ where γ is a closed piecewise analytic graphs and the integers $\vec{n}$ labelling its edges are subject to the gauge invariance constraint

$$\sum_{b(e)=v} n_e - \sum_{f(e)=v} n_e = 0 \tag{2.59}$$

for each vertex v of γ. It is easy to see that

$$T_s(A) = \prod_e A(e)^{n_e} =: A(s) = e^{i\int d^3x A_a X_s^a} \text{ where } X_s^a = \sum_e n_e X_e^a \tag{2.60}$$

If we define

$$X^a_{s_f}(x) = \int d^3y f(x-y) X^a_s(y) \tag{2.61}$$

then by the isomorphism

$$\omega^H_P(\hat{A}(s)) = \omega^H_F(\hat{A}(s_f)) = e^{-\frac{\alpha}{4}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1}X^b_{s_f}\delta_{ab}} \tag{2.62}$$

defines a positive linear functional on the C^*algebra generated by the holonomies and therefore the GNS Hilbert space is of the form $L_2(\overline{\mathcal{A}}, d\mu_f)$ where the measure is uniquely defined via the Riesz representation theorem (on C^* algebras, positive linear functionals are automatically continuous) and certainly depends on f. On the other hand, the Hilbert space $\mathcal{H}_0$ is the GNS Hilbert space arising from the positive linear functional on the C^* algebra defined by $\omega_0(\hat{A}(s)) = \delta_{s,(\emptyset,\vec{0})}$.

The measures μ_f, μ_0 on $\overline{\mathcal{A}}$ are mutually singular with respect to each other, see [15] or appendix B.

Our next task will be to construct, next to the vacuum state Ω^H_F, also coherent states $\Omega^H_{F,m}$ labelled by a point $m \in \mathcal{M}$. Using the definition of being eigenstates of the annihilation operators $P\cdot\hat{z}_a$, we obtain the formal expression

$$\Omega^H_{F,m} = e^{-\frac{1}{2}\int d^3x \bar{z}^a P\cdot z_a}\ e^{\int d^3x z^a(m) P\cdot\hat{z}^\dagger_a}\Omega^H_F \tag{2.63}$$

where $z = z(m)$ as in (2.50). In order to compute the expectation value of the operator $\hat{A}(s_f)$ for instance, we make use of the identity

$$\int d^3x X^a_{s_f}\hat{A}_a = \sqrt{\alpha/2}\int d^3x[(\sqrt[4]{-\Delta})^{-1}X^a_{s_f}][\hat{z}_a + \hat{z}^\dagger_a] \tag{2.64}$$

and use the Baker-Campbell-Hausdorff (BCH) formula in order to normal order $\hat{A}(s_f)$.

$$\begin{aligned} <\Omega^H_{F,m}, \hat{A}(s_f)\Omega^H_{F,m}> &= A(s_f)e^{-\frac{1}{2}[i\sqrt{\alpha/2}\int d^3x((\sqrt[4]{-\Delta})^{-1}X^a_{s_f})\hat{z}^\dagger_a, i\sqrt{\alpha/2}\int d^3x((\sqrt[4]{-\Delta})^{-1}X^a_{s_f}\hat{z}_a]} \\ &= A(s_f)e^{-\frac{\alpha}{4}[\sqrt{\alpha/2}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1}X^a_{s_f}]} \end{aligned} \tag{2.65}$$

so it is just the classical value times a fluctuation correction. Similarly one proceeds with expectation values involving $\hat{E}^a$, one just expresses them in terms of annihilation and creation operators.

We now ask the crucial question: Given the states $\omega^H_{F,m}$ we can construct, again using the isomorphism, a state $\omega^H_{P,m}$, and corresponding GNS data $(\Omega_{P,m}, \mathcal{H}^H_{P,m}, \pi^H_{P,m})$. Is it possible to relate these GNS data to those of ω_0, that is, $(\Omega_0 = 1, \mathcal{H}_0 = L_2(\overline{\mathcal{A}}, d\mu_0), \pi_0)$?

Since the states T_s form an orthonormal basis of $\mathcal{H}_0$ we will make an ansatz of the form

$$\Omega^H_{P,m} = \sum_s c_{s,m}\overline{T_s} \tag{2.66}$$

where the coefficients $c_{s,m}$ are to be determined. Notice that we do not expect (2.66) to be normalizable as it stands with respect to the inner product on $\mathcal{H}_0$, just as it is the case for $\Omega^H_{F,m}$ with respect to $L_2(\mathcal{A}', [dA])$. In order to arrive at the expecation value functional $\omega^H_{P,m}$ we must divide by the (infinite) norm $||\Omega^H_{P,m}||^2_0$ just as we have to divide $[dA]$ by the norm squared of $||\Omega^H_{F,m}||^2$. This is also the reason for additional complex conjugation involved in (2.66) as compared to [14] which is due to the fact that we will not treat $\Omega^H_{P,m}$ as an element of a distributional extension of $\mathcal{H}_0$ but formally as a (non-normalizable) element of $\mathcal{H}_0$. It will become normalizable by dividing by its (infinite)

norm later on, thereby inducing a new inner product. This is similar in spirit to the group averaging proposal, see [4] and references therein. It is also similar in spirit to constructive QFT methods where one first regularizes a formally defined interacting measure by introducing IR and UV cut-offs, thereby obtaining a well-defined measure which is absolutely continuous with respect to a free one, and then takes the limit to arrive at a well-defined interacting measure.

In order to determine the $c_{s,m}$ we will ask that $\Omega^H_{P,m}$ is an annihilation operator eigenstate, but the question is, what annihilation operator should be chosen ? The idea of [14] is to use the basic generators $\hat{A}(p_f), \hat{E}^a$ of $\mathcal{A}_P$, to relate them to to the annihilation operators $\hat{z}_a$ and then to use the isomorphism $\mathcal{I}$. Thus we write

$$\int d^3x X^a_{s_f} A_a = \int d^3x X^a_{s_f} [\sqrt{2\alpha}(\sqrt[4]{-\Delta})^{-1} z_a + i\sqrt{-\Delta}^{-1} E^a] \tag{2.67}$$

so that, by purely algebraic manipulations, we obtain the operator identity

$$\begin{aligned} \hat{A}(s_f) &= e^{i\int d^3x[(\sqrt{2\alpha}(\sqrt[4]{-\Delta})^{-1} X^a_{s_f})\hat{z}_a + i(\sqrt{-\Delta}^{-1} X^a_{s_f})\hat{E}^a]} \\ &= e^{-\frac{\alpha}{2}\int d^3x X^a_{p_f}\sqrt{-\Delta}^{-1} X^a_{s_f}}\ e^{-\int d^3x(\sqrt{-\Delta}^{-1} X^a_{s_f})\hat{E}^a}\ e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1} X^a_{s_f})\hat{z}_a} \end{aligned} \tag{2.68}$$

where we have used again the BCH formula. Thus, our coherent state $\Omega^H_{F,m}$ can be characterized by the requirement that

$$\hat{A}(s_f)\Omega^H_{F,m} = e^{-\frac{\alpha}{2}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1} X^a_{s_f}}\ e^{-\int d^3x(\sqrt{-\Delta}^{-1} X^a_{s_f})\hat{E}^a}\ e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1} X^a_{s_f})z_a(m)}\Omega^H_{F,m} \tag{2.69}$$

Notice that a similar relation like (2.67) in the language of the algebra $\mathcal{A}_P$ is not possible because the distributional form factor X_p is not square integrable. Also the operator valued distribution $\hat{z}_a$, while well-defined on $\mathcal{H}^H_{F,m}$, has no analog on $\mathcal{H}_0$ on which we would like to define our annihilation operator. Thus, a definition of an annihilation operator along these lines fails. Therefore, as a substitute the authors of [14, 16] chose to translate (2.68), guided by the isomorphism $\mathcal{I}$, as follows

$$\hat{A}(s)\Omega^H_{P,m} = e^{-\frac{\alpha}{2}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1} X^a_{s_f}}\ e^{-\int d^3x(\sqrt{-\Delta}^{-1} X^a_{s_f})\hat{E}^a_f}\ e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1} X^a_{s_f})z_a(m)}\Omega^H_{P,m} \tag{2.70}$$

It is crucial to remark, however, that there is a certain amount of arbitrariness in this choice: For instance, should we not reexpress $z(m)$ in terms of $A(p_f), E^a$ in (2.68) and then substitute by $(A(p), E^a_f)$ in (2.70) ? Only for the vacuum state with $m = 0$ there is no such ambiguity. We will come back to this question in the next section where we derive a precise relation between coherent states from the general complexifier principle: It turns out that (2.70) is not quite the correct classical correspondence because the expectation values for $\hat{A}(p), \hat{E}^a_f$ do not come out the right way. This is automatically fixed by the complexifier approach to coherent states.

Let us then solve (2.70). It is a remarkable feature, *again true for Abelean gauge theories only*, that the T_s are eigenfunctions of 3D smeared electric field operators, specifically

$$\hat{E}^a_f(x)T_s = -\alpha X^a_{s_f} T_s \tag{2.71}$$

Now $\overline{T_s} = T_{-s}$ where $-s = (\gamma(s), -\vec{n}(s))$ and $T_s T_{s'} = T_{s+s'}$ where $s + s' = (\gamma(s) \cup \gamma(s'), \vec{n}(s) + \vec{n}(s'))$. The notation means that if e is an edge of $\gamma(s)$ then there is a unique decomposition $e = \prod_{\tilde{e}} \tilde{e}^{\sigma(e,\tilde{e})}$ where $\sigma(e, \tilde{e}) = 1, -1, 0$ if an edge $\tilde{e}$ of $\tilde{\gamma} = \gamma \cup \gamma' = \gamma(s + s')$ is overlapped by e with equal (1) or (-1) opposite orientation or not overlapped at all (0) (in the Abelean case ordering of that composition

is irrelevant). We then define $n_{\tilde{e}}(s) = \sum_e n_e \sigma(e,\tilde{e})$ and similar for the edges of $\gamma(s')$. Thus, (2.70) becomes

$$\begin{aligned} \hat{A}(s^0)\Omega^H_{P,m} &= \sum_s c_{s,m}\overline{T_{s-s^0}} = \sum_s c_{s+s^0,m}\overline{T_s} \\ &= e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f}\sqrt{-\Delta}^{-1}X^a_{s^0_f}} e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s^0_f})z_a(m)} \sum_s c_{s,m} e^{-\alpha\int d^3x(\sqrt{-\Delta}^{-1}X^a_{s^0_f})X^a_{s_f}}\overline{T_s} \end{aligned} \tag{2.72}$$

Since the $\overline{T_s}$ provide a basis we conclude

$$c_{s+s^0,m} = c_{s,m} e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f}\sqrt{-\Delta}^{-1}X^a_{s^0_f}} \quad e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s^0_f})z_a(m)} \quad e^{-\alpha\int d^3x(\sqrt{-\Delta}^{-1}X^a_{s^0_f})X^a_{s_f}} \tag{2.73}$$

Set $c_m = c_{(\emptyset,\vec{0}),m}$ then we find by setting $s=0$ in (2.73) that

$$c_{s^0,m} = c_m e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f}\sqrt{-\Delta}^{-1}X^a_{s^0_f}} \quad e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s^0_f})z_a(m)} \tag{2.74}$$

which now, upon reinsertion into (2.73) leads to the following consistency requirement

$$\begin{aligned} & e^{-\frac{\alpha}{2}\int d^3x X^a_{s+s^0_f}\sqrt{-\Delta}^{-1}X^a_{s+s^0_f}} \quad e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s+s^0_f})z_a(m)} \\ = \; & \big[e^{-\frac{\alpha}{2}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1}X^a_{s_f}} \quad e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s_f})z_a(m)}\big] \times \\ & \times\big[e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f}\sqrt{-\Delta}^{-1}X^a_{s^0_f}} \quad e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s^0_f})z_a(m)} \quad e^{-\alpha\int d^3x(\sqrt{-\Delta}^{-1}X^a_{s^0_f})X^a_{s_f}}\big] \end{aligned} \tag{2.75}$$

which is an identity since $X_{s_f} + X_{s^0_f} = X_{(s+s_0)_f}$. From the present point of view the remarkable identity (2.75) looks like a miracle. We will see however in the next section that it follows trivially from the consistency of the complexifier operator on $\mathcal{H}_0$.

Concluding, we find the unique solution (up to a free function $m \mapsto c_m$ which we set equal to unity since this is the value that is automatically selected by the complexifier method)

$$\Omega^H_{F,m} = \sum_s e^{-\frac{\alpha}{2}\int d^3x X^a_{s_f}\sqrt{-\Delta}^{-1}X^a_{s_f}} \; e^{i\sqrt{2\alpha}\int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{s_f})z_a(m)}\overline{T_s} \tag{2.76}$$

In [16] the authors made the following observation: Define for edges e, e' of a graph γ

$$G^{ee'}_f := \int d^3x X^a_{e_f}\sqrt{-\Delta}^{-1}X^a_{e'_f} \text{ and } z^e_f(m) := \int d^3x(\sqrt[4]{-\Delta})^{-1}X^a_{e_f})z_a(m) \tag{2.77}$$

then (2.76) can be rewritten as

$$\Omega^H_{F,m} = \sum_s e^{-\frac{\alpha}{2}\sum_{e,e'} G^{ee'}_f n_e n_{e'}} e^{i\sqrt{2\alpha}\sum_e z^e_f(m) n_e}\overline{T_s} \tag{2.78}$$

where the sums over edges at given s are over those of $\gamma(s)$. If we denote by Y_e the right invariant vector field on $U(1)$ acting on the degree of freedom $A(e)$ then we may rewrite (2.78) further in the form

$$\Omega^H_{F,m} = \sum_s e^{\frac{\alpha}{2}\sum_{e,e'} G^{ee'}_f Y_e Y_{e'}} e^{-\sqrt{2\alpha}\sum_e z^e_f(m) Y_e}\overline{T_s} \tag{2.79}$$

This formula suggests an immediate generalization to $SU(2)$ by replacing the charge network states T_s by spin network states, replacing $Y_e Y_{e'}$ by $Y^j_e Y^j_{e'}$ and $z^e(m)Y_e$ by $z^e_{j,f}(m)Y^j_e$ where Y^j_e are right invariant vector fields on $SU(2)$. Thus, they propose

$$\Omega^H_{F,m} = \sum_s e^{\frac{\alpha}{2}\sum_{e,e'} G^{ee'}_f Y^j_e Y^j_{e'}} e^{-\sqrt{2\alpha}\sum_e z^e_{j,f}(m) Y^f_e} \overline{T_s} \tag{2.80}$$

However, several remarks are in order:
1)
Since the "edge metric" $G^{ee'}_f$ is not diagonal, the operator $Y^j_e Y^j_{e'}$ is not gauge invariant for $e \neq e'$. In [16] the authors propose to repair this by averaging the states over the $SU(2)$ gauge group. Another, maybe more direct method is to work instead with operators of the form

$$\mathrm{Tr}([\mathrm{Ad}_{\hat{A}(\rho_{b(e)})} Y_e]\ [\mathrm{Ad}_{\hat{A}(\rho_{b(e')})} Y_{e'}]) \tag{2.81}$$

where $x \mapsto \rho_x$ defines a system of paths between the point x and an arbitrary but fixed point x_0. Again, it is *only for Abelean groups* not necessary to do this. However, while the operator in the exponent of (2.80) is negative definite when restricted to functions over the graph in question,

$$\hat{G}^f_\gamma := \sum_{e,e'} G^{ee'}_f Y^j_e Y^j_{e'} = -\int d^3x [\sum_e X^a_{e_f} Y^j_e]^\dagger(x) (\sqrt{-\Delta}^{-1} [\sum_e X^a_{e_f} Y^j_e])(x) \tag{2.82}$$

the path holonomies in (2.81) spoil positivity. They also spoil the convenient fact that (2.82) leaves each of the Hilbert spaces $\mathcal{H}_{0,\gamma}$ separately invariant. Thus, not only the spectral analysis of (2.81) is unclear, also if we are thinking of cut-off states (shadows), they are no longer granted to be normalizable.
2.)
The operators $\hat{G}^f_\gamma$ do not form a consistent system of operators and therefore no inductive limit operator $\hat{G}^f$ exists. This is closely related to the fact that, *unless the gauge group is Abelean*, there is no continuum function, i.e. a complexifier, which gives rise to the states as will be demonstrated in the next section.
3.)
Since there is no classical complexifier function available, it is unclear what the function $z^e_{j,m}$ should be where $m = (A^j_a, E^a_j)$. Abstracting from (2.77) one might guess, e.g. in the case of general relativity the formula

$$\begin{aligned} z^e_{j,f}(m) &:= \int d^3x (\sqrt[4]{-\Delta})^{-1} X^a_{e_f}) z^j_a(m) \\ z^j_a(m) &= \frac{1}{\sqrt{2\alpha}} [\sqrt[4]{-\Delta} A^j_a - \frac{i}{a} (\sqrt[4]{-\Delta})^{-1} E^a_j] \end{aligned} \tag{2.83}$$

where a is a is a constant with unit cm. Also, the Feinstruktur constant α must be exchanged against some dimensionless constant of the form $[\ell_p/a]^n$ for some power of n. Notice that $P \cdot z^j_a$ is not gauge covariant, moreover, since there is no physically interesting non-Abelean Hamiltonian which is bilinear in $z^j_a, \bar{z}^j_a$ there is no good motivation for the proposal (2.83).
4.)
Since there is no underlying complexifier, we have no guarantee that the expectation values of the operators $\hat{A}(e), \hat{E}^a_{f,j}$ have any close relation with their classical values at the phase space point m. What is more, the operator $\hat{E}^a_{j,f}$ fails to be densely defined in the representation $\mathcal{H}_0$ so that it is

impossible to define it (or its dual) on the state $\Omega^H_{P,m}$. Therefore, the commutator of these operators is ill-defined so that the expectation value of the commutators cannot have anything to do with the classical Poisson brackets. From the point of view of the isomorphism $\mathcal{I}$ this is a consequence of the fact that for non-Abelean gauge theories there is no gauge invariant state $\omega^H_{F,m}$ from which we could derive $\Omega^H_{P,m}$ by similar methods, moreover, there is no isomorphism of Poisson algebras because smeared holonomies do not transform gauge covariantly any longer and the Poisson algebra between 3D smeared electric fields and unsmeared holonomies no longer closes.
5.
It is unclear what the definition of a suitable annihilation operator should be.

We conclude this section by showing that in the Abelean case the $\hat{G}_\gamma$ actually form a consistent family. Responsible for this are:
1) The smeared form factors satisfy the following identities

$$X_{(e^{-1})_f} = -X_{e_f},\ X_{(e\circ e')_f} = X_{e_f} + X_{e'_f} \tag{2.84}$$

which implies corresponding identities for $z^e_f(m)$.
2) The vector fields Y_e form an Abelean algebra *only in the Ablean case* and satisfy

$$Y_{e^{-1}} = -Y_e,\ (Y_e)_{|e\circ e'} = (Y_{e'})_{|e\circ e'} = Y_{e\circ e'} \tag{2.85}$$

where the notation in the latter identity means that the vector fields are restricted to functions of $A(e\circ e')$.
3) We have

$$\begin{aligned}\hat{G}^f_\gamma &:= \sum_{e,e'} G^{ee'}_f Y_e Y_{e'} = \int d^3x [\sum_e X^a_{e_f} Y_e](x)(\sqrt{-\Delta}^{-1}[\sum_e X^a_{e_f} Y_e])(x)\\ &=: \int d^3x Y^f_\gamma(x)(\sqrt{-\Delta}^{-1} Y^f_\gamma)(x)\end{aligned} \tag{2.86}$$

Now consistency means that $\hat{U}_{\gamma'\gamma}\hat{G}_{\gamma'} = \hat{G}_\gamma \hat{U}_{\gamma'\gamma}$ for any $\gamma' \subset \gamma$ where we can confine ourselves to the case that for the sets of oriented edges of a graph holds a) $E(\gamma') = E(\gamma) - \{e\}$ does not contain some edge e, b) $E(\gamma') = (E(\gamma) - \{e\}) \cup \{e^{-1}\}$ contains some edge inverted or c) $E(\gamma') = (E(\gamma) - \{e_1, e_2\}) \cup \{e_1 \circ e_2\}$ contains two edges only as their composition. It is straightforward to check, using properties 1),2),3), that consistency already holds for the vector fields Y^f_γ so that consistency for $\hat{G}_\gamma$ follows easily from the Abelean nature of the Y_e. For the same reason also the vector fields

$$Z^f_\gamma = \sum_e z^e_f Y_e \tag{2.87}$$

form a consistent family. All of this will be spoiled in the non-Abelean context where (2.85) is replaced by

$$Y^j_{e^{-1}} = -O_{jk}(A(e))Y^k_e,\ (Y^j_e)_{|e\circ e'} = Y^j_{e\circ e'},\ (Y^j_{e'})_{|e\circ e'} = O_{jk}(A(e))Y^k_{e\circ e'} \tag{2.88}$$

where $O_{jk}(h)\tau_k = \mathrm{Ad}_h(\tau_j)$ is an orthogonal matrix and τ_j is a basis of $Lie(G)$, unless $G^{ee'}_f$ is diagonal as in [9, 10] which of course is not the case here.

Finally, notice the similarity between (2.78) which we write in the more suggestive form

$$\Omega^H_{F,m} = \sum_s e^{-\lambda_s/\hbar} T_{s_f}(A^{\mathbb{C}})\overline{T_s} \tag{2.89}$$

and the complexifier coherent states of the previous subsection. Here $\lambda_s = \frac{\alpha\hbar}{2}\sum_{e,e'} G_f^{ee'} n_e n_{e'}$ and $A^{\mathbb{C}} = A - i\sqrt{-\Delta}^{-1}E$. In the next section we will show the complexifier theoretic origin for all of these facts, the only thing that is disturbing is the appearance of smeared holonomies in (2.89) which suggests that $\Omega^H_{F,m}$ is peaked at $m_f = (A_f, E_f)$ rather than m as we would expect. In fact, this is due to the fact that the translation of the annihilation condition (2.69) into (2.70) is ambiguous without any additional structural principle which will be automatically provided by the complexifier construction.

3 Complexifier Theoretic Origin of Varadarajan States

In order to motivate the complexifier underlying the Varadarajan states we begin again with the usual Fock states $\Omega^H_{F,m}$ but this time we will not derive them by asking to be eigenstates of an annihilation operator, rather we would like to derive them from a complexifier. The corresponding, exponentiated, annihilator will be automatically produced as a byproduct. This line of derivation makes the similarity between the Fock representation and the polymer representation even more transparent.

We claim that the usual Maxwell coherent states follow from the complexifier

$$C^H_F = \frac{1}{2q^2}\int d^3x E^a(x)(\sqrt{-\Delta}^{-1}E^a)(x) \tag{3.1}$$

In order to verify this, it is sufficient to check that

$$\sum \frac{i^n}{n!}\{A_a, C^H_F\}_{(n)} = A_a - i\sqrt{-\Delta}^{-1}E^a = \sqrt{2\alpha}\,\sqrt[4]{-\Delta}z_a = A^{\mathbb{C}}_a(m) \tag{3.2}$$

is indeed the complexified connection that is induced by the Maxwell Hamiltonian. It follows from the rules of canonical quantization that

$$\hat{C}^H_F/\hbar = -\frac{\alpha}{2}\int d^3x \frac{\delta}{\delta A_a(x)}(\sqrt{-\Delta}^{-1}\frac{\delta}{\delta A_a})(x) \tag{3.3}$$

which is formally self-adjoint and positive on the formal Hilbert space $\mathcal{H}^H_F = L_2(\mathcal{A}/\mathcal{G}', [dA])$. Thus we can compute the complexifier coherent states once we have a suitable δ-distribution with respect to the Lebesgue measure $[dA]$ at our disposal. A suitable representation can be formally given as the formal functional integral over transversal momenta k^a

$$\delta^F_{A'}(A) = \int [\frac{d^2k}{(2\pi)^2}] e^{i\int d^3x k^a(x)[A'_a(x) - A_a(x)]} \tag{3.4}$$

Notice the formal similarity between (3.4) and the δ-distribution on $\mathcal{H}_0 = L_2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$ defined in section 2.2 and which specializes for $U(1)$ to

$$\delta^P_{A'}(A) = \sum_s T_s(A')\overline{T_s(A)} \tag{3.5}$$

where the sum is over gauge invariant charge networks. Thus we get correspondences between objects on the Fock side on the one hand and objects on the Polymer side on the other, summarized in the following table

$$\begin{array}{ccc}
\text{Lebesgue measure } [dA] & \leftrightarrow & \text{Uniform measure } d\mu_0 \\
\text{Tempered distributions } \mathcal{A}/\mathcal{G}' & \leftrightarrow & \text{Ashtekar-Isham spectrum } \overline{\mathcal{A}/\mathcal{G}} \\
\text{Functional integral } \int[\frac{d^2k}{(2\pi)^2}] & \leftrightarrow & \text{Discrete sum } \sum_s \\
\text{3D smeared ``holonomy'' } e^{i\int d^3x k^a(x)A_a(x)} & \leftrightarrow & \text{holonomy } A(s) = T_s(A) = e^{i\int d^3x X^a_s A_a}
\end{array} \tag{3.6}$$

Interestingly, while the objects on the Fock side are only formal, the polymer objects are *rigorously defined* !

Our complexifier coherent states on the Fock side become

$$\begin{aligned}\Omega^H_{F,m}(A) &= [(e^{-\hat{C}^H_F/\hbar}\delta_{A'})(A)]_{A'\to A^{\mathbb{C}}(m)} \\ &= (\int [\frac{d^2k}{(2\pi)^2}] e^{-\frac{\alpha}{2}\int d^3x k^a\sqrt{-\Delta}k_a}\; e^{i\int d^3x k^a[A'_a - A_a]})_{A'\to A^{\mathbb{C}}(m)} \\ &= \mathcal{N}(A^{\mathbb{C}}) e^{-\frac{1}{2\alpha}\int d^3x [A^{\mathbb{C}}_a - A_a] P\cdot\sqrt{-\Delta}[A^{\mathbb{C}}_a - A_a]} \end{aligned} \tag{3.7}$$

where $\mathcal{N}(A^{\mathbb{C}})$ is the usual infinite constant coming from the functional determinant in a Gaussian integral. It is easy to check that (3.7) divided by its (infinite) norm coincides with our earlier definition up to a finite phase, similar to the case of a harmonic oscillator.

An interesting object is the exponentiated annihilation operator corresponding to a smeared holonomy

$$\begin{aligned}\hat{g}_{p_f} &:= e^{-\hat{C}^H_F/\hbar}\hat{A}(p_f)e^{\hat{C}^H_F/\hbar} = e^{e^{-\hat{C}^H_F/\hbar}\, i\int d^3x X^a_{p_f}\hat{A}_a\, e^{\hat{C}^H_F/\hbar}} \\ &= e^{i\sum\frac{1}{n!}\int d^3x X^a_{p_f}[\hat{A}_a,\hat{C}^H_F/\hbar]_{(n)}} = e^{i\int d^3x X^a_{p_f}[\hat{A}_a - i\sqrt{-\Delta}^{-1}\hat{E}^a]} \\ &= \hat{A}(p_f)e^{\int d^3x(\sqrt{-\Delta}^{-1}X^a_{p_f})\hat{E}^a]} e^{-\frac{\alpha}{2}\int d^3x X^a_{p_f}\sqrt{-\Delta}^{-1}X^a_{p_f}} \\ &= e^{\int d^3x(\sqrt{-\Delta}^{-1}X^a_{p_f})\hat{E}^a]}\hat{A}(p_f)e^{\frac{\alpha}{2}\int d^3x X^a_{p_f}\sqrt{-\Delta}^{-1}X^a_{p_f}} \end{aligned} \tag{3.8}$$

which can be easily checked to have eigenvalue $e^{i\int d^3x X^a_{pf}A^{\mathbb{C}}_a(m)}$ on $\Omega^H_{F,m}$.

We now turn to the polymer representation. It is easy to see that the operator $\hat{C}^H_F$ is not well-defined on charge network states, specifically, $\hat{C}^H_F/\hbar T_s = [\frac{\alpha}{2}\int d^3x X^a_s\sqrt{-\Delta}^{-1}X_s]T_s$ which blows up due to the too singular behaviour of the form factors. The isomorphism $\mathcal{I}$ suggests to try the smeared **Varadarajan complexifier**

$$C^H_P = \frac{1}{2q^2}\int d^3x E^a_f(\sqrt{-\Delta}^{-1}E^a_f) \tag{3.9}$$

which corresponds to a slightly different complexified connection

$$A^{\mathbb{C}}_{a,f}(m) = A_a - i\int d^3x f(\sqrt{-\Delta}^{-1}E^a_f) \tag{3.10}$$

in which the electric field appears automatically *doubly smeared*. The operator corresponding to (3.9) is in fact a densely defined, positive, essentially self-adjoint operator which is diagonalized by the charge network states, that is, its spectrum is pure point ! Specifically

$$\hat{C}^H_P/\hbar T_s = [\frac{\alpha}{2}\int d^3x X^a_{s_f}(\sqrt{-\Delta}^{-1}X^a_{s_f})]T_s =: \lambda_s/\hbar T_s \tag{3.11}$$

The reader will immediately recognize the eigenvalue λ_s *precisely* as the eigenvalue that appeared in Varadarajan's derivation of the coherent states $\Omega^H_{P,m}$. Our complexifier method now produces the coherent states

$$\begin{aligned}\Omega^H_{P,m}(A) &:= [(e^{-\hat{C}^H_P/\hbar}\delta_{A'})(A)]_{A'\to A^{\mathbb{C}}_f(m)} \\ &= \sum_s e^{-\lambda_s/\hbar}T_s(A^{\mathbb{C}}_f(m))\overline{T_s(A)} \end{aligned} \tag{3.12}$$

Our (exponentiated) annihilation operators are defined in the charge network basis as (notice that $\hat{E}^a_f T_s = -\alpha X^a_{s_f} T_s$)

$$\begin{aligned}
\hat{g}_{s^0} T_s &:= e^{-\hat{C}^H_P/\hbar} \hat{A}(s^0) e^{\hat{C}^H_P/\hbar} T_s = e^{-\lambda_{s+s^0}/\hbar} e^{\lambda_s/\hbar} T_{s+s^0} \\
&= e^{-\lambda_{s^0}/\hbar - \alpha \int d^3x X^a_{s_f} \sqrt{-\Delta}^{-1} X^a_{s^0_f}} T_{s+s^0} \\
&= e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f} (\sqrt{-\Delta})^{-1} X^a_{s^0_f})} \hat{A}(s^0)\, e^{\int d^3x (\sqrt{-\Delta}^{-1} X^a_{s^0_f}) \hat{E}^a_f}\, T_s \\
&= e^{\frac{\alpha}{2}\int d^3x X^a_{s^0_f} (\sqrt{-\Delta})^{-1} X^a_{s^0_f})} e^{\int d^3x (\sqrt{-\Delta}^{-1} X^a_{s^0_f}) \hat{E}^a_f}\, \hat{A}(s^0)\, T_s
\end{aligned} \tag{3.13}$$

from which

$$\begin{aligned}
\hat{g}_{s^0} &= e^{-\frac{\alpha}{2}\int d^3x X^a_{s^0_f} (\sqrt{-\Delta})^{-1} X^a_{s^0_f})} \hat{A}(s^0)\, e^{\int d^3x (\sqrt{-\Delta}^{-1} X^a_{s^0_f}) \hat{E}^a_f} \\
&= e^{\frac{\alpha}{2}\int d^3x X^a_{s^0_f} (\sqrt{-\Delta})^{-1} X^a_{s^0_f})} e^{\int d^3x (\sqrt{-\Delta}^{-1} X^a_{s^0_f}) \hat{E}^a_f}\, \hat{A}(s^0)
\end{aligned} \tag{3.14}$$

which should be compared with formula (3.8). It is easy to check that the $\Omega^H_{P,m}$ are eigenstates for the $\hat{g}_s$ with eigenvalue $T_s(A^{\mathbb{C}}_f(m))$ which by comparison with (3.14) is the correct classical value corresponding to that operator. This is an important difference with the states (2.76) for which the operators $\hat{g}_s$ have eigenvalue $T_{s_f}(A^{\mathbb{C}}(m)$ which is the correct eigenvalue for the operators (3.8) but not for (3.14). Thus we see that the complexifier method, being based on an underlying canonical transformation that complexifies a configuration space, is automatically produces coherent states with the correct classical correspondence. Likewise one can explicitly check that the commutator among the $\hat{g}_s, \hat{g}^\dagger_s$ precisely corresponds to the Poisson brackets between the $g_s, \bar{g}_s$.

Of course, the states (3.12), even in their cut-off form, have not been demonstrated to have all the desired semiclassical properties. In their cut-off form, the methods of [9, 10] could be of help provided the function f is stronly peaked at the origin of the coordinate system, so that the off-diagonal elements of the edge metric are suppressed. For instance, following [29], Varadarajan proposed

$$f(x) = \frac{e^{-\frac{||x||^2}{2r^2}}}{(\sqrt{2\pi} r)^3} \tag{3.15}$$

which becomes a $\delta-$distribution in the limit $r \to 0$. If we parameterize edges as the image of the closed interval in $[0, 1]$ then it is easy to see that the edge metric components become

$$G^{ee'}_f = \frac{1}{2\pi^2 r} \int_0^1 dt \int_0^1 dt' \frac{< \dot{e}(t), \dot{e}'(t') >}{||e(t) - e'(t')||} \int_0^\infty dk\, e^{-k^2} \sin(k \frac{||e(t) - e'(t')||}{r}) \tag{3.16}$$

where $< ., . >$ is the Euclidean inner product on $\mathbb{R}^3$. Let $y \neq 0$, then using the Taylor expansion of the sine function we find

$$\int_0^\infty dk e^{-k^2} \frac{\sin(ky)}{y} = \frac{1}{2} \sum_{n=0}^\infty \frac{(-y^2)^n\, n!}{(2n+1)!} \approx e^{-y^2/4}$$

where we have used Stirling's approximation for the factorials. Thus

$$G^{ee'}_f \approx \frac{1}{(2\pi r)^2} \int_0^1 dt \int_0^1 dt' < \dot{e}(t), \dot{e}'(t') > e^{-\frac{||e(t) - e'(t')||^2}{4r^2}} \tag{3.17}$$

As an example, let us compute (3.17) for $e(t) = u\epsilon t, ;\ e'(t) = N\epsilon w + v\epsilon t$ where u, v, w are unit vectors, ϵ is the typical edge length of a given graph and N denotes the distance between the edges in lattice units. Then

$$G_f^{ee'} \approx \frac{\epsilon^2 < u, v >}{(2\pi r)^2} \int_0^1 dt \int_0^1 dt' e^{-\epsilon^2 \frac{||Nw+vt'-ut||^2}{4r^2}} \text{ and } G_f^{ee} \approx \frac{\epsilon^2}{(2\pi r)^2} \int_0^1 dt \int_0^1 dt' e^{-\epsilon^2 \frac{(t-t')^2}{4r^2}} \tag{3.18}$$

So the off-diagonal elements are suppressed by an order of $e^{-(\frac{N\epsilon}{2r})^2}$ which is small for $N\epsilon \gg 2r$. Thus, since r is fixed, for sufficiently fine graphs an order of $N \approx r/\epsilon$ nearest neighbour elements cannot be neglected which becomes arbitrarily large as $\epsilon \to 0$ so that one will tie $\epsilon \approx r$ in order to control this. Concluding, the Poisson transformation that was the important tool in [9, 10] to arrive at peakedness estimates and Ehrenfest theorems can still be applied but due to the non-diagonal nature of $G_f^{ee'}$ and since for Maxwell theory $\alpha \approx 1/137$ is not extremely small, the estimates will become worse.

Let us now compute the fluctuation of the basic operators $\hat{A}(p), \hat{E}(S)$, on which the polymer Hilbert space $\mathcal{H}_0$ is based, in the state $\Omega_{P,m}^H$ (in the cut-off version these are fine of course since the states are normalizable). In order to compute those the strategy will be to try to express them in terms of the operators $\hat{g}_e, \hat{g}_e^\dagger$. Using (3.14) we find

$$\begin{aligned} \hat{g}_p^\dagger \hat{g}_p &= e^{-\alpha \int d^3x X_{p_f}^a (\sqrt{-\Delta})^{-1} X_{p_f}^a)} e^{2 \int d^3x (\sqrt{-\Delta}^{-1} X_{p_f}^a) \hat{E}_f^a} \\ \hat{A}(p) &= e^{-\frac{\alpha}{2} \int d^3x X_{p_f}^a (\sqrt{-\Delta})^{-1} X_{p_f}^a)} \, [\sqrt{\hat{g}_{p^{-1}}^\dagger \hat{g}_{p^{-1}}}] \, \hat{g}_p \end{aligned} \tag{3.19}$$

The first line represents a function of $\hat{E}$ only from which we would like to extract $\hat{E}(S)$ through a limiting procedure using small loops p.

Notice that all calculations will be performed without recourse to the Fock state $\omega_{F,m}^H$, all calculations are just using techniques from the Hilbert space $\mathcal{H}_0$! This is an important difference to [14, 16] because it shows that *we do not need to first define a completely new representation* (the GNS representation derived from $\omega_{P,m}^H$) which reconfirms $\mathcal{H}_0$ *as the fundamental representation* (at the kinematical level) from which everything else is to be derived.

We begin with the expectation value of $\hat{A}(p)$. In order to get rid of the square root in (3.19) we notice the basic estimate, valid for any $y \geq 0$,

$$\frac{1}{2}[1 + y - (y-1)^2] \leq \sqrt{y} \leq \frac{1}{2}[y+1] \tag{3.20}$$

which could be sharpened further by taking more powers of y into account, however, (3.20) will be sufficient for our purposes. The upper and lower bound in (3.20) coincide the better with the actual value the smaller $|y-1|$. We will exploit this fact in the subsequent estimate. Let us define the normal ordered, positive operator (recall $\hat{g}_p^{-1} = \hat{g}_{p^{-1}}$)

$$\hat{y}_{p^{-1}} : (m) = |g_p(m)|^2 \hat{g}_{p^{-1}}^\dagger \hat{g}_{p^{-1}} \tag{3.21}$$

whose expectation value in the state $\Omega_{P,m}^H$ equals unity. Using the commutation relations it is easy to verify that

$$\hat{y}_{p^{-1}}(m)^2 = |g_p(m)|^4 (\hat{g}_{p^{-1}}^\dagger)^2 (\hat{g}_{p^{-1}})^2 e^{2\alpha \int d^3x X_{p_f}^a \sqrt{-\Delta}^{-1} X_{p_f}^a} \tag{3.22}$$

Writing the holonomy operator in the form

$$\hat{A}(p) = e^{-\frac{\alpha}{2} \int d^3x X_{p_f}^a (\sqrt{-\Delta})^{-1} X_{p_f}^a)} |g_p(m)|^{-1} \sqrt{\hat{y}_{p^{-1}}} \hat{g}_p \tag{3.23}$$

we find for its expectation value

$$\frac{<\Omega^H_{P,m},\hat{A}(p)\Omega^H_{P,m}>}{||\Omega^H_{P,m}||^2}=e^{-\frac{\alpha}{2}\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f}}[A(m)](p)\frac{<\Omega^H_{P,m},\sqrt{\hat{y}}\Omega^H_{P,m}>}{||\Omega^H_{P,m}||^2} \tag{3.24}$$

Thus, putting everything together

$$1-\frac{e^{2\alpha\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f}}-1}{2}\leq e^{\frac{\alpha}{2}\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f}}\frac{<\hat{A}(p)>_m}{[A(m)](p)}\leq 1 \tag{3.25}$$

Thus, for sufficiently short paths p the expectation value is rather close to its classical value and the quantum corrections are a power series in α. Notice that the trick (3.20) can also be applied to other important operators in quantum general relativity which involve square roots, such as area and volume operators. Fluctuations of the holonomy operator for loops p can be reduced to expectation values since $\hat{A}(p)^2=\hat{A}(p^2)$ so formula (3.25) can be applied. We will leave this task to the interested reader.

Next we turn to the expectation value of $\hat{E}(S)$. From (3.19) we find the explicit formula

$$\begin{aligned}\int d^3x(\sqrt{-\Delta}^{-1}X^a_{p_f})\hat{E}^a_f &= \frac{\alpha}{2}\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f}+\ln(|g_p(m)|)+\frac{1}{2}\ln(\hat{y}_p)\\ = \quad -\frac{\alpha}{2}\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f}&+\ln(|g_p(m)|)-\frac{1}{2}\ln(\hat{y}_{p^{-1}})\end{aligned} \tag{3.26}$$

Using the elementary estimate $1-y^{-1}\leq\ln(y)\leq y-1$, valid for all $y>0$, we find

$$|\int d^3x(\sqrt{-\Delta}^{-1}X^a_{p_f})<\hat{E}^a_f>_m-\ln(|g_p(m)|)|\leq\frac{\alpha}{2}\int d^3x X^a_{p_f}(\sqrt{-\Delta})^{-1}X^a_{p_f} \tag{3.27}$$

In order to deduce from this estimate information about $\hat{E}_f$ and then the electric flux itself we must remove the smeared form factors. To do this, let S_p be any surface with $\partial S_p=p$ then upon performing a couple of integrations by parts we obtain

$$E(p_f):=\int d^3x(\sqrt{-\Delta}^{-1}X^a_{p_f})E^a_f=\int_{S_p}dS_a(x)((\sqrt{-\Delta}^{-1}(\nabla\times E)_f)_f)^a(x) \tag{3.28}$$

where the subscript denotes smearing with f as before. Consider now the special circular loops $p^a_\epsilon(x)$ with radius ϵ in the $x^a=const.$ plane with center x and let $S^a_\epsilon(x)$ be the surface in the $x^a=const.$ plane bounded by it. Then

$$\lim_{\epsilon\to 0}\frac{E([p^a_\epsilon(x)]_f)}{\pi\epsilon^2}=((\sqrt{-\Delta}^{-1}(\nabla\times E)_f)_f)^a(x) \tag{3.29}$$

In order to extract E itself from this formula, we must specify the smearing function f. Let us assume, as in [14] that $f(x,y)=e^{-||x-y||^2/(2r^2)}/(\sqrt{2\pi}r)^3$. Then in fact f is the kernel of the smoothening operator $e^{\Delta r^2}$ which has an inverse precisely on functions of the form $e^{\Delta r^2}F$ where F is smooth, say. Thus

$$e^{-\Delta_x r^2}\sqrt{-\Delta_x}e^{-\Delta_x r^2}\lim_{\epsilon\to 0}\frac{E([p^a_\epsilon(x)]_f)}{\pi\epsilon^2}=(\nabla\times E)(x) \tag{3.30}$$

Finally, observing the identity $P=\Delta^{-1}\nabla\times\nabla\times$ for the transversal projection operator we obtain

$$\Delta_x^{-1}\epsilon^{abc}\partial_{x^b}e^{-\Delta_x r^2}\sqrt{-\Delta_x}e^{-\Delta_x r^2}\lim_{\epsilon\to 0}\frac{E([p^c_\epsilon(x)]_f)}{\pi\epsilon^2}=(P\cdot E)^a(x) \tag{3.31}$$

The fact that only gauge invariant holonomies $A_f^{\mathfrak{C}}(p)$ are allowed implies that we can only extract the tranversal piece of the electric field which is precisely what we are interested in.

We can now study the expectation value of $(P\cdot\hat{E})^a_f(x)$ which from the intuition of the isomorphism $\mathcal{I}$ should be a well-defined operator valued distribution, or in other words, for any tranversal smearing field l_a of dimension cm^{-1} the doubly smeared object $\hat{E}_f(l) = \int d^3x l_a \hat{E}^a_f$ should be a well-defined operator with finite fluctuations in the state $\Omega^H_{P,m}$. Already at this point it is intuitively clear that the electric flux operator can be at most an operator valued distribution whose fluctuations therefore should diverge in the coherent states. Let us verify these statements explicitly: We have

$$\hat{E}_f(l) = \int d^3x (\Delta^{-1}\nabla\times l)_a(x)\sqrt{-\Delta_x}e^{-\Delta_x r^2}\lim_{\epsilon\to 0}\frac{\hat{E}([p^a_\epsilon(x)]_f)}{\pi\epsilon^2} \tag{3.32}$$

From (3.27) we find

$$\begin{aligned} 0 &\leq |\int d^3y(\sqrt{-\Delta}^{-1}X^b_{[p^a_\epsilon(x)]_f})(y) < \hat{E}^b_f(y) >_m - \ln(|g_{p^a_\epsilon(x)}(m)|)| \\ &\leq \frac{\alpha}{2}\int d^3y X^b_{[p^a_\epsilon(x)_f}(y)(\sqrt{-\Delta})^{-1}X^b_{[p^a_\epsilon(x)]_f}(y) \\ &= (\pi\epsilon^2)^2\sum_{b\neq a}\int d^3y f_{,b}(u-x)(\sqrt{-\Delta}^{-1}f_{,b})(u-x) + O(\epsilon^5) \end{aligned} \tag{3.33}$$

Thus

$$< \hat{E}_f(l) >_m = [E_f(l)](m) \tag{3.34}$$

has no quantum corrections as expected since in the Fock representation $\hat{E}(l)$ is already normal ordered. In order to compute the fluctuation of $\hat{E}_f(l)$ we could again use formula (3.26) and use estimates for $\ln(y)^2$. It is, however, technically more instructive to use a different, more direct method. We observe that

$$\lim_{\epsilon\to 0}\frac{|g_{p^a_\epsilon(x)}(m)|^2-1}{2\pi\epsilon^2} = ((\sqrt{-\Delta}^{-1}(\nabla\times E)_f)_f)^a(x) \tag{3.35}$$

so that

$$\hat{E}_f(l) = \int d^3x[\Delta^{-1}\nabla\times l]_a[\sqrt{-\Delta_x}e^{-\Delta_x r^2}\lim_{\epsilon\to 0}\frac{\hat{g}^\dagger_{p^a_\epsilon(x)}\hat{g}_{p^a_\epsilon(x)}-1}{2\pi\epsilon^2}] \tag{3.36}$$

where we have judiciously normal ordered the operator. It can be verified once more with (3.36) that (3.35) holds. In computing the square of (3.36) we encounter

$$\begin{aligned} & [\hat{g}^\dagger_{p^a_\epsilon(x)}\hat{g}_{p^a_\epsilon(x)}-1][\hat{g}^\dagger_{p^b_\sigma(y)}\hat{g}_{p^b_\sigma(y)}-1] \\ = \;& \hat{g}^\dagger_{p^a_\epsilon(x)}\hat{g}^\dagger_{p^b_\sigma(y)}\hat{g}_{p^a_\epsilon(x)}\hat{g}_{p^b_\sigma(y)}e^{2\alpha\int d^3z X^c_{[p^a_\epsilon(x)]_f}(z)[\sqrt{-\Delta}^{-1}X^c_{[p^b_\sigma(y)]_f}](z)} \\ & -\hat{g}^\dagger_{p^a_\epsilon(x)}\hat{g}_{p^a_\epsilon(x)} - \hat{g}^\dagger_{p^b_\sigma(y)}\hat{g}_{p^b_\sigma(y)} + 1 \end{aligned} \tag{3.37}$$

which we have written in normal ordered form. The expectation value of (3.37) is thus given by

$$\begin{aligned} & [|g_{p^a_\epsilon(x)}|^2-1][|g_{p^b_\sigma(y)}|^2-1] \\ & +(e^{2\alpha q}-1)\{[|g_{p^a_\epsilon(x)}|^2-1][|g_{p^b_\sigma(y)}|^2-1] + [|g_{p^a_\epsilon(x)}|^2-1] + [|g_{p^b_\sigma(y)}|^2-1] + 1\} \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} q &= \int d^3z X^c_{[p^a_\epsilon(x)]_f}(z)[\sqrt{-\Delta}^{-1}X^c_{[p^b_\sigma(y)]_f}](z) \\ &= \int_{S^a_\epsilon(x)} dS_c(u)\int_{S^b_\sigma(y)} dS_d(v)\int d^3w(\delta^{cd}\delta^{ef}-\delta^{cf}\delta^{de})f_{,e}(w-u)[\sqrt{-\Delta}^{-1}f_{,f}](w-v) \\ &= (\pi\epsilon\sigma)^2\int d^3z(\delta^{ab}\delta^{cd}-\delta^{ad}\delta^{bc})f_{,c}(z-x)[\sqrt{-\Delta}^{-1}f_{,d}](z-y)+O(\epsilon^3\sigma^2,\epsilon^2\sigma^3) \end{aligned} \tag{3.39}$$

We conclude that

$$\begin{aligned} & <\hat{E}_f(l)^2>_m -[E_f(l)](m)^2 \\ = \;& \frac{\alpha}{2}\int d^3x[\Delta^{-1}\nabla\times l]_a(x)\sqrt{-\Delta_x}e^{-\Delta_x r^2}\int d^3y[\Delta^{-1}\nabla\times l]_b(y)\times \\ & \times\sqrt{-\Delta_y}e^{-\Delta_y r^2}\int d^3z(\delta^{ab}\delta^{cd}-\delta^{ad}\delta^{bc})f_{,c}(z-x)[\sqrt{-\Delta}^{-1}f_{,d}](z-y) \\ = \;& \frac{\alpha}{2}\int d^3x l_a(x)[\sqrt{-\Delta}l_a](x) \end{aligned} \tag{3.40}$$

where we used transversality of l. From this formula it is easy to compute the expectation value and fluctuation of the electric flux operator $P\cdot\hat{E}(S)$ by using the distributional smearing field

$$l_a(x) = P_x\cdot e^{-r^2\Delta_x}\int_S dS_a(y)\delta(x-y) \tag{3.41}$$

Upon inserting (3.41) into (3.40) the integral blows up even much worse than on the Fock space side. This is to be expected from the isomorphism $\mathcal{I}$ because the 2D smeared electric field in the polymer algebra has no preimage in the Fock algebra.

What is remarkable about these calculations, however, is that we can perform strong operator limits of the form $\epsilon\to 0$ as in (3.36), as observed in [14], that is, with respect to the new representations defined by coherent states and their excitations. Such limits were not possible with respect to the normalizable states of $\mathcal{H}_0$ which signals that we are dealing here with a new representation upon passing to non-normalizable (distributional) states. In [22] such states were given a home, the algebraic dual Cyl* of the dense subspace of smooth cylindrical functions Cyl$\subset\mathcal{H}_0$. Of course, this is possible only because we are allowed to use the Minkowski background metric all the time which makes it possible that the coefficients of the spin-network expansion of $\Omega^H_{P,m}$ depend smoothly on the graph rather than being diffeomorphism invariant. On the other hand, in order to be of use for the quantization of diffeomorphism invariant operators on $\mathcal{H}_0$ which use 2D smeared electric fluxes in an essential way we must use a complexifier which controls the fluctuations of $\hat{E}(S)$ and $\hat{A}(p)$ simultaneously in such new representations. This brings us to the next section.

4 Electric Flux Operator

In the previous section we have found that the fluctuation of the electric flux operator is ill-defined in the representation $\mathcal{H}^H_{P,m}$. This indicates an obstruction to using states of a similar kind, suitably generalized to $SU(2)$, in order to analyze the semiclassical behaviour of diffeomorphism covariant operators in quantum general relativity such as the Hamiltonian constraint operator. On the other hand, there was no problem with the fluctuation of the holonomy. Could it be that the complexifier is too much adapted to the holonomies but unsufficiently so to electric fluxes ? Can we distribute the fluctuations more democratically so that both become well-defined ?

In this section we examine, still in the context of $U(1)$ to keep things simple, whether a modification of the complexifier can fix this. Specifically we will consider all complexifiers C which are homogeneous bilinear polynomials in the electric fields and which satisfy the following conditions:
1) $A_a^{\mathbb{C}}(m) := A_a - iq^2\delta C/\delta A_a$ or at least $g_p(m) = \exp(i\int_p A^{\mathbb{C}})$ is a well-defined classical function. Distributional Poisson brackets among the $g_p, \overline{g_p}$ will be allowed. This will ensure a suitable correspondence between the coherent states ψ_m and $m \in \mathcal{M}$.
2) $\hat{C}$ is well-defined on $\mathcal{H}_0$. This will make sure that ψ_m is a well-defined distribution in Cyl*.
3) The fluctutions of both $\hat{A}(p), \hat{E}(S)$ are finite. This will ensure that we can compute expectation values of (not necessarily normal ordered) operators more complicated than linear in these two basic operators as it is important for quantum general relativity.

We will see shortly that the set of complexifiers satisfying 1),2),3) is empty. Actually, for $U(1)$ conditions 2) automatically holds due to our restricted ansatz for C and for the same reason 1) holds for all gauge theories. So we just need to consider 3).
Due to our quadratic ansatz for C, the only freedom left is in the choice of a symmetric, positive kernel $K_{ab}(x,y) = K_{ba}(y,x)$ of dimension cm^{-2}, which we require to have an inverse on a suitable function space. Also, for simplicity, we assume that it is an even function of $x-y$ (the analysis for non-translational invariant kernels is similar but more complicated). Then the complexifier becomes

$$C = \frac{1}{2q^2}\int d^3x E^a(x)[K\cdot E]_a(x) := \frac{1}{2q^2}\int d^3x \int d^3y E^a(x) K_{ab}(x,y) E^b(y) \tag{4.1}$$

In the two previous sections we had $K = f\cdot\sqrt{-\Delta}^{-1}\cdot f$. The complexified connection is

$$A_a^{\mathbb{C}}(m) = A_a - i(K\cdot E)_a \tag{4.2}$$

The operator $\hat{C}/\hbar$ is again a positive, essentially self-adjoint operator with pure point spectrum, eigenstates being given by charge network states

$$\hat{C}/\hbar T_s = [\frac{\alpha}{2}\int d^3x X_s^a [K\cdot X_s]_a] T_s =: \frac{\alpha}{2} < X_s, K\cdot X_s > \; T_s \tag{4.3}$$

The annihilation operator becomes

$$\hat{g}_s = e^{-\hat{C}/\hbar}\hat{A}(s)e^{\hat{C}/\hbar} = e^{\alpha<X_s,K\cdot X_s>}e^{<\hat{E},K\cdot X_s>}\hat{A}(s) = e^{-\alpha<X_s,K\cdot X_s>}\hat{A}(s)e^{<\hat{E},K\cdot X_s>} \tag{4.4}$$

corresponding to the classical holonomy

$$g_s(m) = A(s)e^{<E,K\cdot X_s>} \tag{4.5}$$

Let $p_\epsilon^a(x)$ be again the loops of the previous section, then for any surface S we have

$$\begin{aligned}(P\cdot E)(S) &= \int_S dS_a(x)[K^{-1}\cdot\Delta^{-1}\nabla\times\nabla\times\lim_{\epsilon\to 0}\frac{[|g_{p_\epsilon}|^2-1]}{2\pi\epsilon^2}]^a(x)\\ A(p) &= |g_{p^{-1}}|g_p\end{aligned} \tag{4.6}$$

where the x,a dependence of the fraction is the one of $p_\epsilon^a(x)$. We then define

$$\begin{aligned}(P\cdot\hat{E})(S) &:= \int_S dS_a(x)[K^{-1}\cdot P\cdot\lim_{\epsilon\to 0}\frac{[\hat{g}_{p_\epsilon}^\dagger\hat{g}_{p_\epsilon\epsilon}-1]}{2\pi\epsilon^2}]^a(x)\\ \hat{A}(p) &= [\sqrt{\hat{g}_{p^{-1}}^\dagger\hat{g}_{p^{-1}}}]\hat{g}_p\end{aligned} \tag{4.7}$$

Consider the operator

$$\hat{y}_{p^{-1}}(m) = |g_p(m)|^2 \hat{g}^\dagger_{p^{-1}} \hat{g}_{p^{-1}}$$

which has the square

$$\hat{y}_{p^{-1}}(m)^2 = |g_p(m)|^4 (\hat{g}^\dagger_{p^{-1}})^2 (\hat{g}_{p^{-1}})^2 e^{2\alpha < X_p, K\cdot X_p >} \tag{4.8}$$

Then we compute, using our estimates for $\sqrt{y}$ from the previous section,

$$\begin{aligned}
& 1 - \frac{1}{2}[e^{2\alpha < X_p, K\cdot X_p >} - 1] \leq \frac{< \hat{A}(p) >_m}{[A(p)](m)} =< \sqrt{\hat{y}_{p^{-1}}(m)} >_m \leq 1 \text{ and} \\
& < [P\cdot \hat{E}](S)^2 >_m - [[P\cdot E](S)](m)^2 \\
= & \int_S dS_a(x) \int_S dS_b(y) \lim_{\epsilon\to 0} \lim_{\sigma\to 0} \int d^3u \int d^3v [K^{-1}\cdot P]^a_{a'}(x,u)[K^{-1}\cdot P]^b_{b'}(y,v) \times \\
& \times \frac{e^{2\alpha < X_{p^{a'}_\epsilon(u)}, K\cdot X_{p^{b'}_\sigma(v)} >} - 1}{(2\pi\epsilon\sigma)^2} \\
= & \frac{\alpha}{2} \int_S dS_a(x) \int_S dS_b(y) \epsilon^{a'cd} \epsilon^{b'ef} \int d^3u \int d^3v [K^{-1}\cdot P]^a_{a'}(x,u)[K^{-1}\cdot P]^b_{b'}(y,v) \times \\
& \times \partial^2 K_{df}(u,v)/\partial u^c \partial v^e \\
= & \frac{\alpha}{2} \int_S dS_a(x) \int_S dS_b(y) \int d^3u \int d^3v [K^{-1}\cdot P\cdot \nabla\times]^{ac}(x,u)[K^{-1}\cdot P\cdot\nabla\times]^{bd}(y,v) K_{cd}(u,v) \\
= & \frac{\alpha}{2} \int_S dS_a(x) \int_S dS_b(x) \epsilon^{acd}\epsilon^{bef} \partial^2 [P\cdot K^{-1}\cdot P]_{df}(x,y)/\partial x^c \partial y^e
\end{aligned} \tag{4.9}$$

Thus, the necessary and sufficient condition for the kernel K such that the fluctuations of our elementary variables are well-defined is that

$$\begin{aligned}
& \oint_p dx^a \oint_p dx^b K_{ab}(x,y) \leq \infty \text{ and} \\
& \int_S dS_a(x) \int_S dS_b(y) \epsilon^{acd}\epsilon^{bef} \partial^2 [P\cdot K^{-1}\cdot P]_{df}(u,v)/\partial x^c \partial y^e \\
= & \oint_{\partial S} dx^a \oint_{\partial S} dy^b [P\cdot K^{-1}\cdot P]_{ab}(x,y) \leq \infty
\end{aligned} \tag{4.10}$$

for any surface S and any loop p. In terms of form factors we may rewrite (4.10) as

$$< X_p, K\cdot X_p > \leq \infty \text{ and } < X_{\partial S}, P\cdot K^{-1}\cdot P\cdot X_{\partial S} > \leq \infty \tag{4.11}$$

Since for a loop p the form factor is transversal, specifically

$$\partial_a X^a_p(x) = \int_p dy^a \partial_{x^a} \delta(x,y) = -\int_p [d\delta_x](y) = -[\delta_x]_{\partial p} = 0 \tag{4.12}$$

we simply find the compact condition

$$< X_p, K\cdot X_p > \leq \infty \text{ and } < X_p, K^{-1}\cdot X_p > \leq \infty \tag{4.13}$$

for any closed path p. Notice that X_p is a distribution, thus both K and K^{-1} must be non-distributional. We can rewrite (4.13) in terms of its Fourier transform $\tilde{K}_{ab}(k)$ as

$$\int d^3k \tilde{K}_{ab}(k) F^a_p(k) \overline{F^b_p(k)} \leq \infty \text{ and } \int d^3k \tilde{K^{-1}}_{ab}(k) F^a_p(k) \overline{F^b_p(k)} \leq \infty \tag{4.14}$$

where $F^a_p(k) = \int_p dx^a e^{ik_b x^b}$. To see why (4.14) is impossible, consider first the simplified case $\tilde{K}_{ab}(k) = \delta_{ab}\rho(k)$ with a positive function $\rho(k)$ whence (4.14) turns into

$$\int d^3k\rho(k)\sum_a |F^a_p(k)|^2 \leq \infty \text{ and } \int d^3k \frac{1}{\rho(k)}\sum_a |F^a_p(k)|^2 \leq \infty \tag{4.15}$$

Now for a loop $t \mapsto p(t) = x_0 + \epsilon n(t)$ where $n(t)$ is a unit vector and $||x_0|| \gg \epsilon$ we have $F^a_p(k) \approx e^{ik_b x^b_0}\epsilon \int_n dx^a$ so that for such loops (4.15) essentially turns into

$$\int d^3k\rho(k) \leq \infty \text{ and } \int d^3k \frac{1}{\rho(k)} \leq \infty \tag{4.16}$$

which is impossible since the first condition requires $\rho(k)$ to die off at infinity at least as $1/||k||^{3+\epsilon},\ \epsilon > 0$ so that the second integral diverges. The case with non-diagonal $\tilde{K}_{ab}$ is similar and will be left to the reader.

This concludes this section. We have shown that the representations induced by complexifiers of the form (4.1), where $K_{ab}(x,y) = K_{ab}(x-y)$ is a symmetric positive kernel, cannot implement both, the holonomy and electric flux functions, as well defined operators. This indicates an obstruction to using those representations for the semiclassical analysis of quantum general relativity. Thus, although it would be much preferred to use the distributional states (due to their graph independence), at least at the kinematical level it seems that we are forced to work with graph dependent ones.

5 Generalization to Non-Abelean Gauge Groups

So far, Varadarajan type of complexifiers have been defined only for Abelean gauge groups (the proposal in [16] for $SU(2)$ is not based on a complexifier as we will show below). The underlying reason is the following: The associated complexifiers are of the type (4.1). If we want to generalize this to a Non-Abelean compact gauge group then the most general, homogeneous, bilinear ansatz is given by

$$C = \frac{1}{a\kappa}\int d^3x \int d^3y E^a_j(x) K^{jk}_{ab}(x,y) E^b_k(y) \tag{5.1}$$

where $K^{jk}_{ab}(x,y) = K^{kj}_{ba}(y,x)$ is a dimensionfree, symmetric, positive kernel and a is a constant of the appropriate dimension. The first observation is that, unless $K^{jk}_{ab}(x,y)$ is proportional to $\delta_{jk}\delta(x,y)$, then C will not be gauge invariant. This can be fixed only if we allow the kernel to depend non-trivially on the connection as well. For instance, we could fix once and for all a point $x_0 \in \sigma$ and for each $x \in \sigma$ we choose once and for all a path ρ_x with $b(\rho_x) = x_0, f(\rho_x) = x$. Then

$$C = \frac{1}{\kappa}\int d^3x \int d^3y \mathrm{Tr}(\tau_j \mathrm{Ad}_{A(\rho_x)}(E^a(x)))K_{ab}(x,y)\mathrm{Tr}(\tau_k \mathrm{Ad}_{A(\rho_y)}(E^b(x))) \tag{5.2}$$

is gauge invariant, however, the corresponding operator will be very complicated, certainly it does not leave every subspace $\mathcal{H}_{0,\gamma}$ separately invariant so that its spectral resolution will be very complicated, it is not even clear that it will be positive. Even worse, it can not be defined on $\mathcal{H}_0$ *at all*:
To see this, choose $G = SU(2)$ for quantum general reativity. Then E^a_j is dimensionfree so that a has dimension cm^4. Let us apply the naive quantization of (5.1) with $\hat{E}^a_j = \ell_p^2 \delta/\delta A^j_a$ to a spin network function T_s. The result, after proper regularization, is

$$\hat{C}/\hbar T_s = -\frac{\ell_p^2}{a}\Delta_{\gamma(s)} T_s \tag{5.3}$$

where

$$\begin{aligned}\Delta_\gamma &= \sum_{e,e'}\int_0^1 dt\int_0^1 dt' K_{ab}(e(t),e'(t'))\dot{e}^a(t)\dot{e}^{b\prime}(t')O_{jk}(A(e_t))O_{jl}(A(e'_{t'}))Y_e^k Y_{e'}^l \\ &\quad -2\sum_e\int_0^1 dt\int_0^1 dt' K_{ab}(e(t),e(t'))\dot{e}^a(t)\dot{e}^b(t')\mathrm{Tr}(A(e_t)A(e_{t',t})A(e_{t'})^{-1}\tau_k)Y_e^k \\ &=: \sum_{e,e'} G_{kl}^{e,e'}(A)Y_e^k Y_e^l - 2\sum_e G_k^e(A)Y_e^k \end{aligned} \tag{5.4}$$

Here the sums are over the edges of γ, $Y_e^j = Y^j(A(e))$ is the right invariant vector field acting on the group degree of freedom $A(e)$, $e_{t,t'}$ denotes the segment of e between the points $e(t), e(t')$ with $e_t = e_{0,t}$ and $O_{jk}(h)\tau_k := \mathrm{Ad}_h(\tau_j)$ where τ_j is a basis of $su(2)$. Actually, the result (5.4) holds for any G. Notice that the "white noise kernel" $K_{ab}(x,y) = \delta_{ab}\delta(x,y)$ is not allowed due to the only one dimensional integrals.

Since we have started from a well-defined classical expression (5.1) it is not unexpected, but nevertheless *absolutely non-trivial* to check that the family of operators Δ_γ is cylindrically consistent. In particular, the term linear in X_e^k cannot be discarded, it is very essential in order to have a consistent operator family. The corresponding calculations are straightforward but fill pages and will not displayed here. Notice that by cylindrical consistency we mean here that Δ_γ arestricted to functions over a smaller graph γ' coincides with $\Delta_{\gamma'}$, of course, due to its non-trivial $A-$dependence residing in $G_{kl}^{ee'}, G_k^e$ the operator Δ_γ does not preserve the space Cyl_γ of functions over γ.

This sounds promising, but we want more than just a consistent operator, it should be positive definite. Now the explicit form of (5.4) suggests that a formula like

$$\Delta_\gamma = \sum_{e,e'} Y_e^k G_{kl}^{e,e'}(A) Y_{e'}^l \tag{5.5}$$

should hold so that the term linear in X_e^k in (5.4) simply is due to pulling the operator Y_e^k through $G_{kl}^{e,e'}$. In fact, this is precisely the reason for its occurence in the regularized calculation that led to (5.4). If that were the case then positivity could follow because $G_{kl}^{e,e'}(A)$ is a Hermitean matrix. Unfortunately, (5.5) is already ill-defined as it stands because the operator X_e^k only knows how to act on functions over graphs which depend only on the holonomy $A(e)$ but not on the segments $A(e_t)$. Thus, (5.5) *is false*. However, even if it was valid, there is an even worse obstacle: *The distributional nature of $\overline{\mathcal{A}}$ is such that none of the integrals involved in the definition of $G_{kl}^{e,e'}, G_k^e$ make sense* ! Namely, as shown in [23], the map $t \mapsto A(e_t)$ is not dt measurable. *It is only in the Abelean case that the A-dependence of $G_{kl}^{e,e'}$ disappears and $G_k^e = 0$.*

We conclude that, for Non-Abelean gauge groups, a bilinear ansatz of the form (5.1) does not lead to a well-defined operator on $\mathcal{H}_0$.

The question then arises, whether there exist replacements for $G_{kl}^{ee'}, G_k^e$ in (5.4) which depend only on the edges of γ, such that Δ_γ becomes cylindrically defined, consistent and negative definite (i.e. $\hat{C}$ positive definite). In appendix C we will show that the *unique* solution to this problem is the operator

$$\Delta_\gamma = \sum_{e\in E(\gamma)} l_e Y_e^k Y_e^k \tag{5.6}$$

already found in [11] where the positive function $e \mapsto l_e$ satisfies $l_{e\circ e'} = l_e + l_{e'},\ l_{e^{-1}} = l_e$.

Why do we then not use simply (5.6) as the complexifier for non-Abelean gauge groups ? The answer is that, while the operator is consistent and positive definite, it has no classical limit as shown in [8] and thus there is no map $m \mapsto A^{\mathfrak{C}}(m)$ available. This is precisely the reason for why we have struggled in [9] to define a map $m \mapsto A^{\mathfrak{C}}_\gamma(m)$ for each γ separately so that at least the cut-off coherent states $\psi_{\gamma,m}$ are useful for semiclassical analysis. (In [9, 10] we did not bother with the length functions l_e for simplicity but it is trivial to incorporate them into the map $m \mapsto A^{\mathfrak{C}}_\gamma(m)$). However, precisely because the continuum operator underlying the family (5.6) has no classical limit the maps $m \mapsto A^{\mathfrak{C}}_\gamma(m)$ do not come from a single map $m \mapsto A^{\mathfrak{C}}(m)$. Thus, we may say that as far as the class of complexifiers (C.1) is concerned, [9, 10] is the best one can possibly do.

In what follows we define a new class of complexifiers which are free from these drawbacks, namely each of them has the following properties:
1) it has a classical limit, so we obtain a well-defined map (canonical transformation) $m \mapsto A^{\mathfrak{C}}(m)$ and thus make immediate correspondence between classical and quantum theory.
2) it is gauge invariant.
3) it is positive definite.
4) it is cylindrically defined.
5) it has an explicitly known pure point spectrum.
6) it is almost of the type (5.1).

The clue for how to construct a complexifier with all of these properties comes from the observation that for non-Abelean gauge theories whose Hilbert space is based on holonomies the only known, well-defined and cylindrical momentum operators come from electric fluxes

$$E_j(S) = \int_S dS_a(x) E^a_j(x) \tag{5.7}$$

These objects are not gauge invariant, however, there are precisely two basic invariants that one can build from those, namely $E_j(S)E_k(S')\delta^{jk}$ and $E_j(S)E_k(S')E_l(S')\epsilon^{jkl}$ in the limit as the surfaces involved shrink to a single point. The operators on $\mathcal{H}_0$ for which this shrinking process converges to a well-defined operator are precisely the area operator on the one hand and volume – and length operators on the other [25]. We already have discussed the volume operator as a possible complexifier above and, in fact, it seems to be the more natural possibility because we do not need to introduce any other structure, however, since its spectrum is presently only poorly understood, we will turn to the area operator. By definition, the area operator is only supported on a given surface but we must obtain a complexifiers which is supported everywhere in order that a damping factor is produced *for every graph*. Moreover, as we have shown in section 2.1, we must use a power of the area operator which is greater than one in order to arrive at an entire analytic function (convergence) and since with an embedding $X : \check{S} \subset \mathbb{R}^2 \to S$

$$\mathrm{Ar}(S) = \int_{X^{-1}(S)} d^2u \sqrt{\det(X^* q)}(u) = \int_{X^{-1}(S)} d^2u \sqrt{[E^a_j(X(u)) n^S_a(u)]^2} \tag{5.8}$$

where $n^S_a(u) = \epsilon_{abc} X^b_{,u^1} X^c_{,u^2}$ we see easily that $\lim_{S\to x}[\mathrm{Ar}(S)]^2/[E_j(S)E_j(S)] = 1$. Thus, the natural power, from the point of view of [9, 10] which was built on a gauge invariant version of objects of the type $E_j(S)E_j(S)$, is two. We will then approximate a Gaussian decay as closely as we can in the non-Abelean context.

How should we then construct a complexifier built from objects of the kind $[\mathrm{Ar}(S)]^2$ which is supported everywhere in σ ? There are many possibilities and we will present a few of them:

Version 1: *Foliation and Parquet*
Let us introduce D linearly independent foliations X^I_t of σ, that is, for each $t \in \mathbb{R}$ we obtain an embedding of a $D-1$ surface $X^I_t : \check{S}^I_t \subset \mathbb{R}^{D-1} \to \sigma$ whose topology may vary with t, I and linear independence means that at each point $x \in \sigma$ the D "normal" covectors

$$n^I_a(x) := \epsilon_{aa_1..a_{D-1}}[X^{Ia_1}_{,u^1}..X^{Ia_{D-1}}_{,u^{D-1}}]_{X^I_t(u)=x} \tag{5.9}$$

or equivalently the D tangents $[(\partial X^I_t(u))/\partial t]_{X^I_t(u)=x}$ are linearly independent. Within each leaf of the foliation X^I_t fix a *parquet* P^I_t, that is, a partition into smaller $D-1$ surfaces of fixed (say simplicial) topology and we require that for each I the parquet varies smoothly with I. Notice that all of these structures do not refer to a background metric. The parquet is quite similar in nature to the polyhedronal decomposition dual to a graph defined in [9] but it is diferent because it is *graph-independently defined* so that the resulting complexifier can be defined already classically rather than only in quantum theory graph-wise. We then propose

$$C = \frac{1}{2a\kappa}\sum_{I=1}^{D}\int_{\mathbb{R}} dt \sum_{\square\in P^I_t}[\mathrm{Ar}(\square)]^2 \tag{5.10}$$

where a is again an appropriate dimensionful parameter which we could also make dependent on $\square$. For instance for quantum general relativity in $D=3$, a would have dimension cm^2 if we take the parameter t dimension-free.

The corresponding complexified connection would be

$$A^{\mathbb{C}j}_a(x) = A^j_a(x) - i\sum_{I=1}^{D}\Big(\frac{\mathrm{Ar}(\square^I_x)}{|\det(\partial X^I_t/\partial(t,u))|}\frac{E^b_j(x)n^{\square^I_x}_b(t,u)}{\sqrt{[E^c_j(x)n^{\square^I_x}_c(t,u)]^2}}n^{\square^I_x}_a(t,u)\Big)_{X^I_t(u)=x} \tag{5.11}$$

where $\square^I_x \in P^I_{t^I(x)}$, $X^I_{t^I(x)}(u^I(x)) = x$ is the surface containing x. From (5.11) we see why we cannot do without the parquet since then we would have to work with the areas of the whole leaves which would be to an unsufficiently local object. However, even (5.11) only allows us to reconstruct E from $A^{\mathbb{C}}$ only with a precision that is defined by how fine the parquet is.

The spectrum of the corresponding complexifier operator is essentially derived from the known spectrum of the area operator, together with an important key observation which is responsible for making this operator really leave all the Cyl_γ separately invariant. Recall that given an open, analytic, oriented surface S and a graph γ we can always subdivide its edges in such a way that any of them belongs to precisely one of the four disjoint subsets $E_{in}, E_{out}, E_{up}, E_{down}$ of edges of γ where $e \in E_{in} \Rightarrow e \cap S = e$, $e \in E_{out} \Rightarrow e \cap S = \emptyset$, $e \in E_{up} \Rightarrow e \cap S = b(e)$ and e points up, $e \in E_{down} \Rightarrow e \cap S = b(e)$ and e points down. Here "up,down" denote one of the two half spaces of σ that (the analytic extension of) S bounds. Let $P(S,\gamma) = \{b(e),\ e \in E_{up} \cup E_{down}\}$ and given $p \in P(S,\gamma)$ let $X^j_{up}(p) = \sum_{e\in E_{up}(p)} X^j_e, X^j_{down}(p) = \sum_{e\in E_{down}(p)} X^j_e$. The operators $\Delta_{up}(p) = (X^j_{up}(p))^2$, $\Delta_{down}(p) = (X^j_{down}(p))^2$, $\Delta_{updown}(p) = (X^j_{up}(p) + X^j_{down}(p))^2$ are simultaneously diagonazable with (-2 times) total angular momentum spectrum. The area operator is given by

$$[\widehat{\mathrm{Ar}}(S)]_{\mathrm{Cyl}_\gamma} = \frac{\hbar\kappa}{4}\sum_{p\in P(S,\gamma)}\sqrt{-2\Delta_{up}(p) - 2\Delta_{down}(p) + \Delta_{updown}(p)} \tag{5.12}$$

and its spectrum for $SU(2)$ reads explicitly

$$[\mathrm{Spec}(\widehat{\mathrm{Ar}}(S))]_{\mathrm{Cyl}_\gamma} = \frac{\hbar\kappa}{2}\sum_{p\in P(S,\gamma)}\sqrt{2j_u(p)(j_u(p)+1) + 2j_d(p)(j_d(p)+1) - j_{ud}(p)(j_{ud}(p)+1)} \tag{5.13}$$

with $j_u(p)+j_d(p)\geq j_{ud}(p)\geq |j_u(p)-j_d(p)|$

The key point is now that the subdivision of edges of γ into the classes $E_{in}, E_{out}, E_{up}, E_{down}$ *depends on the surface* S ! That is, a given spin-network state T_s is not an eigenstate of a given operator $\widehat{\mathrm{Ar}(S)}$, rather we must subdivide the edges of $\gamma(s)$ adapted to S and then decompose the intertwiners $I(s)$ in such a way that we get eigenfunctions of $\Delta_{up}(p), \Delta_{down}(p), \Delta_{updown}(p)$ for all vertices p of γ respectively. It follows that the function $\widehat{\mathrm{Ar}(S)}T_s$ depends, in the non-Abelean case, generally no longer on the edges of γ but also on the subdivision of the edges of γ as adapted to S. This is dangerous because we are dealing with operators of the form $\int dt[\widehat{\mathrm{Ar}(S_t)}]^2$ for a foliation $t\mapsto S_t$ and the function $[\widehat{\mathrm{Ar}(S_t)}]^2T_s$ therefore depends on the parameter t. If it would depend on a graph γ_t where γ_t dpends on a subdivision of edges according to S_t then the operator $\hat{C}$ would not exist since $[\widehat{\mathrm{Ar}(S_t)}]^2T_s$ is not $dt-$measurable as we showed above. Fortunately this does not happen:

A point $p\in P(\gamma,S_t)$ falls only into one of the two categories: Either it is a vertex of γ in which case the subdivision of edges does not change the graph or p is an *interior point of a single edge*. However, in the latter case a spin-network function is already an eigenfunction: If $e=e_u(t)\circ e_d^{-1}(t)$ denotes the adapted decomposition of the corresponding edge of γ with $p:=S_t\cap e=b(e_u(t))=b(e_d(t))$ then from (5.13) due to gauge invariance at p

$$\sqrt{-2\Delta_{up}(p)-2\Delta_{down}(p)+\Delta_{updown}(p)}T_s=\hbar\kappa\sqrt{j_e(j_e+1)} \tag{5.14}$$

is completely independent of t. We conclude that spin-network functions T_s are simultaneous eigenfunctions of all possible $\widehat{\mathrm{Ar}}(S_t)$ as long as S_t does not contain a vertex of $\gamma(s)$. However, for given T_s the number of vertices of γ is finite and set $\{t\in\mathbb{R}; S_t\cap V(\gamma)\neq\emptyset\}$ is discrete and thus has $dt-$measure zero.

The spectrum of our complexifier operator therefore can easily be computed as follows:
We will assume that the graph $\gamma(s)$ is contained in a region such that each of the embedded surfaces $t\mapsto X^I_t,\ t\in[a,b]$ has topology independent of t with $\gamma\subset\cup_{t\in[a,b]}X^I_t$ for all I. The more general case including topology change just involves introducing more notation and does not lead to new insights and thus will be left to the reader. Our assumptions about the parquet imply then that, given I, we have a corresponding family of surfaces $S^I_{\square,t}$ with a discrete label $\square$. Fix $I,\square$, and a set of intersection numbers $n^{I,\square}_e=0,1,2,..;\ e\in E(\gamma)$ one for each edge of γ and denote by $t^I_\square(\gamma,\vec{n})$ the $dt-$measure of the set $\{t\in[a,b];\ |S^I_{\square,t}\cap e|=n^{\square,I}_e\}$ (notice that we only count isolated intersection points). Then

$$\begin{aligned}
\frac{\hat{C}}{\hbar}T_s &= \frac{\ell_p^2}{a}\{\sum_{I,\square,\vec{n}^{\square,I}(s)} t^I_\square(\gamma(s),\vec{n}^{\square,I}(s))[\sum_{e\in E(\gamma)} n^{\square,I}_e\sqrt{j_e(j_e+1)}]^2\}T_s \\
&= \frac{\ell_p^2}{a}\{\sum_{e,e'\in E(\gamma)}\sqrt{j_e(j_e+1)}\sqrt{j_{e'}(j_{e'}+1)}\sum_{I,\square,\vec{n}^{\square,I}(s)} t^I_\square(\gamma(s),\vec{n}^{\square,I}(s))n^{\square,I}_e n^{\square,I}_{e'}\}T_s \\
&=: \frac{\ell_p^2}{a}\{\sum_{e,e'\in E(\gamma)} G^{e,e'}_s\sqrt{j_e(j_e+1)}\sqrt{j_{e'}(j_{e'}+1)}\}T_s
\end{aligned} \tag{5.15}$$

which displays a suitable, non-Abelean generalization of the edge metric which is automatically consistent because the area operator is.

Interestingly, if the parquet is much finer than the graph then each of the surfaces $S^I_\square$ will typically intersect at most one edge $e^I_\square$ of the graph and if so then only once. Therefore, $t^I_\square(\gamma(s),\vec{n}^{\square,I}(s))n^{\square,I}_e n^{\square,I}_{e'}$ vanishes unless $n^{\square,I}_e=\delta_{e,e^I_\square}$ up to small corrections in the vicinity of vertices. Thus the sum over edges reduces approximately to diagonal contributions and the sum over surfaces and their intersction

numbers at given e reduces approximately to l^I_e, the dt measure of the set $\{t \in [a,b];\ |S^I_t \cap e| = 1\}$. This means that (5.15) is approximated by

$$\frac{\hat{C}}{\hbar} T_s \approx \frac{\ell_p^2}{a} \sum_{e \in E(\gamma)} j_e(j_e+1)[\sum_I l^I_e] T_s =: \frac{\ell_p^2}{a} \sum_{e \in E(\gamma)} j_e(j_e+1) l_e T_s \tag{5.16}$$

which provides a concrete realization and classical interpretation of the numbers l_e of [11]. In other words, at least for parquets much finer than a given graph, the function (5.10) provides a suitable continuum limit of the complexifier used in [9, 10] ! Of course, the exact operator has a non-diagonal edge metric and one has to repeat all the estimates of [9, 10] for this more general case, however, on graphs sufficiently coarse as compared to the parquet the approximation given by (5.16) should be already quite good. We will provide analytic estimates in a future publication.

In appendix D we present more versions of this operator which depend on a background metric and make it possible to introduce the freedom of arbitrary covariances and to make the operator Euclidean invariant (on a flat background).

These examples show that there is sufficient freedom in working with expressions bilinear in the area functionals, in the non-Abelean context, as a substitute for the bilinear expressions in the electric field, in the Abelean context, in order to produce well-defined operators whose expressions actually come arbitrarily close to those of the Abelean case. We can work with the same kernels K that also work in the Abelean case. Properties for various choices of K, f and *for all* backgrounds will be explored in future papers.

Notice that all of this works directly in the polymer representation without recourse to the Fock representation, the only input is the complexifier. We have shown that it can be used, modulo functional analytic niceties, to generate non-Gaussian measures on the "would-be" Fock side. This can be done either directly or, at least in the Abelean case, by starting from a complexifier on the polymer side which is a function of E_f and then translating it by means of the isomorphism $\mathcal{I}^{-1}$ to the Fock side.

Unfortunately, all these representations that we obtain from complexifiers of the form (5.10) still do not control the fluctuations of the electric flux operator as is obvious from the closeness of the expression (5.15) to the corresponding one in the Abelean case. Thus, while we have no complete proof, we have demonstrated strong indications that the new representations induced by this type of complexifier do not support the operator algebra of QGR as presently formulated and are therefore forced to work with cut-off states or have to use completely different kinds of complexifiers.

6 Averaging of Coherent States: Dirichlet-Voronoi Types of Constructions

As we have just seen, the distributional coherent states ψ_m corresponding to a given complexifier do not give rise to well-defined electric flux operators which in turn are basic building blocks for diffeomorphism invariant operators arising in quantum general relativity. However, the corresponding cut-off states $\psi_{\gamma,m}$ associated to a given graph are actually well-defined L_2 elements of $\mathcal{H}_0$ provided that the complexifier satisfies the criteria outlined in section (2.1) and thus we can perform semi-classical analysis with those. The immediate caveat is that $\psi_{\gamma,m}$ depends on the given graph γ and therefore introduces a huge amount of ambiguity. Which graphs should be chosen in order to achieve

good semiclassical behaviour ? Several proposals have been made in order to overcome this:
1) *Averaging*
In [17, 16] one averages over graphs, based on the so-called Dirichlet-Voronoi construction, in order to reduce the graph dependence.
2) *Random Graphs*
In [26] we use random graphs which, while depending on the particulars of the graph on the microscopic scale (with respect to a given background metric), look isotropic and homogeneous on a macroscopic scale (within each close to flat coordinate patch).
3) *Operator-State Correspondence*
Instead of building states that behave semiclassically with respect to a given class of operators we can turn things around and build operators that behave semiclassically with respect to a given class of states. As long as the classical limit of the theory is the desired classical theory both quantization procedures are equally acceptable. This point of view has been put forward in the last reference of [10] which, in particular, means that operators are quantized in a graph dependent way. This still results in consistently defined families of operators but their expression is quite ugly and generically different from the ones that have appeared in the literature already whence this approach is disfavoured.

In this section we will show that, unfortunately, an averaging over graphs of the Drichlet-Voronoi type does not help in order to achieve good semi-classical behaviour in the following sense: While they help to improve the semi-classical behaviour of the three metric q_{ab} as represented by area operators, they destroy the semi-classical behaviour of the extrinsic curvature K_{ab} as represented by holonomy operators and thus, from this point of view, can be called, at best, weaves. In the next section we show how to remove this problem.

In order to show that the Dirichlet-Voronoi type of averaging over graphs destroys the semiclassical behaviour of the holonomy operator it is not necessary to go into the details of [17]. All that we need is:
a) A point $m \in \mathcal{M}$.
b) A subset $\Gamma_m \subset \Gamma$ of compactly supported, piecewise analytic graphs depending on m.
c) A probability measure ν_m on Γ_m for which discrete sets are of measure zero if Γ_m is not countable.
The concrete Dirichlet-Voronoi construction actually only uses the three-metric information contained in m in order to produce a continuous set of graphs Γ_m and a measure ν_m thereon by a particular, beautiful construction. However, the details will not be important in the subsequent argument.

In order to use these structures we assume that we are given a map

$$\psi : \Gamma \times \mathcal{M} \to \mathcal{H}_0; (\gamma, m) \mapsto \psi_{\gamma,m} \tag{6.1}$$

as for instance through the cut-off states of the complexifier coherent state machinery. We then have the following options:

A) *Pure State*
Consider the formal object

$$\text{“ } \psi_m := \int_{\Gamma_m} d\nu_m(\gamma)\psi_{\gamma,m} \text{ ”} \tag{6.2}$$

which looks like a superposition of states in $\mathcal{H}_0$. The object (6.2) should, however, not be confused with the graph independent distributions of section 3 since it still depends on the particular choice

Γ_m. Let us check whether $\psi_m \in \mathcal{H}_0$. If that were the case then $< T_s, \psi_m > \neq 0$ for at most countably many s and $||\psi_m||^2 := \sum_s | < T_s, \psi_m > |^2$. Now

$$< T_s, \psi_m > := \int_{\Gamma_m} d\nu_m(\gamma) < T_s, \psi_{\gamma,m} > \tag{6.3}$$

Making use of the spin-network decomposition of $\psi_{\gamma,m}$ we find
$< T_s, \psi_{\gamma,m} > = \sum_{\gamma' \subset \gamma} c_{s,\gamma'}(m) \delta_{\gamma',\gamma(s)}$ for certain coefficients $c_{s,\gamma'}$. Thus, (6.3) vanishes unless Γ_m is a countable set (which is not the case for the Dirichlet-Voronoi construction) since the integrand is supported on a ν_m measure zero subset. Thus, $\psi_m = 0$ unless Γ_m is a discrete set and ν_m a corresponding, weighted counting measure. One can ask whether we still can use (6.2) if we formally interchange integration over Γ_m with scalar products in $\mathcal{H}_0$. But even then it is easy to see that $||\psi_m||^2 = 0$ unless Γ_m is countable.

B) *Mixed State*
Consider the object

$$\hat{\rho}_m(.) := \int_{\Gamma_m} d\nu_m(\gamma) \psi_{\gamma,m} < \psi_{\gamma,m}, . > \tag{6.4}$$

which looks like a density matrix. Let us check whether (6.4) defines a trace class operator on $\mathcal{H}_0$. If that were the case then $< T_s, \hat{\rho}_m T_s > \neq 0$ for at most finitely many s and $\mathrm{Tr}(\hat{\rho}_m) := \sum_s < T_s, \hat{\rho}_m T_s >$. But

$$< T_s, \hat{\rho}_m T_s > = \int_{\Gamma_m} d\nu_m(\gamma) | < T_s, \psi_{\gamma,m} > |^2 \tag{6.5}$$

vanishes for the same reason as (6.3) does, except if Γ_m is countable.

C) *Expectation Value Functional*
Let us interchange the integral over Γ_m and the trace operation. This leads to a sensible result since $\sum_s | < T_s, \psi_{\gamma,m} > |^2 = ||\psi_{\gamma,m}||^2 = 1$. Thus we simply define a positive linear functional on the operator algebra over $\mathcal{H}_0$ by

$$\omega_m(\hat{O}) := \int_{\Gamma_m} d\nu_m(\gamma) < \psi_{\gamma,m}, \hat{O} \psi_{\gamma,m} > \tag{6.6}$$

Thus, while for continuous sets Γ_m the operator $\hat{\rho}_m = 0$ vanishes, the state ω_m makes sense.

Let now p be a closed path and $\hat{W}_p := \mathrm{Tr}(\hat{A}(p))$ the usual, gauge invariant Wilson loop operator. It is clear that $< \psi_{\gamma,m}, \hat{W}_p \psi_{\gamma,m} > = 0$ unless $p \subset \gamma$. Thus, if Γ_m is a continuous set

$$\omega_m(\hat{W}_p) = 0 \tag{6.7}$$

for each $m \in \mathcal{M}$. Choosing flat initial data $m = (A = 0, E = const.)$ for which $W_p(m) = 2$ we see that (6.7) does not display good semiclassical behaviour with respect to the magnetic degrees of freedom unless the set Γ_m is countable.

One might hope that the situation can be rescued by passing to "thickened loops", i.e. tubes. But, apart from the fact that it is not thickened three-dimensional loops on which $\mathcal{H}_0$ is built, but rather the ususal one-dimensional ones, even this idea does not work: Let $(u^1, u^2) \mapsto p_u$ be a two-parameter congruence of loops so that $\cup_u p_u$ is topologically a solid torus. Then one might consider the altered Wilson operator $\hat{W}'_p := \int d^2u \hat{W}_{p_u}$. The only hope that this operator has a non-vanishing

expectation value is that the congruence of loops is generated by the same process that generates Γ_m. Suppose that this is even the case. Then

$$< \psi_{\gamma,m}, \hat{W}'_p \psi_{\gamma,m} >= \int d^2u < \psi_{\gamma,m}, \hat{W}_{p_u} \psi_{\gamma,m} >= 0 \tag{6.8}$$

since $\{u \in \mathbb{R}^2;\ p_u \subset \gamma\}$ has d^2u-measure zero.

Thus, if averaging should have any chance to produce good semiclassical states (rather than distributions) then Γ_m must be a countable set and the pure state ψ_m or the mixed state $\hat{\rho}_m$ will be semiclassical for $\hat{W}_p$ at most if $p \subset \gamma$ for at least one $\gamma \in \Gamma_m$. For instance, in one dimension we could fix M points in the interval $[-L/2, L/2]$, distributed according to the scale of variation of the one-dimensional metric and we could consider all possible graphs Γ_m that can be formed by using $N \ll M$ of these points as vertices. In the end one could let $L, M, N \to \infty$ keeping the average lattice length $l = L/N$ and the sample number characteristic M/N fixed. If the spatial metric is flat then the M points will be equidistantly distributed and the weight for each of the approximately $\left(\begin{array}{c} M \\ N \end{array} \right)$ configurations will be the same, that is, approximately $1/\left(\begin{array}{c} M \\ N \end{array} \right)$. It is now straightforward to estimate $\omega_m(\hat{W}_p)$ for the case that $G = U(1)$ (and p is an open path for simplicity):
Even in this one dimensional case, where p and γ are contained in the same one-dimensional manifold, only those graphs γ contribute which not only contain p but also contain $b(p), f(p)$ as vertices of its edges (i.e. these points must not be interior points of edges). Thus the number of contributing graphs is at most $\left(\begin{array}{c} M \\ N-2 \end{array} \right)$ since both end points have to be contained in the configuration. Since for each of the individual expectation values holds $| < \psi_m, \hat{W}_p \psi_m > | \leq 1$ we easily find

$$|\omega_m(\hat{W}_p)| \leq \frac{\left(\begin{array}{c} M \\ N-2 \end{array} \right)}{\left(\begin{array}{c} M \\ N \end{array} \right)} \approx (N/M)^2 \tag{6.9}$$

which is a tiny number (in order that we have many configurations), no matter how the ψ_m are chosen, and is thus grossly off the expected value with $|W_p(m)| = 1$. Results in higher dimensions will be even worse because there are relatively even less graphs γ which contain p.

All of this points us to the fact that averaging simply does not work as far as holonomy operators are concerned because the representation $\mathcal{H}_0$ is such that the functional $T_s \mapsto < 1, T_s >_{\mathcal{H}_0}$ is continuous only if the set Γ is assigned the discrete topology.

7 A Resolution: Diffeomorphism Invariant Operators

Up to this point the analysis lets us conclude that for the semiclassical analysis of QGR at best normalizable cut-off states are useful which are therefore necessarily graph-dependent. The non-averaged ones do not approximate area operators well, the unaveraged ones do not approximate at all the holonomy operators. More in detail, graph dependent, unaveraged coherent states $\psi_{\gamma,m}$ were criticized due to the staircase problem, see the last reference in [10]. That is, given e.g. a regular graph γ (say of cubic topology) and a surface S, unless the surface S is adapted to γ (is composed of the surfaces defined by the plaquettes of the cubic graph), the expectation value of the area operator

should be way off its classical value. While this can possibly be overcome using a random graph for which no surface is adapted to its edges so that none is distinguished, given an arbitrary path the expectation value of the corresponding holonomy will be zero again unless p is contained in the graph. This is an improvement over averaged coherent states which do not give good semiclassical results *for any path*, however, it is not good enough.

The observation is now that *Holonomy and electric flux operators for given coordinate paths and surfaces respectively have no invariant physical meaning*! Physical meaning have only gauge invariant, spatially diffeomorphism invariant operators which also (weakly) commute with the Hamiltonian constraint. At this moment we have little control on the Hamiltonian constraint so let us content ourselves with gauge – and spatially diffeomorphism invariant quantities. Then we can define, following [28], for instance, something like a diffeomorphism invariant area operator by using matter degrees of freedom. Thus, in order to measure the area, say of the surface of the sheet of paper that you are reading right now, we must prepare a state which depends on gravitational and electromagnetic degrees of freedom say and which is peaked on an electromagnetic field whose electromagnetic field energy is concentrated on that sheet of paper and which is peaked on a flat gravitational metric. Now we must construct an operator which does not measure a given coordinate surface but rather the surface of any region in space within which the electromagnetic field energy is non-zero. Suppose that we have constructed such an operator $\widehat{\mathrm{Ar}}$ on the Hilbert space $\mathcal{H}_0^E \otimes \mathcal{H}_0^M$, see [2]. Notice that instead of depending on a coordinate surface S, this operator depends on both gravitational and electromagnetic operators. The key point is now that the semiclassical state on $\mathcal{H}_0^E \otimes \mathcal{H}_0^M$ which describes the sheet of paper in an ambient flat spacetime is necessarily of the form $\psi_{\gamma;m_E,m_M}$ [2], where $m = (m_E, m_M)$ is the corresponding point in the combined Einstein-Maxwell phase space, **because matter can only be located where geometry is excited**. In other words, *the surface whose area is to be measured is dynamically automatically adapted to the graph on which the coherent state is based*. Similar statements can be made concerning the holonomy operator.

What we learn from these considerations is that the staircase problem is actually never there if we consider invariant quantities rather than coordinate dependent ones. For instance, one might think that there should be a gravitational semiclassical state peaked on flat space which assigns to the area operator for any possible coordinate surface the same area expectation value as long as those surfaces are only translated and rotated copies of each other. However, physically this is completely unnecessary because there exists no matter configuration which describes, at the same instant of time, an uncountably infinite number of surfaces where electromagnetic field energy is concentrated. Thus, every measurement of one of those translated or rotatetd surfaces corresponds to a different physical situation, a different point $m = (m_E, m_M)$ in phase space and a different measurement and it is therefore physically correct to use a different coherent state for each of them corresponding to different points $m \in \mathcal{M}$ in the combined matter – geometry phase space. We will come back to this point in much detail in a future publication, but what can be said here is that the types of operators which are very well approximated by graph dependent coherent states, whether averaged or not, arise from integrated scalar densities of weight one which is precisely the class of operators in which the Hamiltonian constraint falls. See [26] for first explicit calculations in that direction.

8 Summary

In this article we have summarized the present status of semiclassical states for quantum general relativity.

As far as distributional coherent states are concerned we have shown that the states defined

by Varadarajan in [14] for $G = U(1), U(1)^3$ are *precisely* of the type of the complexifier coherent states defined in [8] which unifies the current proposals under a common principle, the complexifier machinery. Conceptually, we have made a change of viewpoint as compared to [14] in that we showed that the inner product on the distributional coherent states is not an extra ingredient (coming from the isomorphism $\mathcal{I}$ which in turn relies on a Gaussian measure) but *can be derived* from the already existent inner product on the kinematical Hilbert space $\mathcal{H}_0$ for quantum general relativity through a limiting procedure. This is interesting because it confirms the latter Hilbert space as the "common source" of possibly interesting other kinematical representations and thus gives it the status of a "fundamental" representation.

On the other hand, we have shown that those distributional states that come from complexifiers which are bilinear in the electric field operator do not support an electric flux operator. A first reaction might be that such kind of complexifiers are simply not well-defined operators on $\mathcal{H}_0$ if G is not Abelean and that for QGR for wich electric flux operators are basic building blocks we must consider different kinds of complexifiers. We have constructed a particularly simple one which is bilinear in area functionals rather than electric fields and still is close to the one of the Abelean theory. This works because the complexifier machinery equips us with a huge freedom for how to adapt the complexifier to the quantum dynamics and is completely independent of the existence of a corresponding representation on the Fock side.

The corresponding distributional states, although now appropriate for non-Abelean gauge groups, still do not admit well-defined electric flux operators. Thus, the analysis suggests that the problem with electric flux fluctuations has nothing to do with G being Abelean or not but rather lies in the nature of things: The representations induced by the distributional states constructed are somehow too close to Fock like representations for which the preimage of an electric flux operator under Varadrajan's isomorphism $\mathcal{I}$ is an even more singular object than an operator valued distribution.

Fortunately, at least the cut-off versions of these distributional states on given graphs, being elements of $\mathcal{H}_0$, make sense with respect to the QGR operator algebra and the associated distributional state, which can also be considered as a complex measure, should be considered as the "universal" structure underlying those cut-off states. Only these normalizable states thus seem appropriate for the semiclassical analysis of QGR unless we want to redefine the theory from scratch.

In the past the graph dependence of these states was considered to be a bad feature and therefore averaging techniques have been proposed. We have shown here that all of these proposals and natural variations thereof cannot lead to good coherent states either, if coordinate dependent operators are considered as the basic ones, since they assign expectation values to holonomy operators which are (close to) equal to zero. However, we have shown that the graph dependence is actually meaningful once one no longer talks about coordinate curves, surfaces and regions but rather about invariantly defined ones (for instance by matter). This happens because matter can only be where geometry is excited and since matter determines the locus of those curves, surfaces and areas it follows that the graph underlying a coherent state for both gravitational and matter degrees of freedom is *dynamically* automatically adapted to the curves, areas and regions whose holonomy (or length), area or region is to be measured.

Encouraged by the results of [26], we are optimistic that graph dependent coherent states defined through the complexifier machinery, both in their averaged and unaveraged form, provide a suitable starting point for the kinematical semiclassical analysis of QGR, at least as far as providing checks on the Hamiltonian constraint are concerned.

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A Review of the Kinematical Structure of Diffeomorphism Invariant Theories of Connections

The complexifier coherent state machinery can be applied, in particular, to quantum field theories which in their canonical formulation can be described by a phase space of the form $\mathcal{M} = T^*\mathcal{A}$ where $\mathcal{A}$ is a space of smooth connections for a principal $G-$bundle with compact gauge group G over a $D-$dimensional (spatial) manifold σ of arbitrary topology. Moreover, we consider theories which are not only $G-$invariant but also Diff(σ) invariant, such as general relativity in terms of real connection variables [19]. Of course, connections provide only half of the canonical degrees of freedom, there is also a Lie(G) valued vector density E of weight one. If we denote spatial tensor indices by $a, b, c, .. = 1, .., D$ and Lie algebra indices by $j, k, l, .. = 1, .., \dim(G)$ then the fundamental Poisson brackets can be written as

$$\{A_a^j(x), A_b^k(y)\} = \{E_j^a(x), E_k^b(y)\} = 0, \quad \{E_j^a(x), A_b^k(y)\} = \kappa\delta_b^a\delta_j^k\delta^{(D)}(x,y) \tag{A.1}$$

where κ is the coupling constant of the theory.

In what follows we will try to give an outline of an explanation for why it is natural to consider the representation $\mathcal{H}_0$ currently being used in QGR. The basic, fundamental assumption in this approach, as stated by Gambini et. al. in [20] and by Jacobson, Rovelli and Smolin in [21], is that holonomies $A(p)$ of connections $A \in \mathcal{A}$ along paths $p \subset \sigma$ (which for technical reasons are assumed to be piecewise analytic) can be promoted to well-defined quantum operators. This is motivated by two different observations: First of all, holonomies are the simplest functions of A which transform covariantly under local gauge transformations $g : \sigma \to G;\ x \mapsto g(x)$, that is, $A^g(p) = g(b(p))A(p)g(f(p))^{-1}$ where $b(p), f(p)$ respectively denote the beginning and end point of p respectively. Thus, it is easy construct gauge invariant objects like the Wilson loop function $\mathrm{Tr}(A(p))$ for closed paths p and one can show that such functions capture all gauge invariant information about any given A. The second observation is that our assumption implies that wave functions will depend on the $A(p)$ as basic configuration degrees of freedom which in turn means that they will be labelled by arbitrary (piecewise analytic) paths. Now, since the holonomy is (the non-Abelean generalization) of an integral of a $1-$form along a curve, it is *background independently* defined. Therfore the diffeomorphism group acts on the $A(p)$ also covariantly, namely $A^\varphi(p) = A(\varphi^{-1}(p))$ for any $\varphi \in$Diff(σ) it follows that diffeomorphism invariant wave functions should only depend on diffeomorphism classes of graphs, that is, *(generalized) knot classes*. This is very attractive because it means that one can possibly solve the diffeomorphism constraint rather easily and establish a link with topological quantum field theory.

Having motivated to consider the $A(p)$ as our basic configuration variables we need to decide what our basic momentum variables should be. These should be chosen in a background independent way as well such that together with the holonomies they form a closed subalgebra of the Poisson algebra which is non-distributional in order that in the quantum theory we will deal with operators and not

with operator valued distributions. The latter requirement implies that the $E's$, to which we will refer to as electric fields in what follows, have to be smeared in at least $D-1$ dimensions in order to absorb the δ distribution in (A.1) when computing Poisson brackets. The only other option is to smear them in D dimensions, but as one can easily check, unless G is Abelian, $D-$smeared electric fields do not lead to a closed Poisson algebra (the Poisson bracket of two elementary variables is not a polynomial of elementary variables). Thus, in the most interesting case of non-Abelean G we are forced to work with the background independent *electric fluxes*

$$E_j(S) := \frac{1}{(D-1)!}\int_S dx^{a_1}\wedge ..\wedge dx^{a_{D-1}}\epsilon_{a_1..a_D}E_j^{a_D} \tag{A.2}$$

as the basic momentum degrees of freedom, already from a classical point of view. The $E_j(S)$ transform covariantly under diffeomorphisms, $E_j^\varphi(S)=E_j(\varphi^{-1}(S))$ but not under gauge transformations. This can be easily repaired by replacing $E(x)$ under the integral in (A.2) by $\mathrm{Ad}_{A(p_S(x))}(E(x))$ where for each $x\in S$ we have chosen a path $p_S(x)\subset S$ with $b(p_S(x))=x_S=const.$ and $f(p_S(x))=x$ in which case $E^g(S)=\mathrm{Ad}_{g(x_S)}(E(S))$. This, however, is irrelevant for the construction of gauge invariant functions built from the $E(S)$ later on so we do not need not worry about these issues for what follows. The philosophy is that, while holonomy and electric flux operators are not physically interesting observables themselves, the interesting observables become composite operators built from them in the quantum theory which is why it is important that their algebra is supported by the Hilbert space representation.

Having made the assumption to represent the Poisson * algebra generated by $A(p),E(S)$ (with $A(p),E_j(S)$ being $G-$ valued and real-valued respectively) as operators $\hat{A}(p)=\pi(A(p)),\ \hat{E}(S)=\pi(E(S))$ on a Hilbert space we are looking at the representation theory of that Poisson algebra. In particular, we must represent the Abelean Poisson-subalgebra generated by the $A(p)$. Since G is compact, the $A(p)$ generate an Abelean function algebra of bounded functions which therefore is easily completed to an Abelean, unital C^* algebra $\mathcal{B}$ under the sup norm. By basic C^* algebra theorems, the function algebra can always be thought of as the algebra of continuous functions on a compact Hausdorff space $\overline{\mathcal{A}}$ (the so-called spectrum of the original function algebra) which turns out to be a certain distributional extension of $\mathcal{A}$. Our representation π restricted to that $\mathcal{B}$ is non-degenerate (there is no vector in the kerenel of all $\pi(A(p))$) because the C^* algebra contains the identity operator and thus it is an orthogonal sum of cyclic representations. But each cyclic representation comes from a positive linear functional ω on $\mathcal{B}$ via the GNS construction. Finally, since $\overline{\mathcal{A}}$ is a compact Hausdorff space, each of these cyclic representations is unitarily equivalent to a Hilbert space $L_2(\overline{\mathcal{A}},d\mu)$ with respect to some measure μ on $\overline{\mathcal{A}}$ by the Riesz representation theorem. We conclude that $\pi_{|\mathcal{B}}$ is *necessarily* represented on a (direct sum of) Hilbert space(s) of the form $L_2(\overline{\mathcal{A}},d\mu)$ for some probability measure μ, up to unitary equivalence, as an algebra of multiplication operators. Moreover, each of these Hilbert spaces contains a cyclic vector and it is clear that the common cyclic vector is the constant state $1(A)=1\in L_2(\overline{\mathcal{A}},d\mu)$ since from this we obtain all continuous functions on $\overline{\mathcal{A}}$ by acting with the $\pi(A(p))$ which are dense in the L_2 space. On physical grounds we want $\pi_{|\mathcal{B}}$ to be faithful, so the (direct sum of) measures is supposed to be faithful as well.

Now from the representation property $[\pi(E(S)),\pi(A(p))]=i\hbar\pi(\{E(S),A(p)\})$ and the fact that $\{E(S),A(p)\}$ is a linear combination of holonomies it is clear that $\pi(E(S))$ is necessarily of the form $\pi(E(S))=i\hbar\kappa X_S+F_S$ where X_S is a certain differential operator on $\mathcal{B}$ induced from $\{E(S),A(p)\}$ while $F_S\in\mathcal{B}$ is matrix valued and can map between the different sectors. Its purpose is to ensure that $\pi(E(S))$ is self-adjoint (the reality conditions on the $\pi(A(p))$ are automatically satisfied since they are multiplication operators). Since $X_S1=0$ it is clear that F_S must be an L_2 function in order that $\pi(E(S))$ is densely defined. More details will appear in [27].

The simplest case that one can consider is that there is only one cyclic representation and that $F_S = 0$. The unique solution to this problem is the measure Ashtekar – Lewandowski measure μ_0 of [3] as has been demonstrated in [22], in fact, to date there are no known other solutions, although a uniqueness proof is certainly missing. The most appealing additional features of μ_0 besides providing a representation of the canonical commutation relations and the adjointness conditions is that it is $G-$invariant and Diff(σ) invariant. This provides sufficient support for considering the Hilbert space $\mathcal{H}_0 := L_2(\overline{\mathcal{A}}, d\mu_0)$ as a suitable starting point for further analysis. Moreover, it has been demonstrated in [23] that $D-$smeared electric field operators are ill-defined on $\mathcal{H}_0$. Finally, $(D-1)-$smeared electric field operators have led to very appealing and well-defined results concerning the construction of length, area and volume operators, they are ideally suited for defining the Gauss-, Diffeomorphism- and Hamiltonian constraint operators etc., see e.g. [4] for a close to comprehensive list of these results. Finally, the Hilbert space $\mathcal{H}_0$ has an explicitly known basis of spin-network eigenfunctions which span a dense subspace Cyl$\subset \mathcal{H}_0$. The space of algebraic distributions Cyl* on Cyl is a natural home for the space of solutions to all constraints.

Concluding, there is strong motivation to work with holonomies and electric fluxes in diffeomorphism invariant quantum field theories of connections and we will keep it as our postulate that the fundamental theory should be based on them. Kinematical background dependent representations other than the background independent "fundamental" ones of the type $\mathcal{H}_0$ are certainly allowed but there must be a clear relation with $\mathcal{H}_0$ in the sense that one should obtain them in a certain limit (like temperature representations are limits of Fock representations).

B Mutual Singularity of Uniform and Induced Fock Measures

The measures μ_f, μ_0 on $\overline{\mathcal{A}}$ are mutually singular with respect to each other as the following argument shows (see [15] for more details):
The functions T_s are certainly L_2 for both measures because $|T_s| = 1$ and thus they are L_1 for both since the measures are probability measures. Let now $t \mapsto \varphi_t,\ \varphi_t(x) = x + vt$ be a one parameter spatial translation subgroup of the Poincaré group. The measure μ_0 is spatially diffeomorphism invariant while μ_f is spatially translation and rotationally invariant since the covariance of the Gaussian measure depends only on the operator Δ. Now

$$X^a_{[\varphi_t^{-1}(p)]_f}(x) = \int_{\varphi_t^{-1}(p)} dy^a f(x-y) = \int_p dy^a f(x-\varphi_t(y)) = X^a_{p_f}(x-tv)$$

will vanish in the limit $t \to \infty$ because the function f is of rapid decrease and so we check that

$$\begin{aligned}
\lim_{t\to\infty} < T_s, T_{\varphi_t^{-1}(s')} >_{\mu_0} &= \delta_{s,(\emptyset,\vec{0})}\delta_{s',(\emptyset,\vec{0})} =< T_s, 1 >_{\mu_0} < 1, T_{s'} >_{\mu_0} \text{ and} \\
\lim_{t\to\infty} < T_s, T_{\varphi_t^{-1}(s')} >_{\mu_f} &= \lim_{t\to\infty} \omega_F^H(\hat{A}([\varphi_t^{-1}(s') - s]_f) \\
&= < T_s, 1 >_{\mu_f} < 1, T_{s'} >_{\mu_f} \lim_{t\to\infty} e^{\frac{\alpha}{2}\int d^3x X^a_{s'_f}(x-tv)[\sqrt{-\Delta}^{-1} X^a_{s_f}](x)} \\
&= < T_s, 1 >_{\mu_f} < 1, T_{s'} >_{\mu_f}
\end{aligned} \tag{B.1}$$

Thus, the spatial translations are mixing transformations for both measures (see e.g. section III.5 of [4]), in particular, they are ergodic for both. It follows from the definition of ergodicity that

$$\mu_0(T_s) \cdot 1 = \delta_{s,(\emptyset,\vec{0})} \cdot 1$$

$$
\begin{aligned}
&=_{\mu_0-a.e.} \quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt T_{\varphi_t^{-1}(s)}(A) \\
&=_{\mu_f-a.e.} \quad e^{-\frac{\alpha}{4}\int d^3x X^a_{sf}\sqrt{-\Delta}^{-1}X^a_{sf}}\cdot 1 = \mu_f(T_s)\cdot 1
\end{aligned} \tag{B.2}
$$

Since the constants in the first and last line are different, it follows from the a.e. equality with the middle term, which depends on a specific point $A \in \overline{\mathcal{A}}$, that the supports of μ_f, μ_0 are mutually singular, in fact, even a one parameter family of f's each of whose members yields mutually different values of the constant in the last line of (B.2) gives rise to a one parameter family of mutually singular measures.

C Uniqueness of the Laplacian Complexifier

Here we answer the question whether there exist replacements for $G^{ee'}_{kl}, G^e_k$ in (5.4) which depend only on the edges of γ, such that Δ_γ becomes cylindrically defined, consistent and negative definite (i.e. $\hat{C}$ positive definite).

Definiteness leads us to an ansatz of the form

$$
\begin{aligned}
\Delta_\gamma &= \sum_{e,e'\in E(\gamma)} Y^k_e G^{ee'}_{\gamma;kl} Y^l_{e'} = \sum_{e,e'\in E(\gamma)} G^{ee'}_{kl} Y^k_e Y^l_{e'} + \sum_{e\in E(\gamma)}[\sum_{e'\in E(\gamma)}(Y^l_{e'} G^{e'e}_{\gamma;lk})]Y^k_e \\
&=: \sum_{e,e'\in E(\gamma)} G^{ee'}_{\gamma;kl} Y^k_e Y^l_{e'} + \sum_{e\in E(\gamma)} G^e_{\gamma;k} Y^k_e
\end{aligned} \tag{C.1}
$$

where the Hermitean matrix $G^{ee'}_{\gamma;kl}$ may depend on all the $A(e)$ where e runs through the edges of γ. Let us now explore the consequences of cylindrical consistency. Consider a graph $\gamma' \subset \gamma$ where
I) $E(\gamma') = E(\gamma) - \{e_0\}$,
II) $E(\gamma') = (E(\gamma) - \{e_0\}) \cup \{e_0^{-1}\}$,
III) $E(\gamma') = (E(\gamma) - \{e_1, e_2\}) \cup \{e_0 = e_1 \circ e_2\}$.
I)
Condition I) gives on $f_{\gamma'}$

$$
\begin{aligned}
\Delta_\gamma f_{\gamma'} &= \{\sum_{e,e'\in E(\gamma')} G^{ee'}_{\gamma;kl} Y^k_e Y^l_{e'} + \sum_{e\in E(\gamma')} G^e_{\gamma;k} Y^k_e\} f_{\gamma'} \\
&= \Delta_{\gamma'} f_{\gamma'} = \{\sum_{e,e'\in E(\gamma')} G^{ee'}_{\gamma';kl} Y^k_e Y^l_{e'} + \sum_{e\in E(\gamma')} G^e_{\gamma';k} Y^k_e\} f_{\gamma'}
\end{aligned} \tag{C.2}
$$

We conclude that $G^{e,e'}_{\gamma;kl} := G^{e,e'}_{\gamma-\{e_0\},kl}$ for any $e, e' \neq e_0$. Iterating this over all $\gamma' \subset \gamma$ we conclude that $G^{e,e'}_{\gamma;kl} = G^{e,e'}_{kl}$ should only depend on $A(e), A(e')$. Then for any $e \in E(\gamma')$

$$
G^e_{\gamma;k} = \sum_{e'\in E(\gamma')} (Y^l_{e'} G^{e'e}_{lk}) + (Y^l_{e_0} G^{e_0 e}_{lk}) = G^e_{\gamma';k} = \sum_{e'\in E(\gamma')} (Y^l_{e'} G^{e'e}_{lk}) \tag{C.3}
$$

so

$$
Y^l_{e'} G^{e'e}_{lk} = 0 \text{ for any } e' \neq e \tag{C.4}
$$

We conclude that

$$
G^e_k = (Y^l_e G^{ee}_{lk}) \tag{C.5}
$$

only depends on $A(e)$. Thus, condition I) has already restricted our ansatz to the form

$$
\Delta_\gamma = \sum_{e,e'\in E(\gamma)} G^{ee'}_{kl} Y^k_e Y^l_{e'} + \sum_{e\in E(\gamma)} G^e_k Y^k_e \tag{C.6}
$$

Notice that there are indeed solutions to (C.5), for instance $G^{ee'}_{kl} = G^{ee'}O_{kl}(A(e)A(e')^{-1})$ which has the attractive feature that $G^{ee}_{kl} = G^{ee}\delta_{jk}, G^{e}_{k} = 0$ so that we are reduced to bilinear ansatz which for diagonal elements is even gauge invariant. Then, if $G^{ee'}$ is small for $e \neq e'$ we have an almost gauge invariant operator.
II)
We notice $Y^j_{e^{-1}} = -O_{jk}(A(e)^{-1})Y^k_e$ and find

$$\begin{aligned}\Delta_\gamma f_{\gamma'} = \{&\Delta_{\gamma-\{e_0\}} - \sum_{e'\in E(\gamma)-\{e_0\}}[G^{e'e}_{kl}O_{lm}(A(e)^{-1}) + G^{ee'}_{lk}O_{lm}(A(e)^{-1})]Y^k_{e'}Y^m_{e_0^{-1}}\\
&+G^{e_0,e_0}_{kl}[O_{km}(A(e_0)^{-1})O_{ln}(A(e_0)^{-1})Y^m_{e_0^{-1}}Y^n_{e_0^{-1}} + O_{km}(A(e_0)^{-1})(Y^m_{e_0^{-1}}O_{ln}(A(e_0)^{-1}))Y^n_{e_0^{-1}}]\\
&-G^{e_0}_k O_{kl}(A(e_0)^{-1})Y^l_{e_0^{-1}}\}f_{\gamma'}\end{aligned} \tag{C.7}$$

It is easy to see that $O_{km}(A(e_0)^{-1})(Y^m_{e_0^{-1}}O_{ln}(A(e_0)^{-1})) \propto \epsilon_{kln}$ so the fourth term in (C.7) vanishes identically since the diagonal elements G^{ee}_{kl} are symmetric in k, l. From the second term we find the condition

$$G^{e'e}_{km}O_{ml}(A(e)^{-1}) = -G^{e'e^{-1}}_{kl} \text{ for any } e' \neq e \tag{C.8}$$

which also implies that (C.8) holds with $e' \leftrightarrow e, k \leftrightarrow l$ by using Hermiticity and that O_{kl} is real valued. The third term leads to

$$G^{e_0,e_0}_{kl}O_{km}(A(e_0)^{-1})O_{ln}(A(e_0)^{-1}) = G^{e_0^{-1}e_0^{-1}}_{mn} \tag{C.9}$$

which also implies that condition

$$G^{e_0}_k O_{kl}(A(e_0)^{-1}) = -G^{e_0^{-1}}_l \tag{C.10}$$

required by the fifth term is then already satisfied. Thus (C.8) and (C.9) is all that follows from II).
III)
We notice that $Y^j_{e_1}f_{\gamma'} = Y^j_e f_{\gamma'},\ Y^j_{e_2}f_{\gamma'} = O_{jk}(A(e_1))Y^j_e f_{\gamma'}$. Then

$$\begin{aligned}&\Delta_\gamma f_{\gamma'}\\
=\ &\{\Delta_{\gamma-\{e_0\}} + \sum_{e'\in E(\gamma)-\{e_1,e_2\}}[(G^{e'e_1}_{kl} + G^{e'e_2}_{km}O_{ml}(A(e_1))) + (G^{e_1e'}_{lk} + G^{e_2e'}_{mk}O_{ml}(A(e_1)))]Y^k_{e'}Y^l_{e_0}\\
&+[G^{e_1,e_1}_{kl} + [G^{e_2e_1}_{ml} + G^{e_1e_2}_{lm}]O_{mk}(A(e_1)) + G^{e_2e_2}_{mn}O_{mk}(A(e_1))O_{nl}(A(e_1))]Y^k_{e_0}Y^l_{e_0}\\
&+[G^{e_1}_k + G^{e_2}_l O_{lk}(A(e_1))]Y^k_{e_0}\}f_{\gamma'}\end{aligned} \tag{C.11}$$

The second term rquires that

$$G^{e'e_1}_{kl} + G^{e'e_2}_{km}O_{ml}(A(e_1)) = G^{e'e_0}_{kl} \text{ for any } e' \neq e_1, e_2 \tag{C.12}$$

which again also implies the same identity with labels interchanged due to Hermiticity. We cannot extend (C.12) to $e' = e_1, e_2, e_0$ since it is implicit in the definition of $G^{ee'}_{kl}$ that e, e' do not overlap. Thus, from the the third term we find the additional condition that

$$G^{e_1,e_1}_{kl} + [G^{e_2e_1}_{ml} + G^{e_1e_2}_{lm}]O_{mk}(A(e_1)) + G^{e_2e_2}_{mn}O_{mk}(A(e_1))O_{nl}(A(e_1)) = G^{e_0e_0}_{kl} \tag{C.13}$$

and the fourth term requires

$$G^{e_1}_k + G^{e_2}_l O_{lk}(A(e_1)) = G^{e_0}_k \tag{C.14}$$

which does not follow from (C.12),(C.14) any more.
Thus, there are the six conditions (C.4), (C.8), (C.9), (C.12), (C.13), (C.14) to be satisfied for cylindrical consistency to hold.

Let us now look for solutions of this system of six equations. There are obviously two classes of solutions, in the first class we take $G^{ee'}_{jk}$ as independent of A while in the second class there is non-trivial $A-$dependence. Consider first solutions in the first class. Then equation (C.4) is trivially solved. Equation (C.8) requires obviously that $G^{e'e}_{jk} = 0$ since no $A-$dependence is allowed, thus the metric is necessarily *diagonal*. Then also equation (C.14) is already solved since $G^e_k = 0$. Now, equation (C.9) becomes

$$G^{e_0 e_0}_{kl} O_{km}(A(e_0)^{-1}) O_{ln}(A(e_0)^{-1}) = G^{e_0^{-1} e_0^{-1}}_{mn} \tag{C.15}$$

It follows that $G^{ee}_{kl} = l_e \delta_{kl},\ l_e = l_{e^{-1}}$ since the O_{kl} are orthogonal matrices and the only way to get rid of them is to contract them with the unit matrix (use elementary Clebsch-Gordan theory). Finally then equation (C.12) is automatically satisfied while (C.13) implies that $l_{e_1} + l_{e_2} = l_{e_1 \circ e_2}$. Thus, within the first class there is a unique solution, namely

$$\Delta_\gamma = \sum_{e \in E(\gamma)} l_e Y^j_e Y^j_e \tag{C.16}$$

which is the one underlying the heat kernel Segal-Barmann transform of [11].
Now it turns out that there is in fact no solution in class two. To see this, study first equation (C.12). We consider the spin-network decomposition for $e \neq e'$

$$G^{ee'}_{kl} = a^{e,e'}_{kl} + b^{e,e'}_{kl}(e) + c^{e,e'}_{kl}(e') + d^{e,e'}_{kl}(e, e') \tag{C.17}$$

where round brackets indicate non-trivial spin dependence on the respective edge. Inserting into (C.12) shows that $a^{e,e'}_{kl} = a_{kl}, b^{e,e'}_{kl}(e) = b_{kl}(e)$ are universal functions and from here we see by coefficient comparison that $G^{ee'}_{kl} = 0$ for $e \neq e'$ so that again the metric is diagonal. By the same token we conclude from (C.14) that $G^e_k = 0$. Now we set

$$G^{ee}_{kl} = l_e \delta_{kl} + \alpha^e_{kl} + \beta^e_{kl}(e) \tag{C.18}$$

where $\alpha^e_{kl} \not\propto \delta_{kl}$ and find by the same token from equation (C.13) that $\alpha^e_{kl} = \beta^e_{kl}(e) = 0$.

Thus, we have shown that the only positive, cylindrically consistent operator within the class (C.1) is the one defined in [11] already with a diagonal edge metric. In other words, the distributional states for $SU(2)$ proposed in [16] are not of the complexifier type.

D More Non-Abelean Complexifiers

Here we display two more versions of the Non-Abelean complexifier.

Version 2: *Foliation and background dependent smearing*
If we allow ourselves to use a background metric, then we can discard the parquet as follows: Given $x \in \sigma$, determine for each I the leaf $S^I_x := S^I_{t^I(x)}$ of the foliation X^I from the identity $X^I_{t^I(x)}(u^I(x)) = x$. Let f^I_u be a smearing function on $\check{S}^I := (X^I)^{-1}(S^I_t)$ concentrated (with respect to the background metric) at u where we have assumed for simplicity that $\check{S}^I$ is independent of t (again, topology change

can easily dealt with by suitably subdividing the range of t according to the topology type). Then define

$$C := \frac{1}{2a\kappa}\int_\sigma d^3x \int_\sigma d^3y \sum_{I,J} K_{IJ}(x,y)\mathrm{Ar}_f(S^I_x)\mathrm{Ar}_f(S^J_x) \tag{D.1}$$

where $K_{IJ}(x,y)$ is a background dependent, symmetric kernel, a a constant of appropriate dimensionlity and

$$\mathrm{Ar}_f(S^I_x) = \int_{(X^I)-1(S^I_x)} d^2u f^I_{u^I(x)}(u)\sqrt{[E^a_j(X^I_{t^I(x)}(u))n^{S^I_x}_a(u)]^2} \tag{D.2}$$

For instance, if $\sigma = \mathbb{R}^2$ and we consider a flat background metric then we may take $\check{S}^I = \mathbb{R}^2$ and $f^I_u(u') = e^{-||u-u'||^2/(2r^2)}$ where the foliations are the natural ones given by a Cartesian frame. Since $\int d^2u f_{u'}(u) = 2\pi r^2$ we see that (D.2) is concentrated on a disk $\square^I_x$ with center x in the $x^I = const.$ plane of Euclidean area $\approx \pi r^2$ so that in this case $\mathrm{Ar}_f(S^I_x) \approx \mathrm{Ar}(\square^I_x)$. In order to get close to the previous formula (5.10) we could choose the kernel $K_{IJ}(x,y) = \delta_{IJ}\delta(x,y)$ where the sum over surfaces $\square$ inside a leaf S^I_t is then replaced by an integral over the points within the leaf.

The complexified connection is given by

$$A^{j\mathbb{C}}_a(x) = A^j_a(x) - i\kappa\delta C/\delta E^a_j(x) \tag{D.3}$$

and this time allows to reconstruct E from $A^{\mathbb{C}}$ given suitable assumtions about the kernel K. The explicit form of the functional derivative in (D.3) is similar to the one of (5.11) and will be left to the reader.

Similar remarks with respect to the spectrum of the area operator apply as before so that we get

$$\begin{aligned}
\frac{\hat{C}}{\hbar}T_s &= \frac{\ell_p^2}{2a}\{\int d^3x \int d^3y K_{IJ}(x,y) \times \\
&\quad \times[\sum_{e\in E(\gamma)}\sum_{p\in S^I_x\cap e} f^I_{u^I(x)}((X^I_{t^I(x)})^{-1}(p))\sqrt{j_e(j_e+1)}] \times \\
&\quad \times[\sum_{e'\in E(\gamma)}\sum_{p\in S^I_y\cap e} f^J_{u^I(y)}((X^I_{t^I(y)})^{-1}(p))\sqrt{j_{e'}(j_{e'}+1)}]\}T_s \\
&=: \frac{\ell_p^2}{2a}\{\sum_{e,e'} G^{e,e'}_{K,f}\sqrt{j_e(j_e+1)}\sqrt{j_{e'}(j_{e'}+1)}\}T_s
\end{aligned} \tag{D.4}$$

Again, this gives a suitable well-defined generalization of the edge metric to the non-Abelean regime which depends on a foliation and a background metric. Using the fact that the edges e have compact support one can check that the matrix $G^{ee'}$ is finite and positive definite and gives rise to a cylindrically consistent operator.

Version 3: *Background dependent smearing*
The idea is to to introduce, for each point $x \in \sigma$ a two parameter family of surfaces $S_x(\varphi,\theta)$ through x and then to average over these surfaces over the sphere S^2 with respect to some background dependent measure $d\nu(\Omega)$ where $\Omega = (\varphi,\theta)$. For instance, if the background is flat, $d\nu$ would be naturally chosen to be the rotation invariant measure on S^2. Then, for given Ω one would let $n_a(\Omega)$ be the standard unit vector defined in $\mathbb{R}^3$ pointing into the direction Ω defined by the natural Cartesian frame and $S_x(\Omega)$ would be the plane throgh x orthogonal to $n(\Omega)$. Thus the proposal would be in general

$$C := \frac{1}{2a\kappa}\int_\sigma d^3x \int_\sigma d^3y \sum_{I,J} K(x,y)\mathrm{Ar}_f(x)\mathrm{Ar}_f(y) \tag{D.5}$$

where we have again introduced appropriate smearing functions f on the $S_x(\Omega)$ (for simplicity we assume that they can be chosen to be independent of x, Ω) and

$$\mathrm{Ar}_f(x) = \int_{S^2} d\nu(\Omega) \mathrm{Ar}_f(S_x(\Omega)) \tag{D.6}$$

as in (D.2). Expression (D.5) has the advantage that it is *Euclidean invariant* in the case of a flat background if we choose a Euclidean invariant kernel.

The complexified connection is again given by formula (D.3) while the spectrum of the complexifier becomes

$$\begin{aligned}
\frac{\hat{C}}{\hbar} T_s &= \frac{\ell_p^2}{2a} \{ \int d^3x \int d^3y K(x,y) \times \\
&\quad \times [\sum_{e \in E(\gamma)} \int d\nu(\Omega) \sum_{p \in S_x(\Omega) \cap e} f_{u^\Omega(x)}((X_x^\Omega)^{-1}(p)) \sqrt{j_e(j_e+1)}] \times \\
&\quad \times [\sum_{e' \in E(\gamma)} \int d\nu(\Omega) \sum_{p' \in S_y(\Omega) \cap e'} f_{u^\Omega(y)}((X_y^\Omega)^{-1}(p')) \sqrt{j_{e'}(j_{e'}+1)}] \} T_s \\
&=: \frac{\ell_p^2}{2a} \{ \sum_{e,e'} G^{e,e'}_{K,f} \sqrt{j_e(j_e+1)} \sqrt{j_{e'}(j_{e'}+1)} \} T_s
\end{aligned} \tag{D.7}$$

where $X_x^\Omega : \check{S} \subset \mathbb{R}^2 \to S_x(\Omega)$ is an embedding and $X_x^\Omega(u^\Omega(x)) := x$.

References

[1] T. Thiemann, Physics Letters B **380** (1996) 257-264, [gr-qc/960688]

[2] T. Thiemann, Class. Quantum Grav. **15** (1998) 839-73, [gr-qc/9606089]; Class. Quantum Grav. **15** (1998) 875-905, [gr-qc/9606090]; Class. Quantum Gravity **15** (1998) 1207-47, [gr-qc/9705017]; Class. Quantum Grav. **15** (1998) 1249-80, [gr-qc/9705018]; Class. Quantum Grav. **15** (1998) 1281-1314, [gr-qc/9705019]; Class. Quantum Grav. **15** (1998) 1487-1512, [gr-qc/9705021]; Class. Quantum Grav. **15** (1998) 1463-85, [gr-qc/9705020]

[3] A. Ashtekar, C.J. Isham, Class. & Quan. Grav. **9**, 1433 (1992)
A. Ashtekar, J. Lewandowski, "Representation theory of analytic holonomy $C^\star$ algebras", in "Knots and Quantum Gravity", J. Baez (ed), Oxford University Press, Oxford, 1994

[4] T. Thiemann, "Introduction to Modern Canonical Quantum General Relativity", gr-qc/0110034

[5] C. Rovelli, "Loop Quantum Gravity", Review written for electronic journal "Living Reviews", gr-qc/9710008

[6] A. Ashtekar, "Quantum Geometry and Gravity: Recent Advances", gr-qc/0112038
A. Ashtekar, "Quantum Geometry in Action: Big Bang and Black Holes", math-ph/0202008

[7] M. Bojowald, H. Kastrup, Class. Quantum Grav. **17** (2000) 3009 [hep-th/9907042]
M. Bojowald, Class. Quantum Grav. **17** (2000) 1489 [gr-qc/9910103];
Class. Quantum Grav. **17** (2000) 1509 [gr-qc/9910104]; Class. ,Quantum Grav. **18** (2001) 1055-1070 [gr-qc/0008052]; Class. Quantum Grav. **18** (2001) 1071-1088 [gr-qc/0008053]; [gr-qc/0008054]; Phys. Rev. Lett. **86** (2001) 5227-5230 [gr-qc/0102069]; Phys. Rev. Lett. **87** (2001) 121301 [gr-qc/0104072]; Phys.Rev.D64:084018,2001, [gr-qc/0105067]; Class. Quantum Grav. **18** (2001) L109-L116 [gr-qc/0105113]
M. Bojowald, "Quantization Ambiguities in Isotropic Quantum Geometry", to appear

[8] T. Thiemann, Class. Quantum Grav. **18** (2001) 2025-2064, [hep-th/0005233]

[9] T. Thiemann, Class. Quant. Grav. **18** (2001) 3293-3338,2001 [hep-th/0005232]
[10] T. Thiemann, O. Winkler, Class. Quantum Grav. **18** (2001) 2561-2636, [hep-th/0005237]; Class. Quantum Grav. **18** (2001) 4629-4682, [hep-th/0005234]; Class.Quant.Grav. **18** (2001) 4997-5054, [hep-th/0005235]
H. Sahlmann, T. Thiemann, O. Winkler, Nucl. Phys. **B606** (2001) 401-440, [gr-qc/0102038]
[11] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, Journ. Funct. Analysis **135** (1996) 519-551, [gr-qc/9412014]
[12] B. Hall, Journ. Funct. Analysis **122** (1994) 103
B. Hall, J. F. Mitchell, J.Math.Phys.43 (2002) 1211 [quant-ph/0109086]; "The Large Radius Limit for Coherent States on Spheres", quant-ph/0203142
[13] T. Thiemann, "Quantum Spin Dynamics (QSD) VIII. The Classical Limit", in preparation
[14] M. Varadarajan, Phys.Rev.D61:104001,2000, [gr-qc/0001050]
M. Varadarajan, Phys.Rev.D64 (2001) 104003 [gr-qc/0104051]
M. Varadarajan, "Gravitons from a Loop Representation of Linearized Gravity", gr-qc/0204067
[15] J. Velhinho, "Invariance Properties of Induced Fock Measures for U(1) Holonomies", math-ph/0107002
[16] A. Ashtekar, J. Lewandowski, "Relation Between Polymer and Fock Excitations", Class.Quant.Grav.18:L117-L128,2001, [gr-qc/0107043]
[17] L. Bombelli, "Statistical Lorentzian Geometry and the Closeness of Lorentzian Manifolds, J.Math.Phys.41:6944-6958,2000, [gr-qc/0002053]
C. Itzykson, J.-M.Drouffe, "Statistical Field Theory", vol. 2, Cambridge University Press, Cambridge, 1989
[18] T. Thiemann Class. Quantum Grav. **13** (1996) 1383-1403, [gr-qc/9511057]; Acta Cosmologica **21** (1995), 145-167, [gr-qc/9511049]
[19] F. Barbero, Phys. Rev. **D51** (1995) 5507
F. Barbero, Phys. Rev. **D51** (1995) 5498
[20] R. Gambini, A. Trias, Phys. Rev. **D22** (1980) 1380
C. Di Bartolo, F. Nori, R. Gambini, A. Trias, Lett. Nuov. Cim. **38** (1983) 497
R. Gambini, A. Trias, Nucl. Phys. **B278** (1986) 436
[21] T. Jacobson, L. Smolin, Nucl. Phys. **B299** (1988) 295
C. Rovelli, L. Smolin, Nucl. Phys. **B331** (1990) 80
[22] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, T. Thiemann, Journ. Math. Phys. **36** (1995) 6456-6493, [gr-qc/9504018]
[23] J. Mourão, T. Thiemann, J. Velhinho, J. Math. Phys. **40** (1999) 2337-53, [hep-th/9711139]
[24] C. Rovelli, L. Smolin, Phys. Rev. **D52** (1995) 5743
J. Baez, Adv. Math. **117** (1996) 253-272
[25] C. Rovelli, L. Smolin, Nucl. Phys. B **442** (1995) 593, Erratum : Nucl. Phys. B **456** (1995) 734
A. Ashtekar, J. Lewandowski, Class. Quantum Grav. **14** A55-81 (1997); Adv. Theo. Math. Phys. **1** (1997) 388-429
T. Thiemann, Journ. Math. Phys. **39** (1998) 3372-92, [gr-qc/9606092]
[26] H. Sahlmann, T. Thiemann, "Towards the QFT on CST Limit of QGR I. : A General Scheme; II. A Concrete Implementation", to appear
[27] H. Sahlmann, "On Representing the Holonomy – Electric Flux Algebra in Quantum General Relativity", to appear
[28] C. Rovelli, Class. Quantum Grav. **8** (1991) 297; Class. Quantum Grav. **8** (1991) 317; Phys. Rev. **D43** (1990) 442; Phys. Rev. **D43** (1990) 2638
[29] A. Ashtekar, C. Rovelli, L. Smolin, Phys. Rev. **D44** (1991) 1740