Published by Institute of Physics Publishing for SISSA
Received: *October 28, 2005*
Accepted: *December 7, 2005*
Published: *January 13, 2006*

New integrable system of 2dim fermions from strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Luis Fernando Alday,[a] **Gleb Arutyunov**[a*] **and Sergey Frolov**[b†]

[a] *Institute for Theoretical Physics and Spinoza Institute, Utrecht University 3508 TD Utrecht, The Netherlands*
[b] *Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut Am Mühlenberg 1, D-14476 Potsdam, Germany*
E-mail: L.F.Alday@phys.uu.nl, G.Arutyunov@phys.uu.nl, frolovs@aei.mpg.de

Abstract: We consider classical superstrings propagating on $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time. We consistently truncate the superstring equations of motion to the so-called $\mathfrak{su}(1|1)$ sector. By fixing the uniform gauge we show that physical excitations in this sector are described by two complex fermionic degrees of freedom and we obtain the corresponding lagrangian. Remarkably, this lagrangian can be cast in a two-dimensional Lorentz-invariant form. The kinetic part of the lagrangian induces a non-trivial Poisson structure while the hamiltonian is just the one of the massive Dirac fermion. We find a change of variables which brings the Poisson structure to the canonical form but makes the hamiltonian nontrivial. The hamiltonian is derived as an exact function of two parameters: the total S^5 angular momentum J and string tension λ; it is a polynomial in $1/J$ and in $\sqrt{\lambda'}$ where $\lambda' = \frac{\lambda}{J^2}$ is the effective BMN coupling. We identify the string states dual to the gauge theory operators from the closed $\mathfrak{su}(1|1)$ sector of $\mathcal{N} = 4$ SYM and show that the corresponding near-plane wave energy shift computed from our hamiltonian perfectly agrees with that recently found in the literature. Finally we show that the hamiltonian is integrable by explicitly constructing the corresponding Lax representation.

Keywords: Integrable Field Theories, Penrose limit and pp-wave background, AdS-CFT Correspondence.

[*] Also at Steklov Mathematical Institute, Moscow.
[†] Also at SUNYIT, Utica, USA, and Steklov Mathematical Institute, Moscow.

http://jhep.sissa.it/archive/papers/jhep012006078/jhep012006078.pdf

Contents

1. Introduction

Further progress in understanding the AdS/CFT duality [1] in the large N limit requires quantizing superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Even though classical superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is an integrable model [2] it is difficult to quantize it by conventional methods developed in the theory of quantum integrable systems [3]. Action variables are encoded in algebraic curves describing finite-gap solutions of the string sigma-model [4], however, angle variables have not been yet identified.

JHEP01(2006)078

On the other hand, the dilatation operator of $\mathcal{N}=4$ SYM can be viewed as a hamiltonian of an integrable spin chain [5] which at higher loops becomes long-range [6]. Perturbative scaling dimensions of composite operators can be computed by solving the corresponding Bethe ansatz equations [7, 8].[1]

The success of the Bethe ansatz approach in gauge theory hints that the spectrum of quantum strings might also be encoded in a similar set of equations. Indeed, a Bethe type ansatz which captures dynamics of quantum strings in certain asymptotic regimes has been proposed [14]. The quantum string Bethe ansatz (QSBA) describes the spectrum of string states dual to gauge theory operators from the closed $\mathfrak{su}(2)$ sector [14]. The dual gauge theory contains other closed sectors [15], and it is possible to generalize the QSBA to these [16], and even to the complete model [8]. However, it remains unclear how the QSBA can emerge from an exact (non-semiclassical) quantization of strings.

It turns out that classical superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ admits consistent truncations to smaller sectors [17] which contain string states dual to operators from the closed sectors of gauge theory. Apparently, the truncated models are non-critical, and therefore, are expected to loose many important features of the superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ such as conformal invariance and renormalizability. However, they inherit classical integrability of the parent theory, and one might hope that despite their apparent non-renormalizability there would exist a unique quantum deformation which preserves integrability and describes correctly the dynamics of quantum superstrings in these sectors.

As is known [15], $\mathcal{N}=4$ SYM contains three simple closed sectors: $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$. In the full theory they are related to each other by supersymmetry which implies highly nontrivial relations between the spectra of operators from these sectors [16]. The consistent truncations of *classical* superstring theory to the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors describe strings propagating in $\mathbb{R}\times\mathrm{S}^3$ and $\mathrm{AdS}_3\times\mathrm{S}^1$, respectively. A truncation of superstrings to the $\mathfrak{su}(1|1)$ sector is unknown, and finding it is one of the aims of our paper.

The $\mathfrak{su}(1|1)$ sector of the gauge theory seems to be the simplest one, in particular, the one-loop dilatation operator describes a free lattice fermion [18]. In truncated string theory one expects physical excitations to be carried by two complex fermions, and, therefore, one might hope to find an action which is polynomial in the fermionic variables. This would represent a drastic simplification in comparison to the reductions of superstrings to the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors where physical excitations are bosonic and described by non-polynomial Nambu-type actions [19]. Thus, finding quantum deformations in the $\mathfrak{su}(1|1)$ sector might be more feasible.

Independently of the importance of this problem to the AdS/CFT correspondence, finding consistent reductions of the superstring theory provides a way to generate new interesting integrable models. The simplest example of such a kind is the Neumann model [20] describing rigid multi-spin string solitons [21]. Among other examples of new integrable systems is the Nambu-type hamiltonian for physical degrees of freedom of bosonic strings on $\mathrm{AdS}_5\times\mathrm{S}^5$ [19].

[1] Related aspects of integrability of strings on $\mathrm{AdS}_5\times\mathrm{S}^5$ and its gauge theory counterpart were also studied in [9]–[13] and subsequent works.

We start from the classical string action on $AdS_5 \times S^5$ [22, 23] formulated as a sigma-model on the coset PSU(2,2|4)/SO(4,1)×SO(5). It is essential for our approach to parametrize a coset representative by coordinates on which the global symmetry group PSU(2,2|4) is linearly realized. This makes the identification between these string coordinates and the fields of the dual gauge theory transparent. It also allows us to find easily a consistent truncation of the string equations of motion to the $\mathfrak{su}(1|1)$ sector. This procedure involves imposing a so-called uniform gauge [24, 19] which amounts to identifying the global AdS time with the world-sheet time τ and fixing the momentum of an angle variable of S^5 to be equal to the corresponding U(1) charge J. Before the gauge fixing the string lagrangian of the reduced model has two bosons and two complex fermions, and inherits two linearly realized supersymmetries from the parent theory. Imposing the gauge completely removes all the bosons so that the physical excitations are carried only by the fermions while supersymmetries become non-linearly realized. Quite surprisingly, the two complex space-time fermions can be combined into a single Dirac fermion, ψ, and the action can be cast into a manifestly two-dimensional Lorentz-invariant form. Thus, the original Green-Schwarz fermions which are world-sheet scalars transform into world-sheet spinors. This reminds the relation between the flat space light-cone formulations of the Green-Schwarz and NSR superstrings. In addition the lagrangian exhibits the usual U(1) symmetry which is realized by a phase multiplication of the Dirac fermion.

The hamiltonian we obtain coincides with that of the massive Dirac fermion. However, the kinetic part of the lagrangian induces a non-trivial Poisson structure which we explicitly describe. The Poisson bracket is ultra-local, and is an 8-th order polynomial in the fermion ψ and its first derivative. Then, we show that there is a change of variables which brings the Poisson structure to the canonical form but makes the hamiltonian nontrivial. We find the hamiltonian as an exact function of two parameters: the total S^5 angular momentum J and string tension λ. It appears to be a polynomial in $1/J$ and in $\sqrt{\lambda'}$ where $\lambda' = \frac{\lambda}{J^2}$ is the effective BMN coupling.

We can also use our hamiltonian to study the near-plane wave corrections to the energy of the plane-wave states from the $\mathfrak{su}(1|1)$ sector. To this end we keep in the hamiltonian terms up to order $1/J$, and compute the energy shift by using the first-order perturbation theory. The same correction has been already found in [25, 26] by using a light-cone type gauge. The uniform gauge we adopt in our approach is different and that makes a comparison of their hamiltonian with ours difficult. Nevertheless, we demonstrate that the energy of an arbitrary M-impurity plane-wave state computed by using our hamiltonian is in a perfect agreement with the results by [25, 26]. Thus, at the order $1/J$ our hamiltonian leads to equivalent dynamics. Let us also mention that the coherent state description of the $\mathfrak{su}(1|1)$ sector with its further comparison to string theory was considered in [27].

Finally, we show that the Lax representation of the full string sigma-model [2] also admits a consistent reduction to the $\mathfrak{su}(1|1)$ sector. Thus, the hamiltonian of the reduced model is also integrable.

The paper is organized as follows. In section 2 we recall the necessary facts about the Lie superalgebra $\mathfrak{psu}(2,2|4)$, and the construction of the string sigma-model lagrangian. We also discuss our specific choice for the coset representative as well as the global symmetries

JHEP01(2006)078

of the model. In section 3 we identify the consistent truncation to the $\mathfrak{su}(1|1)$ sector and in section 4 we obtain the corresponding lagrangian. In section 5 the hamiltonian and the Poisson structure of the model are found. By redefining the fermionic variables we transform in section 6 the Poisson structure to the canonical form and compute the accompanying hamiltonian. In section 7 the near-plane wave energy shift is computed, and in section 8 the Lax representation for the reduced model is studied. Finally, some technical details and the Poisson structure of the reduced model are collected in five appendices.

2. Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ as the coset sigma-model

Superstring propagating in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time can be described as the non-linear sigma-model whose target space is the following coset [22]

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}\,. \tag{2.1}$$

Here the supergroup $\mathrm{PSU}(2,2|4)$ with the Lie algebra $\mathfrak{psu}(2,2|4)$ is the isometry group of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superspace. The string theory action is the sum of the non-linear sigma-model action and of the topological Wess-Zumino term to ensure κ-symmetry.

In what follows we need to introduce a suitable parametrization for the coset element (2.1). We start by recalling several basic facts about the corresponding Lie superalgebra.

2.1 The superalgebra $\mathfrak{psu}(2,2|4)$

The superalgebra $\mathfrak{su}(2,2|4)$ is spanned by 8×8 matrices M which can be written in terms of 4×4 blocks as

$$M=\begin{pmatrix} A & X\\ Y & D\end{pmatrix}\,. \tag{2.2}$$

These matrices are required to have vanishing supertrace $\mathrm{str}M=\mathrm{tr}A-\mathrm{tr}D=0$ and to satisfy the following reality condition

$$HM+M^{\dagger}H=0\,. \tag{2.3}$$

For our further purposes it is convenient to pick up the hermitian matrix H to be of the form

$$H=\begin{pmatrix} \Sigma & 0\\ 0 & -\mathbb{I}\end{pmatrix}\,, \tag{2.4}$$

where Σ is the following matrix

$$\Sigma=\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1\end{pmatrix} \tag{2.5}$$

and $\mathbb{I}$ denotes the identity matrix of the corresponding dimension. The matrices A and D are even, and X, Y are odd (linearly depend on fermionic variables). The condition (2.3) implies that A and D span the subalgebras $\mathfrak{u}(2,2)$ and $\mathfrak{u}(4)$ respectively, while X and Y are related through $Y = X^\dagger \Sigma$. The algebra $\mathfrak{su}(2,2|4)$ also contains the $\mathfrak{u}(1)$ generator $i\mathbb{I}$ as it obeys eq.(2.3) and has zero supertrace. Thus, the bosonic subalgebra of $\mathfrak{su}(2,2|4)$ is

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)\,. \tag{2.6}$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is defined as the *quotient algebra* of $\mathfrak{su}(2,2|4)$ over this $\mathfrak{u}(1)$ factor; it has no realization in terms of 8×8 supermatrices.

The superalgebra $\mathfrak{su}(2,2|4)$ has a $\mathbb{Z}_4$ grading

$$M = M^{(0)} \oplus M^{(1)} \oplus M^{(2)} \oplus M^{(3)}$$

defined by the automorphism $M \to \Omega(M)$ with

$$\Omega(M) = \begin{pmatrix} KA^tK & -\,KY^tK \\ KX^tK & KD^tK \end{pmatrix}, \tag{2.7}$$

where we choose the 4×4 matrix K satisfying $K^2 = -I$ to be

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.8}$$

The space $M^{(0)}$ is in fact the $\mathrm{so}(4,1) \times \mathrm{so}(5)$ subalgebra, the subspaces $M^{(1,3)}$ contain odd fermionic variables.

The orthogonal complement $M^{(2)}$ of $\mathrm{so}(4,1) \times \mathrm{so}(5)$ in $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$ can be conveniently described as follows. In appendix A we introduce the matrices γ_a and Γ_a, $a = 1, \ldots, 5$, which are the Dirac matrices for SO(4,1) and SO(5) correspondingly. These matrices obey the relations

$$K\gamma_a^t K = -\gamma_a\,, \qquad K\Gamma_a^t K = -\Gamma_a \tag{2.9}$$

and, therefore, they span the orthogonal complements to the Lie algebras so(4,1) and so(5) respectively.

2.2 The lagrangian

Consider now a group element g belonging to $\mathrm{PSU}(2,2|4)$ and construct the following current

$$\mathbf{A} = -g^{-1}\mathrm{d}g = \underbrace{\mathbf{A}^{(0)} + \mathbf{A}^{(2)}}_{\text{even}} + \underbrace{\mathbf{A}^{(1)} + \mathbf{A}^{(3)}}_{\text{odd}}\,. \tag{2.10}$$

Here we also exhibited the $\mathbb{Z}_4$ decomposition of the current. By construction this current has zero-curvature.

The lagrangian density for superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be written in the form [22, 23]

$$\mathscr{L} = -\frac{1}{2}\sqrt{\lambda}\gamma^{\alpha\beta}\mathrm{str}\Big(\mathbf{A}^{(2)}_\alpha \mathbf{A}^{(2)}_\beta\Big) - \kappa\epsilon^{\alpha\beta}\mathrm{str}\Big(\mathbf{A}^{(1)}_\alpha \mathbf{A}^{(3)}_\beta\Big)\,, \tag{2.11}$$

which is the sum of the kinetic and the Wess-Zumino terms. κ-symmetry requires $\kappa = \pm\frac{1}{2}\sqrt{\lambda}$. Here we use the convention $\epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$ is the Weyl-invariant combination of the metric on the string world-sheet with $\det\gamma = -1$.

2.3 Coset representative

Obviously there are many different ways to parametrize the coset element (2.1), all of them related by non-linear field redefinitions. In what follows we find convenient to use the following parametrization for the coset element

$$g = g(\theta,\eta)g(x,y)\,. \tag{2.12}$$

Here $g(x,y)$ describes an embedding of $\mathrm{AdS}_5 \times \mathrm{S}^5$ into $\mathrm{SU}(2{,}2) \times \mathrm{SU}(4)$ and $g(\theta,\eta)$ is a matrix which incorporates the original 32 fermionic degrees of freedom. We take

$$g(x,y) = \underbrace{\exp\tfrac{1}{2}(x_a\gamma_a)}_{g(x)}\,\underbrace{\exp\tfrac{i}{2}(y_a\Gamma_a)}_{g(y)} \tag{2.13}$$

Here the coordinates x_a parametrize the AdS_5 space while y_a stand for coordinates of the five-sphere. It is also understood that $g(x,y)$ is a 8 by 8 block-diagonal matrix with the upper 4 by 4 block equal to $g(x)$, and the lower block equal to $g(y)$.

Finally, the odd matrix is of the form (distinction between θ's and η's will be discussed later)

$$g(\theta,\eta) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\ 0 & 0 & 0 & 0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\ 0 & 0 & 0 & 0 & \theta^1 & \theta^2 & \theta^3 & \theta^4 \\ 0 & 0 & 0 & 0 & \theta^5 & \theta^6 & \theta^7 & \theta^8 \\ \eta_5 & \eta_1 & -\theta_1 & -\theta_5 & 0 & 0 & 0 & 0 \\ \eta_6 & \eta_2 & -\theta_2 & -\theta_6 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_3 & -\theta_3 & -\theta_7 & 0 & 0 & 0 & 0 \\ \eta_8 & \eta_4 & -\theta_4 & -\theta_8 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.14}$$

Here θ^i and η^i are $8+8$ complex fermions obeying the following conjugation rule $\theta^{i\,*} = \theta_i$ and $\eta^{i\,*} = \eta_i$. By construction the element g and, $g(\theta,\eta)$ in particular, belong to the supergroup SU(2,2|4).

It is worth emphasizing that the parametrization of the coset element we choose is different from the one used by Metsaev and Tseytlin [22], in particular we put the matrix containing fermionic variables to the left from the bosonic coset representative. As we will see such a form of the coset element makes the transformation properties of fermions under the global symmetry group transparent and will allow us to easily identify the consistent truncation.

JHEP01(2006)078

The bosonic coset element (2.13) provides parametrization of the $AdS_5 \times S^5$ space in terms of $5+5$ unconstrained coordinates x_a and y_a. It is however more convenient to work with the constrained $6+6$ coordinates which describe the embeddings of the AdS_5 and the five-sphere into $\mathbb{R}^{4,2}$ and $\mathbb{R}^6$ respectively. The latter parametrization was introduced in [20]. Here the AdS and the sphere representatives, $g_a(v)$ and $g_s(u)$, are described by the following matrices

$$g_a(v) = \begin{pmatrix} 0 & -iv_5 - v_6 & v_1 - iv_4 & -iv_2 - v_3 \\ iv_5 + v_6 & 0 & -iv_2 + v_3 & v_1 + iv_4 \\ -v_1 + iv_4 & iv_2 - v_3 & 0 & iv_5 - v_6 \\ iv_2 + v_3 & -v_1 - iv_4 & -iv_5 + v_6 & 0 \end{pmatrix}, \tag{2.15}$$

$$g_s(u) = \begin{pmatrix} 0 & -iu_5 - u_6 & -iu_1 - u_4 & -u_2 + iu_3 \\ iu_5 + u_6 & 0 & -u_2 - iu_3 & -iu_1 + u_4 \\ iu_1 + u_4 & u_2 + iu_3 & 0 & iu_5 - u_6 \\ u_2 - iu_3 & iu_1 - u_4 & -iu_5 + u_6 & 0 \end{pmatrix}. \tag{2.16}$$

The new variables u, v are constrained

$$\begin{aligned} v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_5^2 - v_6^2 &= -1 \\ u_1^2 + u_2^2 + u_3^2 + u_2^2 + u_5^2 + u_6^2 &= 1 \end{aligned} \tag{2.17}$$

which guarantees that $g_a(v)$ and $g_s(u)$ belong to SU(2,2) and SU(4) respectively. On the coordinates (u, v) the conformal and R-symmetry transformations act *linearly* which is not the case for (x, y).

It is not difficult to find the explicit relation between these two different description of the coset space. Taking into account that arbitrary coset elements $g_a(v)$ and $g_s(u)$ of SU(2,2)/SO(4,1) and SU(4)/SO(5) respectively can be represented in the form

$$g_a(v) = g(x)Kg(x)^t\,, \qquad g_s(u) = g(y)Kg(y)^t\,, \tag{2.18}$$

where $g(x)$ and $g(y)$ are SU(2,2) and SU(4) matrices, and choosing them to be given by (2.13), we see that the following relations are satisfied

$$x_a = \frac{|x|}{\sinh|x|} v_a, \qquad |x| = \mathrm{arcosh} v_6\,, \tag{2.19}$$

$$y_a = \frac{|y|}{\sin|y|} u_a, \qquad |y| = \arccos u_6\,. \tag{2.20}$$

Here also

$$|x|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2\,, \qquad |y|^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\,.$$

As was mentioned above, the coordinates (u, v) are very convenient because they transform linearly under the isometry group. In the following we first determine the lagrangian of the theory in terms of the coset element (2.1) and then substitute in the final result the change of variables $(x, y) \to (u, v)$ according to eqs. (2.19), (2.20).

JHEP01(2006)078

2.4 Global symmetries

To identify a consistent truncation to the $\mathfrak{su}(1|1)$ sector we have to analyze the global symmetries in more detail. According to the standard technique of non-linear realizations the isometry group $\mathrm{PSU}(2,2|4)$ acts on the coset representative by multiplication from the left

$$Gg = g' g_c \,. \tag{2.21}$$

Here $G \in \mathrm{PSU}(2,2|4)$, g and g' are the coset representatives before and after the group action and g_c is a *compensating transformation* from SO(4,1)$\times$ SO(5). We will need only infinitesimal transformations generated by the algebra $\mathfrak{psu}(2,2|4)$.

Conformal transformations of bosonic fields. Consider first the bosonic AdS coset element $g(x)$. We note that since a matrix $A \equiv \frac{1}{2} x_a \gamma_a$ obeys the relation $KA^tK = -A$ the element g itself also obeys

$$Kg(x)^t K = -g(x) \,. \tag{2.22}$$

This gives a nice way to describe this coset. The coset element is just a matrix from SU(2,2) group obeying an additional constraint (2.22). An infinitesimal conformal transformation reads

$$\delta g(x) = \Phi g(x) - g(x)\Phi_c \,. \tag{2.23}$$

Here Φ is an arbitrary matrix from the Lie algebra $\mathfrak{su}(2,2)$; it plays the role of the parameter of an infinitesimal conformal transformation. The matrix Φ_c belongs to so(4,1)$\subset \mathfrak{su}(2,2)$ and, therefore, it obeys the relation

$$K\Phi_c^t K = \Phi_c \,. \tag{2.24}$$

The element Φ_c is not independent but should be found for a given Φ by requiring that $\delta g(x)$ also belongs to the coset, in other words,

$$K\delta g(x)^t K = -\delta g(x) \,. \tag{2.25}$$

This equation allows one to find the compensating so(4,1) transformation $\Phi_c \equiv \Phi_c(\Phi, g)$. Actually to determine the transformation law for the variables v the compensating matrix Φ_c is not needed. Indeed, using the formula (2.18) we obtain

$$\delta g_a(v) = \Phi g_a(v) + g_a(v)\Phi^t \,, \tag{2.26}$$

where Φ_c decouples due to eq. (2.24). The explicit form of the transformation rules for the coordinates v can be found in appendix B.

JHEP01(2006)078

In what follows we will be interested in the form of Φ corresponding to translations of the global AdS time coordinate. The corresponding generator is identified with the dilatation operator. As is clear from eq. (2.17), the global AdS time coordinate t can be expressed through v_5 and v_6 as follows

$$e^{it} = iv_5 + v_6 \ .$$

Then the U(1) subgroup which rotates only v_5 and v_6 corresponds to translations of t. The explicit form of Φ can be easily found from the formulas in appendix B, and is given by

$$\Phi = \xi \, \frac{i}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.27}$$

The time coordinate t is shifted by the transformation by ξ: $t \to t' = t + \xi$, and one can easily see by using formulas from appendix A that the dilatation operator that generates the shift is

$$\Phi_t = \tfrac{1}{2}\gamma_5 \, .$$

For our further purposes it is useful to identify the so(4) $\subset \mathfrak{su}(2,2)$ symmetry which linearly rotates $v_1, \ldots, v_4$ but does not affect $v_{5,6}$ directions. It is induced by the following matrix

$$\Phi_{\mathrm{so}(4)} = \begin{pmatrix} i\xi_1 & \alpha_1 + i\beta_1 & 0 & 0 \\ -\alpha_1 + i\beta_1 & -i\xi_1 & 0 & 0 \\ 0 & 0 & i\xi_3 & \alpha_6 + i\beta_6 \\ 0 & 0 & -\alpha_6 + i\beta_6 & -i\xi_3 \end{pmatrix} \tag{2.28}$$

that is a direct sum of two $\mathfrak{su}(2)$'s.

R-symmetry transformations of bosonic fields. A similar analysis goes for the action of the $\mathfrak{su}(4)$ R-symmetry transformations. There are several interesting U(1) subgroups of the $\mathfrak{su}(4)$ algebra. To identify them we notice that the form of g_s in eq. (2.15) suggests to introduce the following three complex scalars

$$Z_1 = u_4 + iu_1 \, , \qquad Z_2 = u_2 + iu_3 \, , \qquad Z_3 = u_6 + iu_5 \, . \tag{2.29}$$

Then from Appendices B and A we deduce that the field Z_3 carries a unit charge under the following $\mathfrak{u}(1)$ of $\mathfrak{su}(4)$ generated by the matrix

$$\Phi_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \tfrac{1}{2}\Gamma_5 \, . \tag{2.30}$$

The fields Z_1 and Z_2 are neutral under this U(1) group.

In the same way we find that Z_1 carries a unit charge and Z_2 and Z_3 are neutral under the $\mathfrak{u}(1)$ generated by

$$\Phi_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and Z_2 carries a unit charge and Z_1 and Z_3 are neutral under the $\mathfrak{u}(1)$ generated by

$$\Phi_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Further we note that the last two $\mathfrak{u}(1)$'s are subalgebras of so(4)=$\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry algebra which rotates only $u_1, \dots u_4$, and is embedded in $\mathfrak{su}(4)$ as

$$\Phi_{\mathrm{so}(4)} = \begin{pmatrix} i\xi_1 & \alpha_1 + i\beta_1 & 0 & 0 \\ -\alpha_1 + i\beta_1 & -i\xi_1 & 0 & 0 \\ 0 & 0 & i\xi_3 & \alpha_6 + i\beta_6 \\ 0 & 0 & -\alpha_6 + i\beta_6 & -i\xi_3 \end{pmatrix}. \qquad (2.31)$$

Conformal and R-symmetry transformations of fermions. Let us now determine the transformation rules for the fermionic variables under conformal and R-symmetry transformations. To simplify the notation we denote $g(\theta, \eta) = \exp\Theta$ and the bosonic coset element by g. Then the infinitezimal action of the symmetry group on fermions can be deduced from the general formula describing the variation of the coset element

$$\delta_\Phi(e^\Theta g) = \Phi e^\Theta g - e^\Theta g \Phi_c \, ,$$

where Φ_c is again a compensating transformation which generically might depend on Φ, g and Θ. Taking into account the expression (2.23) we find the transformation rule for fermionic variables

$$\delta_\Phi \Theta = [\Phi, \Theta] \, . \qquad (2.32)$$

This shows an advantage of our coset parametrization: the symmetries act linearly on fermionic variables, just in the same manner as in the dual gauge theory!

The similarity can be made even more explicit if we use (2.14) to write the fermionic matrix Θ in the block form

$$\Theta = \begin{pmatrix} 0 & \widetilde{\Psi} \\ \Psi & 0 \end{pmatrix},$$

and the conformal and R-symmetry transformations matrix Φ in the block-diagonal form

$$\Phi = \begin{pmatrix} \Phi_a & 0 \\ 0 & \Phi_s \end{pmatrix}.$$

Then, it is easy to see that

$$\delta_\Phi \Psi = \Phi_s \, \Psi - \Psi \, \Phi_a \, , \quad \delta_\Phi \widetilde{\Psi} = \Phi_a \, \widetilde{\Psi} - \widetilde{\Psi} \, \Phi_s \, . \tag{2.33}$$

It is clear from the formula that all columns of Ψ transform in the fundamental representation of $\mathfrak{su}(4)$, and all columns of $\widetilde{\Psi}$ transform in the fundamental representation of $\mathfrak{su}(2,2)$.

The transformation law (2.33) and the form of the dilatation matrix (2.27) can be used to determine that all η^i have charge $\frac{1}{2}$ under the dilatation while the charge of θ^i is $-\frac{1}{2}$. This explains the notational distinction we made for the fermions η's and θ's.

Supersymmetry transformations. For the infinitezimal supersymmetry transformations with fermionic parameter ϵ (comprising 32 supersymmetries) we find (up to the linear order in Θ)

$$\delta_\epsilon g = \tfrac{1}{2}[\epsilon, \Theta] g - g\Phi_c \, , \tag{2.34}$$

$$\delta_\epsilon \Theta = \epsilon \, . \tag{2.35}$$

Here again g is the bosonic coset element and $\Phi_c \equiv \Phi_c(\epsilon, \Omega) \in$so(4,1)$\times$so(5) should be determined from the condition (2.22). For the elements $g_a(v)$ and $g_s(u)$ formula (2.34) implies

$$2\delta_\epsilon \begin{pmatrix} g_a(v) & 0 \\ 0 & g_s(u) \end{pmatrix} = [\epsilon, \Theta] \begin{pmatrix} g_a(v) & 0 \\ 0 & g_s(u) \end{pmatrix} + \begin{pmatrix} g_a(v) & 0 \\ 0 & g_s(u) \end{pmatrix} [\epsilon, \Theta]^t \, .$$

This concludes our discussion of the global symmetry transformations.

3. The $\mathfrak{su}(1|1)$ sector of string theory

We would like to find a consistent truncation of the superstring equations to the smallest sector which should include the states dual to the $\mathfrak{su}(1|1)$ sector of the dual gauge theory. We therefore start with recalling the necessary facts about the $\mathfrak{su}(1|1)$ sector of the gauge theory.

The $\mathfrak{su}(1|1)$ sector of $\mathcal{N} = 4$ SYM comprises gauge invariant composite operators of the type

$$\mathrm{tr}\big(\Psi^M Z^{J-\frac{M}{2}}\big) + \cdots \, . \tag{3.1}$$

In the $\mathcal{N} = 1$ language Z stands for one of the three complex scalar superfields, while Ψ_α is gaugino from the vector multiplet. The field Ψ_α transforms as a spinor under one of the $\mathfrak{su}(2)$'s from the Lorentz algebra $\mathfrak{su}(2,2)$ and is neutral under the other. We use Ψ to denote the highest weight component of Ψ_α. The fields Z and Ψ carry charges 1 and 1/2 under the U(1) subgroup of SU(4) generated by Φ_3 (2.30). By dots in eq. (3.1) we mean all possible operators which can be obtained by permuting the fermions inside the trace. In the free theory the conformal dimension of the operators is $\Delta_0 = J + M$ and the $\mathfrak{su}(4)$ Dynkin labels $[0, J - \frac{M}{2}, M]$.

JHEP01(2006)078

Coming back to string theory we notice that the three complex scalars Z_i are naturally assumed to be dual to scalar superfields of the gauge theory. Thus, reduction to the $\mathfrak{su}(1|1)$ sector requires in particular to put, *e.g.*, $Z_1 = Z_2 = 0$. Also we put $v_1 = \ldots = v_4 = 0$ leaving $v_{5,6}$ corresponding to the global AdS time non-zero. The residual bosonic symmetry algebra is then

$$\mathrm{so}(4) \times \mathrm{so}(4) = \underbrace{\mathfrak{su}(2) \times \mathfrak{su}(2)}_{\text{AdS part}} \times \underbrace{\mathfrak{su}(2) \times \mathfrak{su}(2)}_{\text{sphere}} . \tag{3.2}$$

Taking into account eq. (2.32) together with eq. (2.28) it is easy to see how the original 16 complex fermions are decomposed w.r.t. the residual symmetry. Employing the notation of [25] this decomposition can be described as follows

$$(2,1;2,1) \oplus (2,1;1,2) \oplus (1,2;2,1) \oplus (1,2;1,2) . \tag{3.3}$$

For instance an explicit form of the fermionic matrix carrying irrep $(2,1;1,2)$ is

$$\Theta_{(2,1;1,2)} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \eta^7 & \eta^8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta^3 & \eta^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_8 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) . \tag{3.4}$$

One can show that the string equations of motion admit a consistent reduction to this sector which is governed by two $\mathfrak{su}(2)$-symmetries: one $\mathfrak{su}(2)$ from the Lorentz algebra and another one from the R-symmetry algebra, see also [25]. However, this sector is still not the one to put in correspondence with its counterpart from the dual gauge theory.

As we will see a consistent truncation to a smaller set of fermions exists. It amounts to putting

$$\eta^4 = \eta^7 = \eta_4 = \eta_7 = 0$$

or, alternatively,

$$\eta^3 = \eta^8 = \eta_3 = \eta_8 = 0 .$$

To understand this we notice that in the $\mathcal{N} = 1$ setting only SU(3)×U(1) subgroup of the R-symmetry group is manifest. Since the gauge theory fermion Ψ_α belongs to the vector multiplet it is neutral under $\mathfrak{su}(3)$ which rotates three complex scalars between themselves. On the string side the corresponding Lie algebra element is

$$\Phi_{\mathfrak{su}(3)\times\underline{\mathfrak{u}(1)}} = \begin{pmatrix} i\xi_1 & \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 & 0 \\ -\alpha_1 + i\beta_1 & i\xi_2 & \alpha_4 + i\beta_4 & 0 \\ -\alpha_2 + i\beta_2 & -\alpha_4 + i\beta_4 & i\xi_3 & 0 \\ 0 & 0 & 0 & -i(\xi_1 + \xi_2 + \xi_3) \end{pmatrix} , \tag{3.5}$$

where the $\mathfrak{su}(3)$ part is specified by choosing $\xi_3 = -\xi_1 - \xi_2$. Under the diagonal part of this matrix the fields transform as

$$\delta Z_1 = i(\xi_1 + \xi_3)Z_1\,, \quad \delta Z_2 = i(\xi_2 + \xi_3)Z_2\,, \quad \delta Z_3 = i(\xi_1 + \xi_2)Z_3\,.$$

and, therefore, the $\underline{\mathfrak{u}(1)}$ part under which all Z-fields carry the same charge is specified by $\xi_1 = \xi_2 = \xi_3$. Again, by using eq. (2.32) it is easy to see that the fields which do not transform under $\mathfrak{su}(3)$ are η^8 and η^4. Thus, we would like to put one of these fields in correspondence with the gauge theory fermion Ψ. However, analyzing the structure of the cubic couplings in the lagrangian one can realize that the consistent reduction which keeps non-zero only one fermion, say, η^8 *is not allowed*. An obstruction arises due to the Wess-Zumino term as it contains cubic couplings of two fermions with a single holomorphic field $Z \equiv Z_3$. Switching off η^4 breaks the Lorentz algebra $\mathfrak{su}(2)$ down to $\mathfrak{u}(1)$. Under this $\mathfrak{u}(1)$ the field Z is uncharged while η^8 and η^3 carry opposite charges, and therefore, they can form an invariant cubic coupling of the type

$$e^{it} Z \eta_3 \eta_8 \tag{3.6}$$

with possible τ and σ-derivatives acting on fermionic fields. Moreover, one can easily see that this coupling is also allowed by the $\underline{\mathfrak{u}(1)}$-symmetry as the $\underline{\mathfrak{u}(1)}$ charge of Z is precisely the opposite to the sum of the fermionic charges. With the coupling eq. (3.6) in the lagrangian it is inconsistent to put $\eta_3 = 0$ because the equation of motion for η_3 will then turn into a non-linear constraint involving Z and η_8.

Some comments are in order. In what follows we will loosely refer to the reduction which keeps only two complex non-zero fermions, η^3 and η^8, as to the "$\mathfrak{su}(1|1)$ sector" of string theory. The discussion above clearly demonstrates, however, that consistent reductions of string equations of motion are not the same as closed subsectors of the dual gauge theory as they already contain different number of degrees of freedom. We will return to the question about the relation between gauge and string theory sectors in section 7. In this section we have identified a possible truncation but we have not yet proven its consistency. This will be done in section 8.

4. Lagrangian of the reduced model

In this section we fix a so-called uniform gauge [19] and derive the corresponding hamiltonian. In the uniform gauge approach the reparametrization freedom is used to identify the world-sheet time τ with the global AdS time t and also to distribute a single component $J = J_3$ of the S^5 angular momentum homogeneously along the string. We will obtain the gauged-fixed hamiltonian as an exact function of J and further show that in the large J expansion it reproduces the plane-wave hamiltonian and higher-order corrections to it.

In our reduction we choose to keep two complex fermions, η_3 and η_8. Let us recall the definitions

$$iv_5 + v_6 = e^{it}\,, \qquad Z = iu_5 + u_6 = e^{i\phi}\,.$$

Here t is the global AdS time and ϕ is an angle variable along one of the S^1's embedded in S^5. Since we consider closed strings the embedding fields are assumed to be periodic functions of $0 \leq \sigma \leq 2\pi$. Periodicity then implies

$$\phi(2\pi) - \phi(0) = 2\pi m, \qquad m \in \mathbb{Z}\,. \tag{4.1}$$

The winding number m describes how many times the string winds around the circle parametrized by ϕ.

Substituting the reduction into the lagrangian (2.11) we find the following result

$$\begin{aligned} \mathscr{L} = {}& \tfrac{\sqrt{\lambda}}{2}\gamma^{\tau\tau}\left(\dot{t}^2 - \dot{\phi}^2 + \tfrac{i}{2}(\dot{t}+\dot{\phi})\zeta_\tau\right) + \\ & + \tfrac{\sqrt{\lambda}}{2}\gamma^{\sigma\sigma}\left(t'^2 - \phi'^2 + \tfrac{i}{2}(t'+\phi')\zeta_\sigma\right) + \\ & + \sqrt{\lambda}\gamma^{\tau\sigma}\left(\dot{t}t' - \dot{\phi}\phi' + \tfrac{i}{4}(\dot{t}+\dot{\phi})\zeta_\sigma + \tfrac{i}{4}(t'+\phi')\zeta_\tau\right) + \mathscr{L}_{\rm wz}\,. \end{aligned} \tag{4.2}$$

Here and below dot and prime denote the derivatives w.r.t. τ and σ respectively. We have also introduced the following concise notation for the fermionic contributions

$$\begin{aligned} \zeta_\tau &= \eta_i\dot{\eta}^i + \eta^i\dot{\eta}_i\,, \\ \zeta_\sigma &= \eta_i\eta'^i + \eta^i\eta'_i\,, \end{aligned} \tag{4.3}$$

where $i = 3, 8$. Inspection of the Wess-Zumino term shows that it contains exponential terms of the type $e^{i(t+\phi)}$ which are absent in the kinetic part of the lagrangian. All these exponential terms can be removed by making the following rescaling of fermions[2]

$$\eta_{3,8} = e^{-\frac{i}{2}(t+\phi)}\vartheta_{3,8}\,, \qquad \eta^{3,8} = e^{\frac{i}{2}(t+\phi)}\vartheta^{3,8}\,.$$

The original fermions η were charged under the two $\mathfrak{u}(1)$ symmetries that shift t and ϕ. The new variables ϑ do not carry these charges any more, they appear to be *neutral.* This fact will play an important role in constructing physical states dual to the gauge theory operators from $\mathfrak{su}(1|1)$ sector.

It is worth mentioning that because of the field redefinition the fermions ϑ are periodic if the winding number m is even, and anti-periodic if m is odd.[3]

After the rescaling the lagrangian becomes

$$\begin{aligned} \mathscr{L} = {}& \tfrac{\sqrt{\lambda}}{2}\gamma^{\tau\tau}\left(\dot{t}^2 - \dot{\phi}^2 + \tfrac{i}{2}(\dot{t}+\dot{\phi})\zeta_\tau - \tfrac{1}{2}(\dot{t}+\dot{\phi})^2\Lambda\right) + \\ & + \tfrac{\sqrt{\lambda}}{2}\gamma^{\sigma\sigma}\left(t'^2 - \phi'^2 + \tfrac{i}{2}(t'+\phi')\zeta_\sigma - \tfrac{1}{2}(t'+\phi')^2\Lambda\right) + \\ & + \sqrt{\lambda}\gamma^{\tau\sigma}\left(\dot{t}t' - \dot{\phi}\phi' + \tfrac{i}{4}(\dot{t}+\dot{\phi})\zeta_\sigma + \tfrac{i}{4}(t'+\phi')\zeta_\tau - \tfrac{1}{2}(\dot{t}+\dot{\phi})(t'+\phi')\Lambda\right) + \mathscr{L}_{\rm wz}\,, \end{aligned} \tag{4.4}$$

---

[2] The new rescaled fermions $\vartheta_{3,8}$ should not be mistaken with the original fermions θ_i in Θ that were set to zero to get the $\mathfrak{su}(1|1)$ sector.

[3] The same effect was also found in the analysis of the spectrum of fluctuations around a multi-spin circular string [28].

JHEP01(2006)078

where the Wess-Zumino term has a remarkably simple form[4]

$$\mathscr{L}_{\rm wz} = \tfrac{\kappa}{2}\Omega_\tau(t' + \phi') - \tfrac{\kappa}{2}\Omega_\sigma(\dot{t} + \dot{\phi})\,. \tag{4.5}$$

Here for various fermionic contributions we use the concise notations

$$\begin{aligned} &\zeta_\tau = \vartheta_i\dot{\vartheta}^i + \vartheta^i\dot{\vartheta}_i\,, \qquad \Omega_\tau = \vartheta_3\dot{\vartheta}_8 + \vartheta_8\dot{\vartheta}_3 - \vartheta^3\dot{\vartheta}^8 - \vartheta^8\dot{\vartheta}^3\,, \qquad \Lambda = \vartheta_i\vartheta^i\,, \\ &\zeta_\sigma = \vartheta_i\vartheta'^i + \vartheta^i\vartheta'_i\,, \qquad \Omega_\sigma = \vartheta_3\vartheta'_8 + \vartheta_8\vartheta'_3 - \vartheta^3\vartheta'^8 - \vartheta^8\vartheta'^3\,. \end{aligned} \tag{4.6}$$

To further understand the dynamics of our reduced model we have to identify the true (physical) degrees of freedom. The most elegant way to achieve this goal is to construct the hamiltonian formulation of the model. Let us denote by p_t and p_ϕ the canonical momenta for t and ϕ. Computing from eq. (4.4) the momenta p_t and p_ϕ we recast our lagrangian in the phase space form

$$\begin{aligned} \mathscr{L} = {} & p_t\dot{t} + p_\phi\dot{\phi} + \frac{i}{4}(p_t - p_\phi)\zeta_\tau + \tfrac{\kappa}{2}(t' + \phi')\Omega_\tau - \\ & -\frac{1}{\gamma^{\tau\tau}\sqrt{\lambda}}\Big[\frac{1}{4}(p_t - p_\phi)\Big(p_t(2+\Lambda) + p_\phi(2-\Lambda) + 2\kappa\Omega_\sigma\Big) + \\ & +\frac{\lambda}{4}(t' + \phi')\Big(t'(2-\Lambda) - \phi'(2+\Lambda) + i\zeta_\sigma\Big)\Big] + \\ & + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big[p_t t' + p_\phi\phi' + \frac{i}{4}(p_t - p_\phi)\zeta_\sigma + \frac{\kappa}{2}(t' + \phi')\Omega_\sigma\Big]\,. \end{aligned} \tag{4.7}$$

As is usual in string theory with two-dimensional reparametrization invariance the components of the world-sheet metric enter in the form of the lagrangian multipliers.

The uniform gauge amounts to imposing the following two conditions [19]

$$t = \tau\,, \qquad p_\phi = J\,. \tag{4.8}$$

The equations of motion for the phase space variables follow from eq. (4.7). Upon substitution of the gauge conditions (4.8) some of these equations turn into constraints which we have to solve in order to find the true dynamical degrees of freedom. Let us also note that we do not introduce here the canonical momenta for the fermionic variables. Fermions are not involved in our gauge choice and, therefore, can be treated at the final stage when all the bosonic type constraints have been already solved.

Let us now describe the procedure for finding the physical hamiltonian in more detail. First varying w.r.t. $\gamma^{\tau\sigma}$ we obtain an equation for ϕ':

$$\phi' = -i\frac{p_t - p_\phi}{4p_\phi + 2\kappa\Omega_\sigma}\zeta_\sigma\,. \tag{4.9}$$

Variation w.r.t. to $\gamma^{\tau\tau}$ gives an equation which we solve for p_t. We find two solutions: $p_t = p_\phi$ and

$$p_t = -\frac{p_\phi(2-\Lambda) + 2\kappa\Omega_\sigma}{2+\Lambda}\,. \tag{4.10}$$

[4]We have also omitted a unessential total derivative contribution.

The variable p_t conjugated to the global AdS time t is nothing else as the density of the space-time energy of the string: $p_t = -\mathcal{H}$. Indeed, since we fix $t = \tau$ the Noether charge corresponding to the global time translations should coincide with the hamiltonian H for physical degrees of freedom:

$$\mathrm{H} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \mathcal{H} \,. \tag{4.11}$$

We pick up the second solution (4.10) to proceed because it has the correct bosonic limit: $p_t = -p_\phi$ that is $\mathcal{H} = J$. Thus, we have determined the hamiltonian density

$$\mathcal{H} = \frac{J(2-\Lambda) + 2\kappa\Omega_\sigma}{2+\Lambda} \,. \tag{4.12}$$

Recalling the explicit expressions for Λ and Ω_σ we see that the hamiltonian density does not contain the time derivatives of the fermionic fields. We postpone further discussion of $\mathcal{H}$ till we find solution of all the constraints.

Substituting the solution for p_t into eq. (4.9) we obtain

$$\phi' = \frac{i\zeta_\sigma}{2+\Lambda} \,. \tag{4.13}$$

Integrating over σ and taking into account (4.1) we obtain a constraint

$$\mathcal{V} = i \int \frac{\mathrm{d}\sigma}{2\pi} \frac{\zeta_\sigma}{2+\Lambda} = m \,. \tag{4.14}$$

This constraint is the level-matching condition which we will impose on physical states of the theory. Actually the field ϕ is non-physical. Its evolution equation can be found from (4.4) by varying w.r.t. p_ϕ:

$$\dot{\phi} = \frac{2-\Lambda+i\zeta_\tau}{2+\Lambda} \,. \tag{4.15}$$

Equations (4.15) and (4.13) determine ϕ in terms of fermionic variables. Thus, upon imposing gauge conditions and solving the constraints we obtain that the physical degrees of freedom in the sector we consider are carried by fermionic fields only.

Finally, equations of motion for p_t and ϕ can be solved for the world-sheet metric. We find the following result[5]

$$\gamma^{\tau\tau} = \frac{i}{2\sqrt{\lambda}} \frac{\lambda\zeta_\sigma^2 + 4(2J+\kappa\Omega_\sigma)^2}{(\zeta_\tau - 4i)(2J+\kappa\Omega_\sigma) - \kappa\zeta_\sigma\Omega_\tau} \,, \tag{4.16}$$

$$\gamma^{\tau\sigma} = \frac{i}{2\sqrt{\lambda}} \frac{\lambda\zeta_\sigma(\zeta_\tau - 4i) + 4\kappa(2J+\kappa\Omega_\sigma)\Omega_\tau}{(\zeta_\tau - 4i)(2J+\kappa\Omega_\sigma) - \kappa\zeta_\sigma\Omega_\tau} \,. \tag{4.17}$$

Clearly, due to the grassmanian nature of the fermionic variables these and all the other expressions we obtain are polynomial in fermions.

[5] The metric component $\gamma^{\tau\sigma}$ is determined up to an arbitrary function of τ which we have chosen to be zero. This function plays the role of the lagrangian multiplier to the level-matching constraint, *c.f.* the corresponding discussion in [19].

JHEP01(2006)078

Now substituting solutions of the constraints into (4.4) we obtain the following gauge-fixed lagrangian

$$\mathscr{L} = -J - \frac{iJ\zeta_\tau + 2\kappa\Omega_\sigma - 2J\Lambda}{2+\Lambda} - \frac{i\kappa}{2}\frac{\zeta_\tau\Omega_\sigma - \zeta_\sigma\Omega_\tau}{2+\Lambda}\,. \tag{4.18}$$

This lagrangian exhibits very interesting features which will be discussed in the next section.

5. Hamiltonian and Poisson structure of reduced model

First we introduce a two-component complex (Dirac) spinor ψ by combining the fermions as

$$\psi = \begin{pmatrix} \vartheta_3 \\ \vartheta^8 \end{pmatrix} \tag{5.1}$$

and also define the following Dirac matrices

$$\rho^0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{5.2}$$

These matrices satisfy the Clifford algebra with the flat metric of the Minkowski signature. We also define the Dirac conjugate spinor $\bar\psi = \psi^\dagger\rho^0$. By using various fermionic identities collected in appendix C the lagrangian (4.18) can be written as

$$\begin{aligned}\mathscr{L} =& -J - \frac{J}{2}\left(i\bar\psi\rho^0\partial_0\psi - i\partial_0\bar\psi\rho^0\psi\right) + i\kappa(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi) + J\bar\psi\psi + \\ &+ \frac{J}{4}\left(i\bar\psi\rho^0\partial_0\psi - i\partial_0\bar\psi\rho^0\psi\right)\bar\psi\psi - \frac{i\kappa}{2}(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi)\bar\psi\psi - \frac{1}{2}J(\bar\psi\psi)^2 + \\ &+ \frac{\kappa}{2}\epsilon^{\alpha\beta}(\bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi) + \frac{\kappa}{8}\epsilon^{\alpha\beta}(\bar\psi\psi)^2\partial_\alpha\bar\psi\rho^5\partial_\beta\psi\,,\end{aligned} \tag{5.3}$$

where $\rho^5 = \rho^0\rho^1$. Finally, we note that the lagrangian (5.3) can be further simplified if we perform the following change of variables

$$\psi \to \psi + \frac{1}{4}\psi(\bar\psi\psi)\,, \qquad \bar\psi \to \bar\psi + \frac{1}{4}\bar\psi(\bar\psi\psi)\,. \tag{5.4}$$

Indeed, after this shift we obtain

$$\begin{aligned}\mathscr{L} =& -J - \frac{J}{2}\left(i\bar\psi\rho^0\partial_0\psi - i\partial_0\bar\psi\rho^0\psi\right) + i\kappa(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi) + J\bar\psi\psi + \\ &+ \frac{\kappa}{2}\epsilon^{\alpha\beta}(\bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi) - \frac{\kappa}{4}\epsilon^{\alpha\beta}(\bar\psi\psi)^2\partial_\alpha\bar\psi\rho^5\partial_\beta\psi\,,\end{aligned} \tag{5.5}$$

Clearly, if we now rescale the world-sheet variable σ as

$$\sigma \to -\frac{2\kappa}{J}\sigma \tag{5.6}$$

then the lagrangian density acquires the form

$$\mathscr{L} = J\Big[-1 - \frac{1}{2}\left(i\bar\psi\rho^\alpha\partial_\alpha\psi - i\partial_\alpha\bar\psi\rho^\alpha\psi\right) + \bar\psi\psi - \tag{5.7}$$

$$-\frac{1}{4}\epsilon^{\alpha\beta}(\bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi-\partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi)+\frac{1}{8}\epsilon^{\alpha\beta}(\bar\psi\psi)^2\partial_\alpha\bar\psi\rho^5\partial_\beta\psi\Big]$$

and it defines a *Lorentz-invariant theory of the Dirac fermion on the flat two-dimensional world-sheet*! Original space-time fermions of the Green-Schwarz superstring are combined into spinors of the two-dimensional world-sheet. This is very similar to the well-known relation between the light-cone formulations of the NSR and Green-Schwarz superstrings in the flat space. Our lagrangian is however non-linear and extends up to six order in fermions. If we then combine the prefactor J in eq. (5.7) with the transformation of the measure $\mathrm{d}\sigma\to-\frac{2\kappa}{J}\mathrm{d}\sigma$ under eq. (5.6) we see that rescaling (5.6) is equivalent to restoring the overall $\sqrt{\lambda}$ dependence of the lagrangian; the whole dependence on J goes to the integration bound: $0\leq-\sigma\leq\frac{\pi J}{\kappa}$. Finally, we note that it would be interesting to understand if and how to rewrite the lagrangian above as the covariant theory of the Dirac fermion but on the curved world-sheet with the metric (4.16), (4.17). From now on we fix $\kappa=\frac{\sqrt{\lambda}}{2}$.

The lagrangian (5.5) is also invariant under the global U(1) symmetry $\psi\to e^{i\epsilon}\psi$. In fact this symmetry is nothing else but the U(1) part of the Lorentz SU(2) subgroup left unbroken upon the reduction, *c.f.* the corresponding discussion in the previous section. Computing the corresponding Noether charge Q we find

$$Q=J\int\frac{\mathrm{d}\sigma}{2\pi}\Big(\bar\psi\rho^0\psi-i\frac{\sqrt{\lambda'}}{2}\bar\psi\rho^0\psi(\bar\psi\rho^1\partial_1\psi-\partial_1\bar\psi\rho^1\psi)\Big)\,. \tag{5.8}$$

This symmetry will play a crucial role in constructing the physical states dual to gauge theory operators from the $\mathfrak{su}(1|1)$ sector.

To simplify our further discussion of the hamiltonian and Poisson structure of the reduced model it is convenient to rescale the fermions as $\psi\to\frac{1}{\sqrt{J}}\psi$. The lagrangian (5.5) shows the following structure

$$\mathscr{L}=\mathscr{L}_{\rm kin}-\mathcal{H}\,, \tag{5.9}$$

where the hamiltonian density $\mathcal{H}$ is of a very simple form

$$\mathcal{H}=J-i\frac{\sqrt{\lambda'}}{2}(\bar\psi\rho^1\partial_1\psi-\partial_1\bar\psi\rho^1\psi)-\bar\psi\psi\,, \tag{5.10}$$

i.e. it is just the hamiltonian density for a massive two-dimensional Dirac fermion. The kinetic term $\mathscr{L}_{\rm kin}$ contains time derivatives and it is this term which defines the Poisson structure of the model:

$$\begin{aligned}\mathscr{L}_{\rm kin}&=-\frac{1}{2}\left(i\bar\psi\rho^0\partial_0\psi-i\partial_0\bar\psi\rho^0\psi\right)-\\&-\frac{\sqrt{\lambda'}}{2J}(\bar\psi\partial_1\psi\ \bar\psi\rho^5\partial_0\psi-\partial_1\bar\psi\psi\ \partial_0\bar\psi\rho^5\psi)-\frac{\sqrt{\lambda'}}{8J^2}\epsilon^{\alpha\beta}(\bar\psi\psi)^2\partial_\alpha\bar\psi\rho^5\partial_\beta\psi\,.\end{aligned} \tag{5.11}$$

Let us now explain how to find the corresponding Poisson bracket. Obviously, the canonical momentum conjugate to ψ does not depend on $\dot\psi$ and, therefore, implies the (second-class) constraints between the phase-space variables. The standard way to determine the Poisson structure in this case is to construct the corresponding Dirac bracket. We, however, will solve this problem in a simpler but equivalent way. Indeed, the equations of motion that follow from eq. (5.5) can be schematically represented as

$$\Omega_{ij}\dot{\chi}_j = \frac{\delta \mathrm{H}}{\delta \chi_i}\,. \tag{5.12}$$

Here the index i runs from 1 to 4 and we introduced the four-component fermion $\chi = (\psi_1, \psi_2, \psi_1^*, \psi_2^*)$. Denote by Ω^{-1} the inverse matrix. Then, eq. (5.12) can be written as

$$\dot{\chi}_i = \Omega^{-1}_{ij}\frac{\delta \mathrm{H}}{\delta \chi_j} \equiv \{\mathrm{H}, \chi_i\}\,. \tag{5.13}$$

Clearly, Ω^{-1} defines the Poisson tensor which we are interested in. Thus, all what we need to do is to compute from $\mathscr{L}_{\rm kin}$ the 4×4 matrix Ω and then to invert it. Performing the corresponding computation we find the following Poisson structure

$$\{\psi_i(\sigma), \psi_j(\sigma')\} = -i\frac{\sqrt{\lambda'}}{4J}(\psi_k\psi_l)'\delta_{ij}\epsilon_{kl}\delta(\sigma-\sigma') + \cdots \tag{5.14}$$

$$\{\psi_i(\sigma), \psi_j^*(\sigma')\} = \Big[i\delta^{ij} + i\frac{\sqrt{\lambda'}}{2J}(\epsilon_{ik}\delta_{jl}\psi_l'\psi_k^* + \epsilon_{jk}\delta_{il}\psi_k\psi_l'^*)\Big]\delta(\sigma-\sigma') + \cdots\,, \tag{5.15}$$

where $\epsilon_{12} = 1$. The Poisson bracket appears rather non-trivial, it extends up to the 8th order in fermion ψ and its derivative ψ', we refer the reader to appendix E where the complete expression for the bracket is presented.

6. Canonical Poisson structure and hamiltonian

In the previous section we formulated our dynamical system in such a way that it has a rather simple hamiltonian but a relatively complicated Poisson structure. In this section we find a further transformation of the fermionic variables which brings the Poisson structure of the model to the canonical form. Of course, the prize we pay for simplification of the Poisson brackets is that under this transformation the hamiltonian becomes rather non-trivial. The key idea is to find such a non-linear redefinition of the fermionic variables which transforms the kinetic term in eq. (5.11) to the canonical form

$$\mathscr{L}_{\rm kin} = -\frac{i}{2}\left(\bar{\psi}\rho^0\partial_0\psi - \partial_0\bar{\psi}\rho^0\psi\right)\,. \tag{6.1}$$

Indeed, the kinetic term (6.1) implies the standard symplectic structure

$$\{\psi_\alpha^*(\sigma), \psi_\beta(\sigma')\} = i\delta_{\alpha\beta}\delta(\sigma-\sigma')\,. \tag{6.2}$$

The proper redefinition can be found order by order in powers of fermions. For the sake of simplicity we omit the corresponding calculations and refer the reader to appendix D, where we give the final and explicit form of the required change of variables. Substituting the found redefinition of the fermions, eqs.(D.1), into eq. (5.10) we obtain the following hamiltonian

$$\mathcal{H} = J \,-\, i\frac{\sqrt{\lambda'}}{2}(\bar{\psi}\rho^1\partial_1\psi - \partial_1\bar{\psi}\rho^1\psi) - \bar{\psi}\psi +$$

$$
\begin{aligned}
&+\frac{1}{J}\Big[\frac{\lambda'}{2}\Big((\bar\psi\partial_1\psi)^2+(\partial_1\bar\psi\psi)^2\Big)-i\frac{\sqrt{\lambda'}}{2}\,\bar\psi\psi(\bar\psi\rho^1\partial_1\psi-\partial_1\bar\psi\rho^1\psi)\Big]+\\
&+\frac{1}{J^2}\Big[-i\frac{\lambda'^{\frac{3}{2}}}{8}(\bar\psi\psi)^2\Big(\partial_1\bar\psi\rho^1\partial_1^2\psi-\partial_1^2\bar\psi\rho^1\partial_1\psi\Big)-\frac{3\lambda'}{8}(\bar\psi\psi)^2\;\partial_1\bar\psi\partial_1\psi+\\
&\qquad\qquad+i\frac{\lambda'^{\frac{3}{2}}}{2}\bar\psi\psi(\bar\psi\partial_1\psi-\partial_1\bar\psi\psi)\partial_1\bar\psi\rho^1\partial_1\psi\Big]-\\
&-\frac{1}{J^3}\Big[\frac{\lambda'^2}{2}(\bar\psi\psi)^2(\partial_1\bar\psi\partial_1\psi)^2\Big]\,. \qquad (6.3)
\end{aligned}
$$

Thus, our dynamical system is described now by the hamiltonian (6.3) supplied with the canonical Poisson bracket (6.2). Therefore, in the following we will refer to eq. (6.3) as to the *canonical* hamiltonian.

The expression (6.3) provides the canonical hamiltonian of the consistently truncated $\mathfrak{su}(1|1)$ subsector of the classical superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. It was derived as an exact function of J. We have rearranged the final result (6.3) in the form of the large J expansion with $\lambda'=\frac{\lambda}{J^2}$ kept fixed.[6]

Thus, the first line in eq. (6.3) is the well-known plane-wave hamiltonian [29] and the second one encodes the near-plane wave correction to it. It is rather intriguing that $1/J$ expansion of $\mathcal{H}$ terminates at order $1/J^3$. This does not happen, for instance, for the bosonic $\mathfrak{su}(2)$ subsector of string theory, where the uniform-gauge hamiltonian is of the Nambu (square root) type. Apriori one could expect the appearance of higher derivative terms in eq. (6.3) that would lead to higher-order terms in $1/J$ (and also in λ') expansion. Such a property of the $1/J$ expansion might have certain implications for the dual gauge theory. We note, however, that in spite of the fact that the classical hamiltonian terminates at order $1/J^3$, the $1/J$-corrections to the classical energy obtained through the semiclassical (perturbative) quantization procedure will not terminate at a certain order.

To conclude this section we note that under redefinition (D.1) the level-matching constraint (4.14) becomes very simple

$$
\mathcal{V}=\int\frac{\mathrm{d}\sigma}{2\pi}\frac{i}{2}(\bar\psi\rho^0\partial_1\psi-\partial_1\bar\psi\rho^0\psi)=i\int\frac{\mathrm{d}\sigma}{2\pi}\psi_i^*\psi_i' \qquad (6.4)
$$

and it just generates the rigid shifts $\sigma\to\epsilon\sigma$. Also the generator Q of the U(1) charge (5.8) simplifies to

$$
Q=\int\frac{\mathrm{d}\sigma}{2\pi}\bar\psi\rho^0\psi=\int\frac{\mathrm{d}\sigma}{2\pi}\psi_i^*\psi_i\,. \qquad (6.5)
$$

This simplification of the level-matching constraint and the U(1) charge can be also considered as an independent non-trivial check of redefinitions (D.1).

7. Near-plane wave correction to the energy

The near-plane wave correction to the energy of the plane-wave states from the $\mathfrak{su}(1|1)$ sector has been already found in [25, 26]. The corresponding computation was based on

[6] The rearrangement of the $1/\sqrt{\lambda}$ expansion in the form of the large J expansion with λ' fixed is a generic fact valid also for the expansion around multi-spin string configurations [28].

finding the $1/J$ correction to the plane-wave hamiltonian in a specific light-cone type gauge. The uniform gauge we adopt in our approach is different. Due to the complicated nature of the results of [25] we were not able to compare directly their hamiltonian with the $1/J$-term in eq. (6.3). Moreover, we see that this comparison will definitely require finding a redefinition of our fermionic variables to that of [25]. Nevertheless it is possible to make a comparison in a simple way. In this section we compute the $1/J$ correction to the energy of arbitrary M-impurity plane-wave states from our hamiltonian (6.3) and find perfect agreement with the results in [25, 26]. The simplicity of the corresponding calculation is rather remarkable.

To create string states dual to the gauge theory operators from the $\mathfrak{su}(1|1)$ subsector we need to choose a proper representation of the anti-commutation relations for fermions. Writing ψ as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{7.1}$$

and expanding the fermions in Fourier modes

$$\psi_\alpha(\sigma) = \sum_{n=-\infty}^{\infty} e^{in\sigma}\psi_{\alpha,n} , \qquad \psi^\dagger_\alpha(\sigma) = \sum_{n=-\infty}^{\infty} e^{-in\sigma}\psi^\dagger_{\alpha,n} , \tag{7.2}$$

we introduce the following creation and annihilation operators

$$\begin{aligned} \psi_{1,n} &= f_n a_n^+ + g_n b_n^- , \qquad & \psi_{2,n} &= f_n b_n^- + g_n a_n^+ , \\ \psi^\dagger_{1,n} &= f_n a_n^- - g_n b_n^+ , \qquad & \psi^\dagger_{2,n} &= f_n b_n^+ - g_n a_n^- , \end{aligned} \tag{7.3}$$

where we have defined the functions

$$f_n = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{1+\lambda' n^2}}} , \qquad g_n = \frac{i\sqrt{\lambda'}\, n}{1+\sqrt{1+\lambda' n^2}}\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{1+\lambda' n^2}}} .$$

In terms of the oscillators, the free lagrangian which is the first line in eq. (5.3) takes the form

$$\mathcal{L} = -J + \sum_{n=-\infty}^{\infty} \left[-i\left(a_n^+ \dot{a}_n^- + b_n^+ \dot{b}_n^-\right) - \omega_n \left(a_n^+ a_n^- + b_n^+ b_n^-\right)\right] , \tag{7.4}$$

where $\omega_n = \sqrt{1+\lambda' n^2}$. We thus see that (a^-, a^+) and (b^-, b^+) are pairs of canonically conjugated operators. The SYM operators from the $\mathfrak{su}(1|1)$ subsector are dual to states obtained by acting by operators a_n^+ on the vacuum. In general, however, such a state with M excitations ("impurities"), $a_{n_1}^+ \cdots a_{n_M}^+$ can be also multiplied by a function of $a_k^+ b_m^+$ because the combination $a_k^+ b_m^+$ is neutral. It does not matter at the first order in the $1/J$ expansion.

The level matching condition has the usual form

$$\mathcal{V} = \frac{1}{J}\sum_{n=-\infty}^{\infty} \left(n\, a_n^+ a_n^- - n\, b_n^+ b_n^-\right) , \tag{7.5}$$

and therefore the sum of a-modes should be equal to the sum of b-modes. For the states dual to SYM operators from the $\mathfrak{su}(1|1)$ subsector the sum of modes should vanish.

Now we can compute the energy shift at order $1/J$. The relevant part of the hamiltonian (6.3) is

$$\mathcal{H} = J - i\frac{\sqrt{\lambda'}}{2}(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi) - \bar\psi\psi + \\ + \frac{1}{J}\Big[\frac{\lambda'}{2}\Big((\bar\psi\partial_1\psi)^2 + (\partial_1\bar\psi\psi)^2\Big) - i\frac{\sqrt{\lambda'}}{2}\;\bar\psi\psi(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi)\Big].$$

We need to substitute here the representation for fermions, eqs.(7.3), and switch off the b-oscillators. The normal-ordered hamiltonian is

$$\mathrm{H} = J + \sum_n \omega_n a_n^+ a_n^- + \frac{\sqrt{\lambda'}}{2J}\sum_{n_1,n_2,n_3,n_4} \delta_{n_1-n_2+n_3-n_4}(f_{n_1}f_{n_2} + g_{n_1}g_{n_2}) \times \\ \times \big[i(n_3+n_4)(f_{n_4}g_{n_3} + f_{n_3}g_{n_4}) - \sqrt{\lambda'}(n_1n_3 + n_2n_4)(f_{n_3}f_{n_4} + g_{n_3}g_{n_4})\big]\; a_{n_4}^+ a_{n_2}^+ a_{n_3}^- a_{n_1}^- \,.$$

A state carrying M units of the U(1) charge Q is

$$|M\rangle = a_{n_1}^+ \dots a_{n_M}^+ |0\rangle\,. \tag{7.6}$$

Since all fermions ψ_α are neutral under the U(1) subgroup rotating the bosonic field Z, any such a state carries the same J units of the corresponding charge for any number of excitations M. That means that an M-impurity string state should be dual to the field theory operator of the form

$$\mathrm{tr}\big(\Psi^M Z^{J-\frac{M}{2}}\big) + \cdots\,.$$

We can see from this formula that at $M = 2J$ there should exist only one string state which is dual to the operator

$$\mathrm{tr}\,\Psi^{2J}\,.$$

Such a restriction cannot be seen in the $1/J$ perturbation theory but would play an important role in the exact (finite J) quantization of the model.

It is trivial to compute the matrix element

$$\langle M|a_{n_4}^+ a_{n_2}^+ a_{n_3}^- a_{n_1}^-|M\rangle = \frac{1}{2}\sum_{i,j=1}^{M}\Big(\delta_{n_1,n_j}\delta_{n_3,n_i} - \delta_{n_1,n_i}\delta_{n_3,n_j}\Big)\Big(\delta_{n_2,n_i}\delta_{n_4,n_j} - \delta_{n_2,n_j}\delta_{n_4,n_i}\Big)\,,$$

where n_i and n_j are some indices which occur in (7.6). With this formula at hand we can easily find the energy shift ($\omega_i \equiv \omega_{n_i}$)

$$\langle M|\mathrm{H}|M\rangle = J + \sum_{i=1}^{M}\omega_i - \frac{\lambda'}{4J}\sum_{i\neq j}^{M}\frac{n_i^2 + n_j^2 + 2n_i^2n_j^2\lambda' - 2n_in_j\omega_i\omega_j}{\omega_i\omega_j}\,. \tag{7.7}$$

This *precisely* reproduces the $1/J$ correction to the M-impurity plane-wave states obtained in [26], which up to order λ'^2 agrees with the gauge theory result [16].

8. Lax representation

In this section we discuss the Lax representation of the equations of motion corresponding to the truncated lagrangian. Our starting point is the Lax pair found in [2]. It is based on the two-dimensional Lax connection $\mathscr{L}$ with components

$$\mathscr{L}_\alpha = \ell_0 A_\alpha^{(0)} + \ell_1 A_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A_\rho^{(2)} + \ell_3 Q_\alpha^+ + \ell_4 Q_\alpha^- \,, \tag{8.1}$$

where ℓ_i are constants and $Q^\pm = A^{(1)} \pm A^{(3)}$. The connection $\mathscr{L}$ is required to have zero curvature

$$\partial_\alpha \mathscr{L}_\beta - \partial_\beta \mathscr{L}_\alpha - [\mathscr{L}_\alpha, \mathscr{L}_\beta] = 0 \tag{8.2}$$

as a consequence of the dynamical equations and the flatness of A_α. This requirement of zero curvature also leads to determination of the constants ℓ_i. First we find

$$\ell_0 = 1, \qquad \ell_1 = \frac{1+x^2}{1-x^2}\,,$$

where x is a *spectral parameter*. Then for the remaining ℓ_i we obtain the following solution

$$\ell_2 = s_1 \frac{2x}{1-x^2}\,, \qquad \ell_3 = s_2 \frac{1}{\sqrt{1-x^2}}\,, \qquad \ell_4 = s_3 \frac{x}{\sqrt{1-x^2}}\,. \tag{8.3}$$

Here $s_2^2 = s_3^2 = 1$ and

$$s_1 + s_2 s_3 = 0 \qquad \text{for} \qquad \kappa = \frac{\sqrt{\lambda}}{2}\,, \tag{8.4}$$

$$s_1 - s_2 s_3 = 0 \qquad \text{for} \qquad \kappa = -\frac{\sqrt{\lambda}}{2}\,. \tag{8.5}$$

Thus, for every choice of κ we have four different solutions for ℓ_i specified by the choice of $s_2 = \pm 1$ and $s_3 = \pm 1$, *c.f.* the corresponding discussion in [17]. As explained in [17], the Lax connection (8.1) can be explicitly realized in terms of 8×8 supermatrices from the Lie algebra $\mathfrak{su}(2,2|4)$. In the algebra $\mathfrak{su}(2,2|4)$ the curvature (8.2) of $\mathscr{L}_\alpha$ is not exactly zero, rather it is proportional to the identity matrix (anomaly) with a coefficient depending on fermionic variables. However, in $\mathfrak{psu}(2,2|4)$ the curvature is regarded to be zero since $\mathfrak{psu}(2,2|4)$ is the factor-algebra of $\mathfrak{su}(2,2|4)$ over its central element proportional to the identity matrix [17, 30]. In the following we consider the Lax connection which corresponds to the choice $\kappa = \frac{\sqrt{\lambda}}{2}$.

Now we are ready to show that the Lax connection (8.1) for the general $\mathfrak{psu}(2,2|4)$ model can be consistently reduced to a Lax connection encoding the equations of motion of physical fields from the $\mathfrak{su}(1|1)$ sector. The fact that the reduction holds at the level of the matrix equations formulated in terms of 8×8 matrices is rather non-trivial and should be regarded as a proof of consistency of the reduction procedure.

We start with the projection $A_\alpha^{(0)}$. As was already discussed, in our reduction we keep non-zero only the Dirac fermion ψ and solve for the world-sheet metric $\gamma^{\alpha\beta}$ and unphysical fields t, and ϕ in terms of ψ by using our uniform gauge conditions and the constraints. Let us now compute the components $A_\alpha^{(0)}$ on our reduction and further perform the shift (5.4).

We find

$$
\begin{aligned}
A_\sigma^{(0)} &= \tfrac{1}{4}(1+\bar\psi\psi)(\bar\psi\psi' - \bar\psi'\psi)\ \mathrm{diag}\Big(1,-1,0,0;0,0,-1,1\Big)\,,\\
A_\tau^{(0)} &= \Big[\tfrac{1}{4}(1+\bar\psi\psi)(\bar\psi\dot\psi - \dot{\bar\psi}\psi) + \tfrac{i}{2}\bar\psi\rho^0\psi\Big]\ \mathrm{diag}\Big(1,-1,0,0;0,0,-1,1\Big)\,.
\end{aligned}
$$

Thus, the component $A^{(0)}$ appears to be a diagonal matrix, the first (last) four eigenvalues correspond to the AdS (sphere) part of the model. These matrices have four zero's in the middle and this suggests that the whole Lax connection for the reduced sector can be formulated in terms of 4×4 matrices rather than 8×8. Computation of the other components of the Lax connection shows that this is indeed the case. Therefore, in what follows we compute the components of the reduced Lax connection as traceless 8×8 matrices and then throw away from all the matrices the 4×4 block sitting in the middle (i.e. the corresponding rows and columns). This block appears to be non-trivial only for $A_\alpha^{(2)}$, however, one can show that it leads to redundant equations which are satisfied due to the equations of motions for fermions followed from other matrix elements. To simplify our treatment in what follows we present the reduced Lax connection in terms of the 4×4 matrices whose dynamical variables are those from the lagrangian (5.5). It is convenient to introduce two diagonal matrices

$$
\mathbf{I} = \mathrm{diag}\big(1,-1,-1,1\big)\,, \qquad \mathbf{J} = \mathrm{diag}\big(1,1,-1,-1\big)\,. \tag{8.6}
$$

Then we find the following bosonic currents for the reduced model

$$
\begin{aligned}
A_\sigma^{(0)} &= \tfrac{1}{4}(1+\bar\psi\psi)(\bar\psi\psi' - \bar\psi'\psi)\ \mathbf{I}\,,\\
A_\tau^{(0)} &= \Big[\tfrac{1}{4}(1+\bar\psi\psi)(\bar\psi\dot\psi - \dot{\bar\psi}\psi) + \tfrac{i}{2}\bar\psi\rho^0\psi\Big]\ \mathbf{I}\,,\\
A_\sigma^{(2)} &= \tfrac{1}{8}\zeta_\sigma\ \mathbf{J}\,,\\
A_\tau^{(2)} &= \tfrac{1}{8}(\zeta_\tau + 2i\bar\psi\psi - 4i)\ \mathbf{J}\,.
\end{aligned} \tag{8.7}
$$

Here for reader's convenience we recall that

$$
\zeta_\tau = \bar\psi\rho^0\dot\psi - \dot{\bar\psi}\rho^0\psi\,, \qquad \zeta_\sigma = \bar\psi\rho^0\psi' - \bar\psi'\rho^0\psi\,. \tag{8.8}
$$

Notice also that the coefficient of $A_\sigma^{(2)}$ is proportional to the density of the level-matching condition. The odd matrices $Q_\alpha^\pm$ appears on our reduction are precisely skew-diagonal. Introducing the matrices

$$
\Theta = (1+\tfrac{1}{4}\bar\psi\psi)\begin{pmatrix} & & & \psi_2 \\ & & \psi_1^* & \\ & \psi_1 & & \\ \psi_2^* & & & \end{pmatrix}, \qquad \hat\Theta = i(1+\tfrac{1}{4}\bar\psi\psi)\begin{pmatrix} & & & -\psi_1 \\ & & -\psi_2^* & \\ & \psi_2 & & \\ \psi_1^* & & & \end{pmatrix} \tag{8.9}
$$

the components $Q_\alpha^\pm$ can be written as

$$
Q_\alpha^+ = [A_\alpha^{(2)}, \Theta] - \partial_\alpha\Theta\,, \qquad Q_\alpha^- = [A_\alpha^{(2)}, \hat\Theta] + \partial_\alpha\hat\Theta\,.
$$

The original Lax connection (8.1) also involves the following terms

$$\gamma_{\tau\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = -\gamma^{\sigma\sigma}A^{(2)}_\sigma - \gamma^{\sigma\tau}A^{(2)}_\tau\,,$$
$$\gamma_{\sigma\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = \gamma^{\tau\tau}A^{(2)}_\tau + \gamma^{\tau\sigma}A^{(2)}_\sigma\,. \tag{8.10}$$

Substituting here the solution for the metric, eqs.(4.16), (4.17), we obtain remarkably simple formulae

$$\gamma_{\tau\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = \frac{i}{8}\Omega_\tau\,\mathbf{J}\,, \tag{8.11}$$
$$\gamma_{\sigma\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = \frac{i}{4\sqrt{\lambda}}(J+\mathcal{H})\,\mathbf{J}\,, \tag{8.12}$$

where

$$\mathcal{H} = J - i\frac{\sqrt{\lambda}}{2}(\bar\psi\rho^1\psi' - \bar\psi'\rho^1\psi) - J\bar\psi\psi$$

is the hamiltonian obtained from the lagrangian (5.5).

By using the equations of motion following from (5.5) one can prove the following on-shell relation

$$\Omega_\tau = -i\frac{\sqrt{\lambda}}{J}\zeta_\sigma\,. \tag{8.13}$$

Thus, we finally get

$$\gamma_{\tau\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = \frac{\sqrt{\lambda}}{8J}\zeta_\sigma\,\mathbf{J}\,, \tag{8.14}$$
$$\gamma_{\sigma\beta}\epsilon^{\beta\rho}A^{(2)}_\rho = \frac{i}{4\sqrt{\lambda}}(J+\mathcal{H})\,\mathbf{J}\,. \tag{8.15}$$

In this way we completely excluded the metric in favor of dynamical variables from the Lax representation.

Now putting all the pieces of the Lax connection together we check that the zero-curvature condition (8.2) is indeed satisfied as the consequence of the dynamical equations for fermions derived from the lagrangian (5.5). This proves that the model of two-dimensional Dirac fermions defined by the lagrangian (5.5) is integrable. Eigenvalues of the monodromy matrix

$$\mathrm{T}(x) = \mathscr{P}\exp\int_0^{2\pi} \mathrm{d}\sigma\mathscr{L}_\sigma \tag{8.16}$$

are the integrals of motion. Finally we note that to get a connection with the lagrangian (5.9) one has to rescale the fermion as $\psi\to\frac{1}{\sqrt{J}}\psi$.

Acknowledgments

We are grateful to V. Bazhanov, N. Beisert, M. Staudacher, A. Tseytlin, M. Zamaklar and K. Zarembo for interesting discussions. We would like to thank V. Kazakov and all the organizers of the *Ecole Normale Supérior Summer Institute* in August 2005 where this work was completed for an inspiring conference, and their warm hospitality. The work of G.A. was supported in part by the European Commission RTN programme HPRN-CT-2000-00131, by RFBI grant N05-01-00758 and by the INTAS contract 03-51-6346. The work of S.F. was supported in part by the EU-RTN network *Constituents, Fundamental Forces and Symmetries of the Universe* (MRTN-CT-2004-005104).

A. Gamma-matrices

Introduce the following five 4×4 matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} = -i\gamma_1\gamma_2\gamma_3\gamma_4 \,.$$

These matrices satisfy the SO(4,1) Clifford algebra

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \,, \qquad a = 1, \ldots, 5.$$

where $\eta = \mathrm{diag}(1, 1, 1, 1, -1)$. Further, the matrices γ_a belong to the Lie algebra $\mathfrak{su}(2, 2)$ as they satisfy the relation

$$\Sigma\gamma_a + \gamma_a^\dagger\Sigma = 0 \,, \qquad \Sigma = \mathrm{diag}(1, 1, -1, -1). \tag{A.1}$$

Analogously, the SO(5) Dirac matrices are

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

They satisfy the SO(5) Clifford algebra

$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\delta_{ab} \,.$$

Moreover, all of them are hermitian, so that $i\Gamma_a$ belongs to $\mathfrak{su}(4)$.

JHEP01(2006)078

We represent the generators of the superconformal group by the $\mathfrak{su}(2,2)$ matrices. In particular, the generator of scaling transformations is chosen to be

$$\mathrm{D} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\tfrac{i}{2}\gamma_5 = \tfrac{1}{2}\Gamma_5\,. \tag{A.2}$$

The generators of translations are given by

$$\mathrm{P}^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{P}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The conformal boosts are defined as

$$\mathrm{K}^i = (\mathrm{P}^i)^t, \quad \text{for} \;\; i = 0,3; \qquad \mathrm{K}^i = -(\mathrm{P}^i)^t, \quad \text{for} \;\; i = 1,2\,. \tag{A.3}$$

We also have

$$\mathrm{P}^0 + \mathrm{K}^0 = -\gamma_3\,, \qquad \mathrm{P}^3 + \mathrm{K}^3 = -\gamma_1\,, \tag{A.4}$$

$$\mathrm{P}^1 + \mathrm{K}^1 = -\gamma_2\,, \qquad \mathrm{P}^2 + \mathrm{K}^2 = -\gamma_4\,. \tag{A.5}$$

B. Global symmetry transformations

Conformal transformations. If we parametrize the $\mathfrak{su}(2,2)$ matrix Φ parametrizing infinitezimal conformal transformation as

$$\Phi = \begin{pmatrix} i\xi_1 & \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 & \alpha_3 + i\beta_3 \\ -\alpha_1 + i\beta_1 & i\xi_2 & \alpha_4 + i\beta_4 & \alpha_5 + i\beta_5 \\ \alpha_2 - i\beta_2 & \alpha_4 - i\beta_4 & i\xi_3 & \alpha_6 + i\beta_6 \\ \alpha_3 - i\beta_3 & \alpha_5 - i\beta_5 & -\alpha_6 + i\beta_6 & -i(\xi_1 + \xi_2 + \xi_3) \end{pmatrix}. \tag{B.1}$$

then eq. (2.26) implies the following transformation rules for the coordinates v:

$$\begin{aligned}
\delta v_1 &= (\beta_1 + \beta_6)v_2 + (\alpha_1 - \alpha_6)v_3 + (\xi_1 + \xi_3)v_4 + (\beta_3 - \beta_4)v_5 + (\alpha_3 - \alpha_4)v_6 \\
\delta v_2 &= -(\beta_1 + \beta_6)v_1 - (\xi_2 + \xi_3)v_3 - (\alpha_1 + \alpha_6)v_4 + (-\alpha_2 + \alpha_5)v_5 + (\beta_2 - \beta_5)v_6 \\
\delta v_3 &= (-\alpha_1 + \alpha_6)v_1 + (\xi_2 + \xi_3)v_2 + (\beta_1 - \beta_6)v_4 + (\beta_2 + \beta_5)v_5 + (\alpha_2 + \alpha_5)v_6 \\
\delta v_4 &= -(\xi_1 + \xi_3)v_1 + (\alpha_1 + \alpha_6)v_2 + (-\beta_1 + \beta_6)v_3 + (\alpha_3 + \alpha_4)v_5 - (\beta_3 + \beta_4)v_6 \\
\delta v_5 &= (\beta_3 - \beta_4)v_1 + (-\alpha_2 + \alpha_5)v_2 + (\beta_2 + \beta_5)v_3 + (\alpha_3 + \alpha_4)v_4 + (\xi_1 + \xi_2)v_6 \\
\delta v_6 &= (\alpha_3 - \alpha_4)v_1 + (\beta_2 - \beta_5)v_2 + (\alpha_2 + \alpha_5)v_3 - (\beta_3 + \beta_4)v_4 - (\xi_1 + \xi_2)v_5\,.
\end{aligned}$$

R-symmetry transformations. If we parametrize the $\mathfrak{su}(4)$ matrix Φ parametrizing infinitezimal R-symmetry transformation as

$$\Phi_{\mathfrak{su}(4)} = \begin{pmatrix} i\xi_1 & \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 & \alpha_3 + i\beta_3 \\ -\alpha_1 + i\beta_1 & i\xi_2 & \alpha_4 + i\beta_4 & \alpha_5 + i\beta_5 \\ -\alpha_2 + i\beta_2 & -\alpha_4 + i\beta_4 & i\xi_3 & \alpha_6 + i\beta_6 \\ -\alpha_3 + i\beta_3 & -\alpha_5 + i\beta_5 & -\alpha_6 + i\beta_6 & -i(\xi_1 + \xi_2 + \xi_3) \end{pmatrix}. \tag{B.2}$$

then we find the following transformation rules for the coordinates u_i:

$$\begin{aligned} \delta u_1 &= (\beta_1 + \beta_6)u_2 + (\alpha_1 - \alpha_6)u_3 + (\xi_1 + \xi_3)u_4 + (\alpha_3 - \alpha_4)u_5 + (\beta_4 - \beta_3)u_6 \\ \delta u_2 &= -(\beta_1 + \beta_6)u_1 - (\xi_2 + \xi_3)u_3 - (\alpha_1 + \alpha_6)u_4 + (\beta_2 - \beta_5)u_5 + (\alpha_2 - \alpha_5)u_6 \\ \delta u_3 &= (-\alpha_1 + \alpha_6)u_1 + (\xi_2 + \xi_3)u_2 + (\beta_1 - \beta_6)u_4 + (\alpha_2 + \alpha_5)u_5 - (\beta_2 + \beta_5)u_6 \\ \delta u_4 &= -(\xi_1 + \xi_3)u_1 + (\alpha_1 + \alpha_6)u_2 + (-\beta_1 + \beta_6)u_3 - (\beta_3 + \beta_4)u_5 - (\alpha_3 + \alpha_4)u_6 \\ \delta u_5 &= (-\alpha_3 + \alpha_4)u_1 + (-\beta_2 + \beta_5)u_2 - (\alpha_2 + \alpha_5)u_3 + (\beta_3 + \beta_4)u_4 + (\xi_1 + \xi_2)u_6 \\ \delta u_6 &= (\beta_3 - \beta_4)u_1 + (-\alpha_2 + \alpha_5)u_2 + (\beta_2 + \beta_5)u_3 + (\alpha_3 + \alpha_4)u_4 - (\xi_1 + \xi_2)u_5 . \end{aligned}$$

Clearly, these transformations obey the constraint $u_i \delta u_i = 0$.

C. Fermionic identities and conjugation rules

Here we collect some formulas involving fermionic expressions. Introducing the Dirac fermion ψ, see eq. (5.1), we find that

$$\zeta_\tau = \bar\psi\rho^0\partial_0\psi - \partial_0\bar\psi\rho^0\psi\,, \tag{C.1}$$

$$\Omega_\sigma = -i\left(\bar\psi\rho^1\partial_1\psi - \partial_1\bar\psi\rho^1\psi\right)\,, \tag{C.2}$$

$$\Lambda = \bar\psi\psi\,. \tag{C.3}$$

We have a few important identities which allow us to simplify the structure of the lagrangian. They include

$$\begin{aligned} &\epsilon^{\alpha\beta}\Big[-\bar\psi\psi\ \partial_\alpha\bar\psi\rho^5\partial_\beta\psi + \partial_\alpha\bar\psi\partial_\beta\psi\ \bar\psi\rho^5\psi + \bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi\Big] = \\ &= \epsilon^{\alpha\beta}\Big[-\partial_\alpha\left(\bar\psi\psi\ \bar\psi\rho^5\partial_\beta\psi + \partial_\beta\bar\psi\psi\ \bar\psi\rho^5\psi\right) + 2\left(\bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi\right)\Big] \end{aligned}$$

and

$$\begin{aligned} &\epsilon^{\alpha\beta}\Big[\bar\psi\psi\left(-\bar\psi\psi\ \partial_\alpha\bar\psi\rho^5\partial_\beta\psi + \partial_\alpha\bar\psi\partial_\beta\psi\ \bar\psi\rho^5\psi + \bar\psi\partial_\alpha\psi\ \bar\psi\rho^5\partial_\beta\psi - \partial_\alpha\bar\psi\psi\ \partial_\beta\bar\psi\rho^5\psi\right)\Big] \\ &= -(\bar\psi\psi)^2\epsilon^{\alpha\beta}\partial_\alpha\bar\psi\rho^5\partial_\beta\psi\,. \end{aligned} \tag{C.4}$$

The following identity is valid

$$\bar\psi\rho^0\partial_0\psi\ \bar\psi\rho^1\partial_1\psi = -\bar\psi\partial_0\psi\ \bar\psi\rho^5\partial_1\psi \tag{C.5}$$

In addition the properties of the ρ-matrices imply the following complex conjugation rules:

$$(\bar\psi\partial_\alpha\psi)^* = \partial_\alpha\bar\psi\psi, \qquad (\bar\psi\rho^5\partial_\alpha\psi)^* = -\partial_\alpha\bar\psi\rho^5\psi \tag{C.6}$$

D. Change of variables

We have found that in order to bring the kinetic term (5.11) to the canonical form (6.1) the following non-linear shift of the fermions in eq. (5.11) should be performed (here $\kappa = \frac{\sqrt{\lambda'}}{2}$)

$$\begin{aligned}
\psi \to \psi &+ \frac{i\kappa}{J^2}\rho^1\psi\ (\partial_1\bar\psi\psi) - \frac{i\kappa}{8J^3}\rho^1\partial_1\psi\ (\bar\psi\psi)^2 - \frac{\kappa^2}{4J^4}\partial_1^2\psi\ (\bar\psi\psi)^2 - \\
&- \frac{\kappa^2}{2J^4}\partial_1\psi\ (\partial_1\bar\psi\psi - \bar\psi\partial_1\psi)\bar\psi\psi + \frac{\kappa^2}{2J^4}\rho^0\psi\ (\partial_1\bar\psi\rho^0\partial_1\psi)\bar\psi\psi - \\
&- \frac{5}{4}\frac{i\kappa^3}{J^6}\rho^1\partial_1\psi(\partial_1\bar\psi\partial_1\psi)\ (\bar\psi\psi)^2\,, \\
\bar\psi \to \bar\psi &- \frac{i\kappa}{J^2}\bar\psi\rho^1\ (\bar\psi\partial_1\psi) + \frac{i\kappa}{8J^3}\partial_1\bar\psi\rho^1\ (\bar\psi\psi)^2 - \frac{\kappa^2}{4J^4}\partial_1^2\bar\psi\ (\bar\psi\psi)^2 + \\
&+ \frac{\kappa^2}{2J^4}\partial_1\bar\psi\ (\partial_1\bar\psi\psi - \bar\psi\partial_1\psi)\bar\psi\psi + \frac{\kappa^2}{2J^4}\bar\psi\rho^0\ (\partial_1\bar\psi\rho^0\partial_1\psi)\bar\psi\psi + \\
&+ \frac{5}{4}\frac{i\kappa^3}{J^6}\partial_1\bar\psi\rho^1(\partial_1\bar\psi\partial_1\psi)\ (\bar\psi\psi)^2\,.
\end{aligned} \tag{D.1}$$

Under this shift the hamiltonian (5.10) transforms into expression (6.3). Below we also give the formulae for the transformation inverse to (D.1):

$$\begin{aligned}
\psi \to \psi &- \frac{i\kappa}{J^2}\rho^1\psi\ (\partial_1\bar\psi\psi) + \frac{i\kappa}{8J^3}\rho^1\partial_1\psi\ (\bar\psi\psi)^2 - \frac{\kappa^2}{4J^4}\partial_1^2\psi\ (\bar\psi\psi)^2 - \\
&- \frac{\kappa^2}{2J^4}\partial_1\psi\ (\partial_1\bar\psi\psi + \bar\psi\partial_1\psi)\bar\psi\psi - \frac{\kappa^2}{2J^4}\rho^0\psi\ (\partial_1\bar\psi\rho^0\partial_1\psi)\bar\psi\psi - \\
&- \frac{i\kappa^3}{4J^6}\rho^1\partial_1\psi(\partial_1\bar\psi\partial_1\psi)\ (\bar\psi\psi)^2\,, \\
\bar\psi \to \bar\psi &+ \frac{i\kappa}{J^2}\bar\psi\rho^1\ (\bar\psi\partial_1\psi) - \frac{i\kappa}{8J^3}\partial_1\bar\psi\rho^1\ (\bar\psi\psi)^2 - \frac{\kappa^2}{4J^4}\partial_1^2\bar\psi\ (\bar\psi\psi)^2 - \\
&- \frac{\kappa^2}{2J^4}\partial_1\bar\psi\ (\partial_1\bar\psi\psi + \bar\psi\partial_1\psi)\bar\psi\psi - \frac{\kappa^2}{2J^4}\bar\psi\rho^0\ (\partial_1\bar\psi\rho^0\partial_1\psi)\bar\psi\psi + \\
&+ \frac{i\kappa^3}{4J^6}\partial_1\bar\psi\rho^1(\partial_1\bar\psi\partial_1\psi)\ (\bar\psi\psi)^2\,.
\end{aligned} \tag{D.2}$$

E. Poisson structure

As was discussed in section 5, the Poisson structure of the model which corresponds to the simple hamiltonian (5.10) is rather involved and extends up to the 8th order in the variables ψ, ψ'. It is unclear for the moment if and how this Poisson bracket could be related to the known algebraic structures appearing in the inverse scattering method or in the theory of Kac-Moody algebras. In view of future understanding and further applications we give below the complete list of the Poisson relations (5.14),(5.15) for ψ, ψ^* variables.

$$\{\psi_1, \psi_1\} = -\ \frac{i\sqrt{\lambda'}}{2J}(\psi_1\psi_2)' +$$

$$
\begin{aligned}
&+\ \frac{i\sqrt{\lambda'}}{4J^2}(\psi_1\psi_2)'(\psi_2\psi_2^* - 2\sqrt{\lambda'}\psi_2\psi_1'^* + \sqrt{\lambda'}\psi_2^*\psi_1') + \\
&+\ \frac{i\lambda'^{\frac{3}{2}}}{8J^3}(\psi_1\psi_2)'^2(\psi_1^*\psi_2'^* + 3\psi_2^*\psi_1'^*)\,, \\
\{\psi_2,\psi_2\} = &-\ \frac{i\sqrt{\lambda'}}{2J}(\psi_1\psi_2)' - \\
&-\ \frac{i\sqrt{\lambda'}}{4J^2}(\psi_1\psi_2)'(\psi_1\psi_1^* - 2\sqrt{\lambda'}\psi_1\psi_2'^* + \sqrt{\lambda'}\psi_1^*\psi_2') - \\
&-\ \frac{i\lambda'^{\frac{3}{2}}}{8J^3}(\psi_1\psi_2)'^2(3\psi_1^*\psi_2'^* + \psi_2^*\psi_1'^*)\,, \\
\{\psi_1,\psi_2\} = &\quad \frac{i\sqrt{\lambda'}}{8J^2}(\psi_1\psi_2)'(\psi_1\psi_2^* + \psi_1^*\psi_2 + 2\sqrt{\lambda'}(\psi_1\psi_1^* - \psi_2\psi_2^*)') - \\
&-\ \frac{i\lambda'}{4J^3}(\psi_1\psi_2)'^2(-\psi_1^*\psi_2^* + \sqrt{\lambda'}(\psi_1^*\psi_1'^* - \psi_2^*\psi_2'^*))\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\psi_1,\psi_2^*\} = &-\ \frac{i\sqrt{\lambda'}}{2J}(\psi_1\psi_1'^* + \psi_2^*\psi_2') + \\
&+\ \frac{i\sqrt{\lambda'}}{8J^2}\big[(-\psi_1\psi_2\psi_1^*\psi_2'^* + \psi_2\psi_1^*\psi_2^*\psi_1' + 2\psi_1\psi_2\psi_2^*\psi_1'^* - 2\psi_1\psi_1^*\psi_2^*\psi_2') + \\
&\qquad + 2\sqrt{\lambda'}(-\psi_1\psi_1^*\psi_2'\psi_1'^* + \psi_1\psi_2^*\psi_1'\psi_1'^* + \psi_1\psi_2^*\psi_2'\psi_2'^* - \psi_2\psi_2^*\psi_2'\psi_1'^*)\big] - \\
&-\ \frac{i\lambda'^{\frac{3}{2}}}{4J^3}\big[\psi_1\psi_2\psi_1^*\psi_1'\psi_1'^*\psi_2'^* + \psi_2\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_2'^*\big]\,, \\
\{\psi_2,\psi_1^*\} = &\quad \frac{i\sqrt{\lambda'}}{2J}(\psi_1^*\psi_1' + \psi_2\psi_2'^*) + \\
&+\ \frac{i\sqrt{\lambda'}}{8J^2}\big[(\psi_1\psi_2\psi_2^*\psi_1'^* - \psi_1\psi_1^*\psi_2^*\psi_2' - 2\psi_1\psi_2\psi_1^*\psi_2'^* + 2\psi_2\psi_1^*\psi_2^*\psi_1') + \\
&\qquad + 2\sqrt{\lambda'}(-\psi_1\psi_1^*\psi_1'\psi_2'^* + \psi_2\psi_1^*\psi_1'\psi_1'^* + \psi_2\psi_1^*\psi_2'\psi_2'^* - \psi_2\psi_2^*\psi_1'\psi_2'^*)\big] + \\
&+\ \frac{i\lambda'^{\frac{3}{2}}}{4J^3}\big[\psi_1\psi_2\psi_2^*\psi_2'\psi_1'^*\psi_2'^* + \psi_1\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_1'^*\big]\,. \\
\{\psi_1,\psi_1^*\} = i &+\ \frac{i\sqrt{\lambda'}}{2J}(\psi_2\psi_1'^* - \psi_2^*\psi_1') + \\
&+\ \frac{i\sqrt{\lambda'}}{8J^2}\big[\psi_2\psi_2^*(\psi_1\psi_2'^* - \psi_1^*\psi_2') + \\
&\qquad + 2\sqrt{\lambda'}(\psi_1\psi_2\psi_1'^*\psi_2'^* - \psi_1\psi_1^*\psi_1'\psi_1'^* + \psi_1\psi_1^*\psi_2'\psi_2'^* - \psi_1\psi_2^*\psi_2'\psi_1'^* - \\
&\qquad\qquad - \psi_2\psi_1^*\psi_1'\psi_2' - \psi_2\psi_2^*\psi_1'\psi_1'^* - \psi_2\psi_2^*\psi_2'\psi_2'^* + \psi_1^*\psi_2^*\psi_1'\psi_2')\big] + \\
&+\ \frac{i\lambda'}{4J^3}\big[\psi_1\psi_2\psi_1^*\psi_2^*(-\psi_1'\psi_1'^* + 2\psi_2'\psi_2'^*) + \\
&\qquad + \sqrt{\lambda'}(-\psi_1\psi_2\psi_2^*\psi_1'\psi_1'^*\psi_2'^* + \psi_2\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_1'^* + \\
&\qquad\qquad + 2\psi_1\psi_2\psi_1^*\psi_2'\psi_1'^*\psi_2'^* - 2\psi_1\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_2'^*)\big] - \\
&-\ \frac{3i\lambda'^2}{4J^4}\psi_1\psi_2\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_1'^*\psi_2'^*\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\psi_2,\psi_2^*\} = i &- \frac{i\sqrt{\lambda'}}{2J}(\psi_1\psi_2'^* - \psi_1^*\psi_2') + \\
&+ \frac{i\sqrt{\lambda'}}{8J^2}\big[(\psi_1\psi_1^*(\psi_2\psi_1'^* - \psi_2^*\psi_1') + \\
&\qquad + 2\sqrt{\lambda'}(\psi_1\psi_2\psi_1'^*\psi_2'^* - \psi_1\psi_1^*\psi_1'\psi_1'^* - \psi_1\psi_1^*\psi_2'\psi_2'^* - \psi_1\psi_2^*\psi_2'\psi_1'^* - \\
&\qquad\qquad - \psi_2\psi_1^*\psi_1'\psi_2' + \psi_2\psi_2^*\psi_1'\psi_1'^* - \psi_2\psi_2^*\psi_2'\psi_2'^* + \psi_1^*\psi_2^*\psi_1'\psi_2')\big] + \\
&+ \frac{i\lambda'}{4J^3}\big[\psi_1\psi_2\psi_1^*\psi_2^*(-2\psi_1'\psi_1'^* + \psi_2'\psi_2'^*) + \\
&\qquad + \sqrt{\lambda'}(\psi_1\psi_2\psi_1^*\psi_2'\psi_1'^*\psi_2'^* - \psi_1\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_2'^* - \\
&\qquad\qquad - 2\psi_1\psi_2\psi_2^*\psi_1'\psi_1'^*\psi_2'^* + 2\psi_2\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_1'^*)\big] - \\
&- \frac{3i\lambda'^2}{4J^4}\psi_1\psi_2\psi_1^*\psi_2^*\psi_1'\psi_2'\psi_1'^*\psi_2'^* \,.
\end{aligned}
$$

The Poisson bracket is ultra-local. Here for the sake of simplicity we have omitted on the r.h.s. the overall delta function $\delta(\sigma - \sigma')$.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[3] L.D. Faddeev and L.A. Takhtajan, *hamiltonian methods in the theory of solitons*, Springer, 1987;
L.D. Faddeev, *Integrable models in* $(1+1)$*-dimensional quantum field theory* in *Recent advances in field theory and statistical mechanics*, Proceedings of the Les Houches Summer School 1982, North-Holland, Amsterdam 1984, pp. 561–608.

[4] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[5] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208];
N. Beisert and M. Staudacher, *The $N = 4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[6] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060];
N. Beisert, *The su(2—3) dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252];
D. Serban and M. Staudacher, *Planar $N = 4$ gauge theory and the Inozemtsev long range spin chain*, *JHEP* **06** (2004) 001 [hep-th/0401057].

[7] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[8] N. Beisert and M. Staudacher, *Long-range psu(2,2—4) Bethe ansatze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

JHEP01(2006)078

[9] B.C. Vallilo, *Flat currents in the classical $AdS_5 \times S^5$ pure spinor superstring*, *JHEP* **03** (2004) 037 [hep-th/0307018].

[10] L. Dolan, C.R. Nappi and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, *JHEP* **10** (2003) 017 [hep-th/0308089].

[11] L.F. Alday, *Non-local charges on $AdS_5 \times S^5$ and pp-waves*, *JHEP* **12** (2003) 033 [hep-th/0310146].

[12] A. Das, J. Maharana, A. Melikyan and M. Sato, *The algebra of transition matrices for the $AdS_5 \times S^5$ superstring*, *JHEP* **12** (2004) 055 [hep-th/0411200].

[13] A. Agarwal, *Comments on higher loop integrability in the su(1—1) sector of $N = 4$ SYM: lessons from the* SU(2) *sector*, hep-th/0506095.

[14] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[15] N. Beisert, *The dilatation operator of $N = 4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [hep-th/0407277].

[16] M. Staudacher, *The factorized S-matrix of CFT/ads*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[17] L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in $AdS_5 \times S^5$*, *JHEP* **07** (2005) 002 [hep-th/0502240].

[18] C.G. Callan Jr., J. Heckman, T. McLoughlin and I.J. Swanson, *Lattice super Yang-Mills: a virial approach to operator dimensions*, *Nucl. Phys.* **B 701** (2004) 180 [hep-th/0407096].

[19] G. Arutyunov and S. Frolov, *Integrable hamiltonian for classical strings on $AdS_5 \times S^5$*, *JHEP* **02** (2005) 059 [hep-th/0411089].

[20] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191];
G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[21] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255]; *Rotating string solutions: adS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [hep-th/0306143].

[22] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 533** (1998) 109 [hep-th/9805028].

[23] R. Roiban and W. Siegel, *Superstrings on $AdS_5 \times S^5$ supertwistor space*, *JHEP* **11** (2000) 024 [hep-th/0010104].

[24] M. Kruczenski and A.A. Tseytlin, *Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory*, *JHEP* **09** (2004) 038 [hep-th/0406189].

[25] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [hep-th/0307032];
C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Holography beyond the Penrose limit*, *Nucl. Phys.* **B 694** (2004) 115 [hep-th/0404007]; *Higher impurity AdS/CFT correspondence in the near-BMN limit*, *Nucl. Phys.* **B 700** (2004) 271 [hep-th/0405153].

[26] T. McLoughlin and I.J. Swanson, *N-impurity superstring spectra near the pp-wave limit*, *Nucl. Phys.* **B 702** (2004) 86 [hep-th/0407240].

[27] R. Hernandez and E. Lopez, *Spin chain sigma models with fermions*, *JHEP* **11** (2004) 079 [hep-th/0410022];
J. Stefanski, B. and A.A. Tseytlin, *Super spin chain coherent state actions and $AdS_5 \times S^5$ superstring*, *Nucl. Phys.* **B 718** (2005) 83 [hep-th/0503185].

[28] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226]; *Quantizing three-spin string solution in $AdS_5 \times S^5$*, *JHEP* **07** (2003) 016 [hep-th/0306130].

[29] R.R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[30] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, hep-th/0502226.

JHEP01(2006)078