FLOWING MAPS TO MINIMAL SURFACES: EXISTENCE AND UNIQUENESS OF SOLUTIONS

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Abstract. We study the new geometric flow that was introduced in [11] that evolves a pair of map and (domain) metric in such a way that it changes appropriate initial data into branched minimal immersions. In the present paper we focus on the existence theory as well as the issue of uniqueness of solutions. We establish that a (weak) solution exists for as long as the metrics remain in a bounded region of moduli space, i.e. as long as the flow does not collapse a closed geodesic in the domain manifold to a point. Furthermore, we prove that this solution is unique in the class of all weak solutions with non-increasing energy. This work complements the paper [11] of Topping and the author where the flow was introduced and its asymptotic convergence to branched minimal immersions is discussed.

1. Introduction

Let M be a smooth closed orientable surface and let (N, G_N) be a (fixed) closed smooth Riemannian manifold of arbitrary dimension that we view as being isometrically immersed in $\mathbb{R}^K$ for some $K \in \mathbb{N}$.

For g a Riemannian metric on M and a map $u : (M, g) \to (N, G_N)$ the Dirichlet energy is defined as

$$E(u, g) := \frac{1}{2} \int_M |du|^2 \, dv_g.$$

We remark that (u, g) is a critical point of E if and only if u is harmonic and weakly conformal, i.e. a branched minimal immersion or a constant map. In the present paper we establish the existence theory for the natural gradient flow of E (considered as a function of both the map and the domain metric) which was introduced in [11]. We refer to this joint paper of Topping and the author for the construction and the geometric background of this flow, but for convenience here recall the main points that led to the definition in [11].

We consider the negative gradient flow of E considered as a function of both the map and the domain metric, but taking into account the symmetries of E, that is the invariance under conformal variations of the domain as well as under the pull-back by diffeomorphisms applied simultaneously to the metric and the map component. That is we consider E and its gradient flow on the set

$$\mathcal{A} = \{[(u, g)];\ g \in \mathcal{M}_c, u \in C^\infty(M, N)\}$$

of equivalence classes where we identify $(u, g) \sim (u \circ f, f^* g)$ for smooth diffeomorphisms $f : M \to M$ homotopic to the identity. Here $\mathcal{M}_c$ stands for the set of smooth metrics of constant (Gauss-)curvature $c = 1, 0, -1$ for surfaces of genus $\gamma = 0, 1$ respectively $\gamma \geq 2$, with unit area in case $\gamma = 1$.

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The tangent space of $\mathcal{M}_c$ splits orthogonally into a horizontal part consisting of the real parts of holomorphic quadratic differentials and a vertical part along the fibers of the action of diffeomorphisms on $\mathcal{M}_c$, i.e. the space of Lie-derivatives of the metric, compare Lemma 2.5 below. This canonical splitting allows us in [11] to represent solutions of the L^2-negative gradient flow of E on $\mathcal{A}$ by the solutions of the system

$$\partial_t u = \tau_g(u) \tag{1.1a}$$

$$\frac{dg}{dt} = \frac{\eta^2}{4}\mathrm{Re}(P_g^H(\Phi(u,g))). \tag{1.1b}$$

Here $\tau_g(u) = \mathrm{tr}_g(\nabla_g(du)) = \Delta_g u + A_g(u)(\nabla u, \nabla u)$, A the second fundamental form of $N \hookrightarrow \mathbb{R}^K$, denotes the tension field of $u : (M,g) \to (N, G_N)$ and $\Phi(u,g)$ stands for the Hopf-differential, i.e. the quadratic differential given in conformal coordinates $z = x+iy$ of (M,g) as $\Phi(u,g) = \phi dz^2$ for $\phi = |u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y\rangle$. Furthermore P_g^H denotes the L^2-orthogonal projection from the space of quadratic differentials onto the finite dimensional subspace of *holomorphic* quadratic differentials on (M,g). Finally $\eta > 0$ is a free coupling constant related to the choice of L^2-metric on $\mathcal{A}$.

As the main result of this paper we prove the following existence and uniqueness theorem

Theorem 1.1. *To any given initial data $(u_0, g_0) \in C^\infty(M,N) \times \mathcal{M}_c$ there exists a weak solution (u,g) of (1.1) defined on a maximal interval $[0,T)$, $T \leq \infty$, that satisfies the following properties*

(i) The solution (u,g) is smooth away from at most finitely many singular times $T_i \in (0,T)$ at which 'harmonic spheres bubble off'. More precisely as $t \nearrow T_i$ energy concentrates at a finite number of points $S(T_i) \subset M$ and suitable rescalings of the maps $u(t)$ around points in $S(T_i)$ converge as $t \nearrow T_i$ to (a bubble-tree of) non-trivial harmonic maps from $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ to N.

(ii) As $t \to T_i$ the maps $u(t)$ converge weakly in H^1 and smoothly away from the set $S(T_i)$ to a limit $u(T_i) \in H^1(M,N)$. Furthermore, the metrics $g(t)$ converge smoothly to an element $g(T) \in \mathcal{M}_c$; in fact, the flow of metrics is Lipschitz-continuous with respect to all C^m metrics on $\mathcal{M}_c$ across singular times.

(iii) The energy $t \mapsto E(u(t), g(t))$ is non-increasing.

(iv) The solution exists as long as the metrics do not degenerate in moduli space; i.e. either $T = \infty$ or the length $\ell(g(t))$ of the shortest closed geodesic in $(M, g(t))$ converges to zero as $t \nearrow T$.

Furthermore, the solution is uniquely determined by its initial data in the class of all weak solutions with non-increasing energy.

Definition 1.2. We call $(u,g) \in H^1_{loc}(M \times [0,T), N) \times C^0([0,T), \mathcal{M}_{-1})$ a weak solution of (1.1) if u solves (1.1a) in the sense of distributions and if g is piecewise C^1 (viewed as a map from $[0,T)$ into the space of symmetric $(0,2)$ tensors equipped with any C^k metric, $k \in \mathbb{N}$) and satisfies (1.1b) away from times where it is not differentiable.

We remark that the assumption on the initial data in Theorem 1.1 can be weakened to $u_0 \in H^1(M,N)$ with the resulting solution being smooth away from finitely many times, possibly including $T_1 = 0$.

On intervals where the obtained solution (u,g) is smooth, the energy decays by

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t),g(t)) = -\int_M |\tau_g(u)|^2\, dv_g - \frac{\eta^2}{16}\|Re[P_g^H(\Phi(u,g))]\|^2_{L^2(M,g(t))}, \tag{1.2}$$

so that if $T = \infty$, both the tension field as well and the holomorphic part of the Hopf-differential converge to zero as $t \to \infty$ suitably. In the joint paper [11] of Topping and the author we indeed prove that if the metric does not degenerate even as $t \to \infty$, then the full Hopf-differential (sub)converges to zero, resulting in a limit that is both harmonic and weakly conformal and thus, if non-constant, a minimal immersion away from at most finitely many branch-points [6]. More precisely, in [11], we prove

Theorem 1.3 ([11], Thm 1.4)**.** *In the setting of Theorem 1.1, if the length $\ell(g(t))$ of the shortest closed geodesic of $(M, g(t))$ is uniformly bounded below by a positive constant, then there exist a sequence of times $t_i \to \infty$ and a sequence of orientation-preserving diffeomorphisms $f_i : M \to M$ such that*

$$f_i^* g(t_i) \to \bar{g} \ \text{ and } \ u(t_i) \circ f_i \to \bar{u}$$

converge to a metric $\bar{g} \in \mathcal{M}_c$ and a branched minimal immersion $\bar{u}$ or a constant map. Here the convergence of metrics is smooth, while the maps converge weakly in $H^1(M, N)$ and strongly in $W^{1,p}_{loc}(M \backslash S)$ for any $p \in [1, \infty)$ away from a finite set of points where energy concentrates.

For suitable initial data, such as incompressible maps, a degeneration of metrics can be excluded so that the flow (sub)converges (up to reparametrisations) to a branched minimal immersion. In [11] we thus recover the well known results on the existence of branched minimal immersions with given action on the level of fundamental groups of Schoen-Yau [14] and Sacks-Uhlenbeck [13] with a flow approach.

Solutions of (1.1) that degenerate in moduli space will be analysed in a forthcoming paper [12] by Topping, Zhu and the author.

Remark 1.4. *For surfaces of genus less than two the structure of the flow (1.1) is simplified considerably and the existence of solutions is known; for spheres the space of holomorphic quadratic differentials is trivial so (1.1) reduces to the harmonic map flow of Eells and Sampson [3] for which existence of global weak solutions was proven in the seminal paper of Struwe [15]. For maps from a surface of genus 1 it is shown in [11] that (1.1) agrees with a flow that was introduced and studied by Ding, Li and Liu in [1]. In this special case the flow of metrics is reduced to two scalar ODEs for parameters describing a global* horizontal *submanifold of the space of metrics. Furthermore, the completeness of Teichmüller space prevents a degeneration of the metric at finite times, leading to the existence of global (weak) solutions for all initial data as obtained in [1].*

In this paper we thus focus on the analysis of the flow from general surfaces of genus $\gamma \geq 2$.

Outline of the paper

The paper consists of three main parts. In the first section we study the properties of *horizontal curves*, i.e. curves that move in the direction of the real part of holomorphic quadratic differentials. Using ideas from Teichmüller theory, we obtain strong estimates for all horizontal curves, and thus in particular for the metric component of the flow, under the sole condition that we stay away from the boundary of moduli space.

In the second section we prove the existence of solutions as claimed in Theorem 1.1. First we obtain short-time existence of smooth solutions based on the properties of horizontal curves derived in the first section. In a second step we then analyse the possible finite time singularities of the flow. On the one hand, we prove that the only way for the metric component to become singular is by a degeneration in moduli space. On the other hand, we obtain that as long as the metric component remains regular, the behaviour of solutions to (1.1a) is similar to the one of solutions of the harmonic map flow as described by Struwe in [15]; the singularity is caused by the *bubbling off* of harmonic spheres and the flow can be continued past the singular time by a weak solution.

Finally we consider the question of uniqueness. We show uniqueness not just for solutions of (1.1) satisfying properties (i)-(iii) of Theorem 1.1 but in the general class of weak solutions with non-increasing energy. This represents the analogue of the uniqueness results [4] and [5] of Freire for the harmonic map flow.

Remark 1.5. For general curves within $\mathcal{M}_{-1}$, satisfying an L^2 bound on the velocity such as (1.2), singularities can form without the metrics degenerating in moduli space. For the flow

$$\partial_t u = \tau_g(u), \quad \frac{dg}{dt} = \frac{\eta^2}{4}\mathrm{Re}(\Phi(u,g)), \tag{1.3}$$

which we would obtain if we were to consider the gradient flow of E without taking into account the symmetries, we thus would not have a characterisation of the maximal existence time of solutions as statement (iv) of Theorem 1.1. For (1.3) we thus could not expect to obtain the global solutions needed to evolve pairs (u, g) to critical points of the energy, i.e. to branched minimal immersions, even for incompressible initial data.

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2. Horizontal curves

We consider general horizontal curves, that is curves moving in the direction

$$\frac{\mathrm{d}}{\mathrm{d}t}g = \mathrm{Re}(\Psi(t))$$

of holomorphic quadratic differentials $\Psi(t) = \psi(t)dz^2$ on $(M, g(t))$, $z = z(t)$ a complex coordinate on $(M, g(t))$. Key for the analysis of such curves is a good understanding of the dependence on the metric $g \in \mathcal{M}_{-1}$ of the horizontal space

$$H(g) := \{\mathrm{Re}(\Phi) : \ \Phi = \phi dz^2 \text{ holomorphic quadratic differential on } (M, g)\}$$

and of the corresponding L^2-orthogonal projection P_g^H. What we essentially need is a quantified version of the idea that a smooth variation of the metric leads to a smooth variation of the complex structure, which in turn results in a smooth change of the space of holomorphic quadratic differentials and of P_g^H.

We remark that there are several equivalent points of view that one can take to study horizontal tensors and curves as well as to study the flow (1.1). Here we follow the differential geometrical approach to Teichmüller theory as presented in the book of Tromba [17]. We view the space of horizontal tensors as a subspace $H(g)$ of the space $Sym^2(M)$ of all real symmetric $(0, 2)$ tensors of class L^2, with $H(g)$ characterised by

$$H(g) = \{h \in Sym^2(M) : \mathrm{tr}_g(h) = 0 \text{ and } \delta_g h = 0\},$$

δ_g the divergence operator (induced by the Levi-Cività connection ∇_g). We then consider the projection

$$P_g : Sym^2(M) \to H(g)$$

that is orthogonal with respect to the $L^2(M, g)$-inner product

$$\langle k, h\rangle_{L^2(M,g)} := \int_M g^{ij}g^{lm}k_{il}h_{jm}\, dv_g.$$

This projection P_g, for which we shall derive an explicit formula later on, is related to the projection P_g^H from the space of quadratic differentials to the space of holomorphic quadratic differentials by

$$P_g(Re(\psi dz^2)) = Re(P_g^H(\psi dz^2)) \tag{2.1}$$

for any quadratic differential ψdz^2 on (M, g).

We first remark that the set of hyperbolic metrics $\mathcal{M}_{-1}$ with smooth coefficients (in the given coordinate charts) is not a manifold. On the other hand for any $s > 3$ the set $\mathcal{M}^s_{-1}$ of hyperbolic metrics with coefficients in the Sobolev space $H^s(M)$ is a smooth submanifold of the (half-)space of all H^s metrics on M, see [17] Theorem 1.6.1.

We shall think of the projection P_g as a map from this Banach manifold $\mathcal{M}^s_{-1}$ into the space $L(Sym^2(M), T\mathcal{M}^s_{-1})$ of linear functions mapping symmetric $(0,2)$-tensors into the tangent bundle of $\mathcal{M}^s_{-1}$ and prove that it is locally Lipschitz.

Proposition 2.1. *For any smooth hyperbolic metric $g_0 \in \mathcal{M}_{-1}$ and every $s > 3$ there exists a neighbourhood W of g_0 in the Banach manifold $\mathcal{M}^s_{-1}$ and a constant $C = C(g_0, s) < \infty$ such that the following holds true: For every tensor $k \in Sym^2(M)$ and every curve $g \in C^1([0,T), \mathcal{M}^s_{-1})$ contained in W we have*

$$\|P_g(k)\|_{H^s} \leq C \cdot \|k\|_{L^2(M,g)} \tag{2.2}$$

and

$$\|\frac{\mathrm{d}}{\mathrm{d}t} P_{g(t)}(k)\|_{H^s} \leq C \cdot \|\frac{\mathrm{d}}{\mathrm{d}t} g(t)\|_{H^s} \cdot \|k\|_{L^2(M,g)}. \tag{2.3}$$

Here and in the following the Sobolev norms $\|\cdot\|_{H^s}$ are to be computed in *fixed* local coordinate charts of M.

Based on this local statement about the projection P_g, we then derive the following result for horizontal curves contained in compact regions of moduli space.

Proposition 2.2. *For every $\varepsilon > 0$ and every $s > 3$ there exists a number $\theta = \theta(\varepsilon, s) > 0$ such that the following holds true. Let $g_0 \in \mathcal{M}^s_{-1}$ be any hyperbolic metric of class H^s for which the length $\ell(g_0)$ of the shortest closed geodesic in (M, g_0) is no less than ε. Then there is a number $C = C(g_0, s) < \infty$ such that for any horizontal curve $g \in C^1([0,T), \mathcal{M}^s_{-1})$ with $g(0) = g_0$ and of L^2-length*

$$L(g) = \int_0^T \|\frac{\mathrm{d}}{\mathrm{d}t} g(t)\|_{L^2(M,g(t))} dt \leq \theta$$

we have

$$\|\frac{d}{dt} g(t)\|_{H^s} \leq C \|\frac{d}{dt} g(t)\|_{L^2(M,g(t))} \text{ for every } t \in [0,T). \tag{2.4}$$

For tori the corresponding result is obtained as a consequence of the existence of a *smooth global horizontal slice*, i.e. of a finite dimensional smooth submanifold of $\mathcal{M}_0$, parametrised over Teichmüller space, whose tangent space at each point is horizontal and which thus contains all horizontal curves passing through g_0.

While for surfaces of genus $\gamma \geq 2$ the space of horizontal tensors $H(g)$ is still finite dimensional, $\dim_{\mathbb{R}}(H(g)) = 6\gamma - 6$ by the Riemann-Roch theorem, the distribution $g \mapsto H(g)$ is no longer integrable, compare [17], section 5.3, so Proposition 2.2 cannot be reduced to a statement about curves on a finite dimensional manifold.

Proof of Proposition 2.1. We prove Proposition 2.1 in two steps; we show first that estimates of the form (2.2) and (2.3) hold true for metrics contained in a so called *slice* and then in a second step pull-back these estimates to give the claim of Proposition 2.1 for general metrics in a neighbourhood of g_0. To do so we make use of ideas from Teichmüller theory as explained in the book of Tromba [17], chapter 2.

So let $g_0 \in \mathcal{M}_{-1}$ be any given metric and let $s > 3$ be fixed. Following [17] we define a small *slice* around g_0 by

$$S := \{g = \rho(h) \cdot (g_0 + h) : h \in U \subset H(g_0)\} \subset \mathcal{M}_{-1} \tag{2.5}$$

for $U = U(g_0, s)$ a suitably small neighbourhood of $0 \in H(g_0)$ chosen later on. Here the function $\rho(h) : M \to \mathbb{R}$ is to be chosen such that $\rho(h) \cdot (g_0 + h)$ has constant curvature -1 and is uniquely determined by this property according to Poincaré's theorem.

The key feature of this finite dimensional submanifold of $\mathcal{M}^s_{-1}$ is that it provides a local model of $\mathcal{M}^s_{-1}/\mathcal{D}^{s+1}_0$, with $\mathcal{D}^{s+1}_0$ the set of H^{s+1}-diffeomorphisms that are homotopic to the identity

Theorem 2.3 ([17], Thm 2.4.3)**.** *For any number $s > 3$, any $g_0 \in \mathcal{M}_{-1}$ and $S = S(g_0, s)$ a sufficiently small slice around g_0, there are neighbourhoods $W \subset \mathcal{M}^s_{-1}$ of g_0 and $V \subset \mathcal{D}^{s+1}_0$ of id for which the map*

$$S \times V \ni (g, f) \mapsto f^* g \in W$$

is a diffeomorphism.

For a proof of this theorem as well as for further insight into Teichmüller theory we refer to the book of Tromba [17]. We remark that the above result remains valid if we replace the slice S by a smaller slice defined by (2.5), for appropriate new neighbourhoods of id in $\mathcal{D}^{s+1}_0$ and of g_0 in $\mathcal{M}^s_{-1}$, but that the theorem does not give the existence of a uniform slice for which the statement is valid for all numbers $s > 3$. We furthermore stress that the theorem demands that the metric g_0 is not only in $\mathcal{M}^s_{-1}$ but smooth; this in turn implies that all metrics contained in a small slice S are smooth and thus satisfy stronger estimates than just H^s bounds, in particular

Lemma 2.4. *For a sufficiently small slice S around $g_0 \in \mathcal{M}_{-1}$ there exists a constant $C = C(s, g_0) < \infty$ such that for all metrics $g_{1,2} \in S$*

$$\|g_1 - g_2\|_{H^{s+1}} \leq C \cdot d_S(g_1, g_2). \tag{2.6}$$

Here we denote by d_S the H^s metric on S, i.e. consider S as a submanifold of the Banach manifold $\mathcal{M}^s_{-1}$.

Apart from the finite dimensionality of $H(g_0)$, and thus of S, the essential observation leading to the above estimate is that the conformal factor $\rho(h)$ can be characterised as the unique solution of an elliptic PDE, compare [17] section 1.5, leading to a smooth dependence of $\rho(h)$ on $h \in U$.

Based on these stronger estimates on elements of the slice, we can analyse the dependence of P_g on $g \in S$ using an explicit formula for P_g that we shall derive now.

We first recall the following canonical splitting of the tangent space $T_g\mathcal{M}^s_{-1}$ into the horizontal and vertical space, see Theorem 2.4.1 of [17].

Lemma 2.5. *For any $g \in \mathcal{M}_{-1}$ the tangent space $T_g\mathcal{M}^s_{-1}$ splits L^2-orthogonally into $H(g)$ and the space $\{L_X g\}$ of Lie-derivatives. More precisely, given any $k \in T_g\mathcal{M}^s_{-1}$ there is a unique vector field X (of class H^{s+1}) such that*

$$tr_g(k - L_X g) = 0 \text{ and } \delta_g(k - L_X g) = 0$$

and X can be characterised as the unique solution of the elliptic PDE

$$\delta_g \delta_g^* X = -\delta_g k, \tag{2.7}$$

$\delta_g^* X = -L_X g$ the $L^2(M, g)$-adjoint of δ_g.

In order to define the orthogonal projection of a general symmetric $(0, 2)$ tensor k onto the horizontal space $H(g)$, we first map k onto an element of $T_g\mathcal{M}^s_{-1}$ using

Lemma 2.6. *For any $g \in \mathcal{M}_{-1}$ and any symmetric $(0,2)$ tensor k of class H^s there exists a unique function $\mu \in H^s(M,\mathbb{R})$ such that*

$$k - \mu \cdot g \in T_g\mathcal{M}^s_{-1}.$$

The function μ is characterised as the unique solution of the equation

$$-\Delta_g \mu + 2\mu = 2DR(g)(k), \tag{2.8}$$

$R(g)$ the Gauss curvature of (M,g).

Given any $g \in \mathcal{M}_{-1}$, we now claim that the orthogonal projection $P_g : Sym^2(M) \to H(g)$ is given by

$$P_g(k) := k - \mu(k,g) \cdot g - L_{X(k-\mu(k,g)\cdot g, g)}g \tag{2.9}$$

where $X(\cdot)$ and $\mu(\cdot)$ stand for the corresponding solutions of (2.7) and (2.8). Indeed, $P_g|_{H(g)} = id$ and for general $k \in Sym^2(M)$ the tensor given by (2.9) is well defined and divergence- as well as trace-free with respect to g, i.e. an element of $H(g)$. Furthermore, $k - P_g(k)$ stands orthogonal to any $h \in H(g)$ as

$$\begin{aligned}\langle h, k - P_g(k)\rangle_{L^2(M,g)} &= \langle h, \mu \cdot g\rangle_{L^2} + \langle h, L_X g\rangle_{L^2} = \int_M \mu \cdot tr_g(h)\, dv_g + \langle h, -\delta_g^* X\rangle_{L^2} \\ &= -\langle \delta_g h, X\rangle_{L^2} = 0.\end{aligned}$$

To analyse the dependence of P_g on g we now use that X and μ are characterised by elliptic PDEs for which the following uniform estimates apply

Lemma 2.7. *Let $s > 3$ and $g_0 \in \mathcal{M}_{-1}$ be given and let $S = S(g_0, s)$ be a sufficiently small slice. Then there exists a constant $C = C(s, g_0) < \infty$ such that the following claims hold true for every $g \in S$. For every vector field Y there is a unique solution of the equation*

$$\delta_g \delta_g^* X = Y \tag{2.10}$$

and for any $0 \leq l \leq s+1$ we have

$$\|X\|_{H^l} \leq C \cdot \|Y\|_{H^{l-2}}.$$

Similarly, the unique solution μ of

$$-\Delta_g \mu + 2\mu = f \in H^{l-2}(M,\mathbb{R}), \tag{2.11}$$

satisfies

$$\|\mu\|_{H^l} \leq C \cdot \|f\|_{H^{l-2}}, \quad 0 \leq l \leq s.$$

We remark that the occurring Sobolev norms with negative exponent are to be understood as the norms of the coefficients in the dual spaces $H^{-k}(\Omega) = (H^k_0(\Omega))^*$, $\Omega \subset \mathbb{R}^2$.

The reason why the solution X of (2.10) is unique is that we work on a surface that has negative curvature. Thus the kernel of $\delta_g\delta_g^*$, which agrees with the space of Killing-fields, is trivial, see e.g. [7], Thm. 5.3. Elliptic regularity theory combined with the Fredholm alternative theorem then immediately gives the estimates for each individual $g \in S$. These estimates are indeed uniform since all metrics in S are contained in a small (H^s) neighbourhood of g_0.

We can now give the proof of Proposition 2.1, first for metric contained in the slice.

Let $g_0 \in \mathcal{M}_{-1}$, $s > 3$ and let $S = S(g_0, s)$ be a small slice as defined above. Combining the elliptic estimates of Lemma 2.7 with (2.9) and the bounds on g given in Lemma 2.4, we find that for every $0 \leq l \leq s$ and every $k \in Sym^2(M)$

$$\begin{aligned}\|P_g(k)\|_{H^l} &\leq \|k\|_{H^l} + C\|\mu\|_{H^l} + C\|X\|_{H^{l+1}} \\ &\leq \|k\|_{H^l} + C(\|DR(g)(k)\|_{H^{l-2}} + \|\delta_g k\|_{H^{l-1}}) \leq C\|k\|_{H^l}.\end{aligned} \tag{2.12}$$

Here and in the following we crucially use that Lemma 2.4 gives bounds on $s+1$ derivatives of g so that we may estimate the H^s and not just the H^{s-1} norm of Lie-derivatives $L_X g$.

Similarly, given any C^1 curve g in the slice, we differentiate the corresponding equations (2.7) and (2.8) characterising X and μ. This leads to elliptic PDEs of the form (2.10) and (2.11) for $\frac{\mathrm{d}}{\mathrm{d}t}X(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mu(t)$. Applying Lemma 2.7 and making use of the bound $\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^{s+1}} \le C\cdot\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}$ of Lemma 2.4, we obtain

$$\|\frac{d}{dt}P_{g(t)}(k)\|_{H^l} \le C\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}\cdot\|k\|_{H^l} \tag{2.13}$$

for any (sufficiently smooth) tensor $k \in Sym^2(M)$ and any $0 \le l \le s$.

In order to establish the estimates (2.2) and (2.3) claimed in Proposition 2.1 we now need to prove that the two estimates (2.12) and (2.13) obtained above remain valid with the H^l norm on the right hand side replaced with the L^2 norm. We use

Claim: There exists $C < \infty$ such that for all $g \in S$ and all $h \in H(g)$

$$\|h\|_{H^s} \le C\|h\|_{L^2(M,g)}.$$

Proof of Claim: The estimate trivially holds true for $g = g_0$ (or indeed for any one fixed metric) since $H(g_0)$ is a finite dimensional space of smooth tensors. For general $g \in S$ we can parametrize $H(g)$ over $H(g_0)$ by restricting the projection P_g onto $H(g_0)$. Using estimate (2.12), we then get

$$\|P_g(k)\|_{H^s} \le C\|k\|_{H^s} \le C\|k\|_{L^2} \text{ for every } k \in H(g_0), g \in S, \tag{2.14}$$

with $\|\cdot\|_{L^2}$ denoting one of the equivalent $L^2(M,g)$ norms, $g \in S$, say $\|\cdot\|_{L^2(M,g_0)}$.

On the other hand, integrating (2.13) for $l = 0$ along a suitable curve of metrics connecting g_0 to g and making use of the fact that $P_{g_0}|_{H(g_0)} = id$, we obtain that for any $k \in H(g_0)$

$$\begin{aligned}\|k\|_{L^2} &\le \|P_g(k) - k\|_{L^2} + \|P_g(k)\|_{L^2} \le Cd_S(g,g_0)\cdot\|k\|_{L^2} + \|P_g(k)\|_{L^2}\\ &\le \frac{1}{2}\|k\|_{L^2} + \|P_g(k)\|_{L^2}\end{aligned}$$

provided the slice is chosen small enough. Combined with estimate (2.14) this implies the claim for tensors in the image $P_g(H(g_0)) \subset H(g)$ which must agree with $H(g)$ because $P_g|_{H(g_0)}$ is injective and $\dim(H(g)) = \dim(H(g_0))$.

Combining this claim with the estimate (2.12) for $l = 0$ we have thus proved the first claim (2.2) of Proposition 2.1 for general tensors $k \in Sym^2(M)$ and for metrics $g \in S$ in the slice.

To obtain an improved version of (2.13), we write

$$P_{g(t)}(k) = P_{g(t)}\big(P_{g(t_0)}(k)\big) + P_{g(t)}\big(P_{g(t)}(k) - P_{g(t_0)}(k)\big)$$

and estimate the derivative of the right hand side at $t = t_0$. Estimate (2.13), applied first for $l = s$ and then for $l = 0$, combined with the estimate (2.2) we just proved then implies that for any $k \in Sym^2(M)$

$$\begin{aligned}\|(\frac{\mathrm{d}}{\mathrm{d}t}P_{g(t)}(k))(t_0)\|_{H^s} &\le \|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}\cdot\|P_{g(t_0)}(k)\|_{H^s} + \|P_{g(t_0)}(\frac{\mathrm{d}}{\mathrm{d}t}P_{g(t)}(k))\|_{H^s}\\ &\le C\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}\cdot\|k\|_{L^2} + \|\frac{\mathrm{d}}{\mathrm{d}t}P_{g(t)}(k)\|_{L^2} \le C\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}\cdot\|k\|_{L^2}.\end{aligned}$$

This completes the proof of Proposition 2.1 for metrics g contained in the slice. We now pull back these estimates to the full H^s neighbourhood W of g_0 given by the slice-theorem

2.3. The key observation allowing us to do so is that the projection onto the horizontal space commutes with the pull-back

$$f^*P_g(k) = P_{f^*g}(f^*k).$$

Thus, given a C^1 curve g in W, we write it (uniquely) in the form $g(t) = f(t)^*\bar{g}(t)$, for $f(t) \in V \subset \mathcal{D}_0^{s+1}$ and $\bar{g}(t) \in S$ and recall that $\|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}\|_{H^s}$ and $\|\frac{\mathrm{d}}{\mathrm{d}t}f\|_{H^{s+1}}$ are controlled by $\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s}$, see Theorem 2.3. Indeed, since the diffeomorphisms f are contained in a neighbourhood of the identity, also $\|\frac{\mathrm{d}}{\mathrm{d}t}f^{-1}\|_{H^{s+1}}$ is bounded in this way. Applying estimates (2.2) and (2.3) for $\bar{g} \in S$, we thus find

$$\|P_g(k)\|_{H^s} = \|f^*(P_{\bar{g}}((f^{-1})^*k)\|_{H^s} \leq C \cdot \|P_{\bar{g}}((f^{-1})^*k)\|_{H^s} \leq C\|k\|_{L^2}$$

as well as

$$\begin{aligned}
\|\frac{\mathrm{d}}{\mathrm{d}t}P_{g(t)}(k)\|_{H^s} &\leq C \cdot \|\frac{\mathrm{d}}{\mathrm{d}t}f\|_{H^{s+1}} \cdot \|P_{\bar{g}}((f^{-1})^*k)\|_{H^s} + C\|\frac{\mathrm{d}}{\mathrm{d}t}(P_{\bar{g}(t)}((f(t)^{-1})^*k)\|_{H^s} \\
&\leq C \cdot \left(\|\frac{\mathrm{d}}{\mathrm{d}t}f\|_{H^{s+1}} + \|\frac{\mathrm{d}}{\mathrm{d}t}\bar{g}\|_{H^s} + \|\frac{\mathrm{d}}{\mathrm{d}t}f^{-1}\|_{H^{s+1}}\right) \cdot \|k\|_{L^2} \\
&\leq C \cdot \|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s} \cdot \|k\|_{L^2}
\end{aligned}$$

for any tensor $k \in Sym^2(M)$ and any curve in W as claimed in Proposition 2.1. $\square$

Proof of Proposition 2.2. For any number $s > 3$ we define a function $\theta : \mathcal{M}_{-1}^s \to [0, \infty]$ as follows. For any metric $g_0 \in \mathcal{M}_{-1}^s$ we let $\theta(g_0)$ be the supremum of all numbers $\theta \geq 0$ such that there exists a number $C < \infty$ for which estimate (2.4) holds true for all (piecewise) horizontal curves in $\mathcal{M}_{-1}^s$ of length $L_{L^2}(g) \leq \theta$ and with $g(0) = g_0$. We stress that both this constant C, as well as the constant in Proposition 2.2, are allowed to depend on the metric g_0.

We first claim that the function θ is strictly positive for all *smooth* metrics. So let $g_0 \in \mathcal{M}_{-1}$ and let W be the neighbourhood of g_0 in $\mathcal{M}_{-1}^s$ for which Proposition 2.1 applies. Writing the velocity of any horizontal curve as $\frac{\mathrm{d}}{\mathrm{d}t}g = P_g(\frac{\mathrm{d}}{\mathrm{d}t}g)$ and applying Proposition 2.1 we find that

$$\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{H^s} \leq C\|\frac{\mathrm{d}}{\mathrm{d}t}g\|_{L^2(M,g)}$$

for as long as the curve is contained in W. But W is an H^s neighbourhood, so this estimate implies that any curve of small enough L^2 length and with $g(0) = g_0$ is fully contained in W and thus that indeed $\theta(g_0) > 0$.

Secondly, we observe that θ is invariant under the pull-back by diffeomorphisms. More precisely let $\mathcal{D}^{s+1}$ be the set of all diffeomorphism of class H^{s+1} (not necessarily homotopic to the identity). Then we claim that for any $g \in \mathcal{M}_{-1}^s$ and any $f \in \mathcal{D}^{s+1}$

$$\theta(f^*g_0) = \theta(g_0).$$

Indeed, pulling-back any horizontal curve g in $\mathcal{M}_{-1}^s$ by a fixed diffeomorphism $f \in \mathcal{D}^{s+1}$ results in another horizontal curve of the same L^2-length and with velocity bounded by $\|\frac{\mathrm{d}}{\mathrm{d}t}(f^*g(t))\|_{H^s} \leq C \cdot \|\frac{\mathrm{d}}{\mathrm{d}t}g(t)\|_{H^s}$, with $C < \infty$ a constant depending on f. But we defined $\theta(g)$ asking only for an estimate of the form (2.4) to be satisfied for *some* constant $C < \infty$, allowed to depend on the considered metric, so the claim follows.

We conclude that θ induces a positive map $\bar{\theta}$ on moduli space $\mathcal{M}_{-1}/\mathcal{D}$ and now want to prove that this function is continuous with respect to the Weyl-Peterson metric d_{WP}.

We recall that the length of a C^1 curve $[g]$ in moduli space (with respect to the Weyl-Peterson metric) is given by

$$L_{WP}([g]) = \frac{1}{2} L_{L^2}(\tilde{g})$$

for $\tilde{g}$ a 'horizontal lift' of $[g]$, that is a *horizontal* curve $\tilde{g} \in \mathcal{M}_{-1}$ with $[\tilde{g}(t)] = [g(t)]$ for each t.

Given any two points $[g_1]$ and $[g_2]$ in $\mathcal{M}_{-1}/\mathcal{D}$ we now claim that

$$\bar{\theta}([g_2]) \geq \bar{\theta}([g_1]) - 2 \cdot d_{WP}([g_1], [g_2]),$$

and thus switching the roles of $[g_1]$ and $[g_2]$ that $\bar{\theta}$ is Lipschitz continuous on moduli space $(\mathcal{M}_{-1}/\mathcal{D}, d_{WP})$. So let $\delta > 0$ be any fixed number and choose a (piecewise) horizontal path $\tilde{g}$ of L^2-length less than $2 \cdot d_{WP}([g_1], [g_2]) + \delta/2$ that connects a representative f^*g_1 of $[g_1]$ with g_2. Let now g be any given (piecewise) horizontal curve with $g(0) = g_2$ and of length $L_{L^2}(g) \leq \bar{\theta}([g_1]) - 2d_{WP}([g_1], [g_2]) - \delta$. Precomposing it with $\tilde{g}$ we obtain a curve G of length $L_{L^2}(G) \leq \bar{\theta}([g_1]) - \delta/2 = \theta(f^*g_1) - \delta/2$ and with starting point $G(0) = f^*g_1$. By definition of $\theta(f^*g_1)$, the estimate (2.4) is satisfied for the extended curve G and thus in particular for g itself, with a constant C depending on f^*g_1 and possibly δ but not on g. We obtain the claim since $\delta > 0$ can be chosen arbitrarily small.

Given any number $\varepsilon > 0$ we now consider the subset K_ε of moduli space consisting of the equivalence classes of smooth metrics with shortest closed geodesic of length no less than ε. This set K_ε is compact by the Mumford compactness theorem, see e.g. [17], p.75. As a positive and continuous function, $\bar{\theta}$ is thus bounded away from zero uniformly on K_ε which implies Proposition 2.2 for smooth metrics.

For non-smooth metrics $g \in \mathcal{M}^s_{-1} \setminus \mathcal{M}_{-1}$, we finally obtain the claim of Proposition 2.2 using the invariance of θ under H^{s+1} diffeomorphisms as well as

Lemma 2.8. *Given any $g \in \mathcal{M}^s_{-1}$ there exists a smooth metric $\bar{g} \in \mathcal{M}_{-1}$ and a diffeomorphism f of class H^{s+1} such that*

$$g = f^*\bar{g}.$$

For the sake of completeness we provide a proof of this fact in the appendix.

For most arguments in the rest of the paper the estimates of Proposition 2.1 and 2.2, controlling the L^2-orthogonal projection in terms of the L^2 norms of the involved tensors, would be sufficient, though would in some cases lead to slightly weaker regularity results. For the proof of uniqueness of weak solutions carried out in section 4 it is however crucial that we can extend P_g continuously onto the space of tensors with finite L^1 norm

Lemma 2.9. *For any $g_0 \in \mathcal{M}_{-1}$ and any $s > 3$ there exists a neighbourhood W of g_0 in $\mathcal{M}^s_{-1}$ such that the following holds true. The map P_g is Lipschitz-continuous as a map from W to the space of linear maps from $(Sym^2(M), \|\|_{L^1})$ to the tangent bundle $T\mathcal{M}^s_{-1}$, i.e. there exists a constant $C = C(g_0, s) < \infty$ such that for all $g_1, g_2 \in W$ and $k \in Sym^2(M)$*

$$\|P_{g_1}(k)\|_{H^s} \leq C \cdot \|k\|_{L^1} \text{ and } \|P_{g_1}(k) - P_{g_2}(k)\|_{H^s} \leq C \cdot d_{\mathcal{M}^s_{-1}}(g_1, g_2) \cdot \|k\|_{L^1}. \tag{2.15}$$

We remark that there is no need to specify with respect to which metric $g \in W$ the L^1 norm is computed as all metrics in W are equivalent.

We prove these refined estimates on P_g using the following consequence of Proposition 2.1

Lemma 2.10. *For any $g_0 \in \mathcal{M}_{-1}$ and any $s > 3$ there exists a neighbourhood W of g_0 in $\mathcal{M}^s_{-1}$ and a constant $C < \infty$ so that we can assign to each metric g in W an $L^2(M, g)$-orthonormal basis $\{\Theta^j(g)\}_{j=1}^{6\gamma-6}$ of $H(g)$ satisfying*

$$\|\Theta^j(g_1) - \Theta^j(g_2)\|_{H^s} \le C \cdot d_{\mathcal{M}^s_{-1}}(g_1, g_2), \quad g_{i,2} \in W, \quad j = 1 \dots 6\gamma - 6 = dim(H(g))$$

Lemma 2.9 then immediately follows from $P_g(k) = \sum_j \langle k, \Theta^j(g)\rangle_{L^2(M,g)}\Theta^j(g)$.

Proof of Lemma 2.10. Let $g_0 \in \mathcal{M}^s_{-1}$ and let W be the neighbourhood of g_0 given by Proposition 2.1. We fix any $L^2(M, g_0)$-orthonormal basis $\Theta^j(g_0)$, $j = 1 \dots 6\gamma - 6$, of $H(g_0)$ and define

$$\Theta^j_0(g) := P_g(\Theta^j(g_0)).$$

According to Proposition 2.1 this auxiliary family of tensors depends continuously on g,

$$\|\Theta^j_0(g_1) - \Theta^j_0(g_2)\|_{H^s} \le C \cdot d_{\mathcal{M}^s_{-1}}(g_1, g_2)$$

so that $\{\Theta^j_0\}$ is a basis of $H(g)$ provided the neighbourhood W is chosen sufficiently small. Furthermore, as the map asigning to each metric g the inner products

$$g \mapsto \langle\Theta^j_0(g), \Theta^k_0(g)\rangle_{L^2(M,g)}$$

is also Lipschitz-continuous on W, so are the coefficients a^j_i of the orthonormal basis $\Theta^j(g) = \sum_{i=1}^j a^j_i(g)\Theta^i_0(g)$ of $H(g)$ obtained by Gram-Schmidt orthogonalisation and thus the basis itself. □

3. Existence of solutions

In this section we establish the existence of weak solutions to (1.1) satisfying the properties claimed in Theorem 1.1, in particular existing for all times unless the metric component degenerates in moduli space. As a first step, we prove the following short-time existence result

Lemma 3.1. *For any initial metric $g_0 \in \mathcal{M}_{-1}$ and any initial map $u_0 \in C^\infty(M, N)$ there exists a smooth solution (u, g) of equation (1.1) to initial data $(u(0), g(0)) = (u_0, g_0)$ defined on an interval $[0, T)$, $T = T(u_0, g_0) > 0$.*

Proof of Lemma 3.1. We first recall that the metric evolves by

$$\frac{dg}{dt} = \frac{\eta^2}{4} Re(P^H_g(\Phi(u, g))) = \frac{\eta^2}{4} P_g(k(u, g)), \tag{3.1}$$

where $k(u, g) = Re(\Phi(u, g))$, compare (1.1b) and (2.1).

To simplify notations and without loss of generality, we shall from now on consider the flow with coupling constant $\eta = 2$. We also remark that computing the variation

$$\frac{d}{ds}E(u, g + sl)|_{s=0} = -\frac{1}{4}\langle Re(\Phi(u, g)), l\rangle_{L^2} \text{ for all } l \in Sym^2(M)$$

in local coordinate charts, allows us to write the real part of the Hopf-differential in general (not necessarily conformal) coordinate charts as

$$k(u, g) = Re(\Phi(u, g)) = 2u^*G_N - 2e(u, g)g,$$

$e(u, g) = \frac{1}{2}|\nabla u|^2_g = \frac{1}{2}g^{ij}\partial_{x_i}u \cdot \partial_{x_j}u$ the energy density.

Using the results of the previous section we can consider equation (1.1) as a system consisting of a semilinear parabolic PDE coupled with a differential equation on a Banach manifold $\mathcal{M}^s_{-1}$ that is defined by a locally Lipschitz continuous vector field. In such a setting we obtain the existence of a classical solution on a short time interval using a standard iteration argument, which, for the sake of completeness, we outline in the appendix. Given any $(u_0, g_0) \in C^{2,\alpha}(M,N) \times \mathcal{M}_{-1}$ and any number $s > 3$ we obtain a solution

$$(u,g) \in C^{2,1,\alpha}([0,T_s) \times M, N) \times C^1([0,T), \mathcal{M}^s_{-1})$$

of (1.1), defined on a maximal interval $[0, T_s)$. This interval might a priori depend not only on (u_0, g_0) but also on the Banach manifold $\mathcal{M}^s_{-1}$ on which we solve (1.1b). Indeed, the key step needed to prove that the obtained solution (u,g) is actually smooth is to show that this is not the case. So suppose that for some $3 < s_1 < s_2$ we have $T_{s_1} \neq T_{s_2}$. Since classical solutions of (1.1) are uniquely determined by their initial data, compare section 4, we remark that the two solutions obtained for the different values of s agree for as long as they both exist, that is until time $T_{s_2} < T_{s_1}$. Since the metric component is continuous (as a map into $\mathcal{M}^{s_1}_{-1}$) up to time T_{s_1} there exists a number $\varepsilon > 0$ such that the length $\ell(g(t))$ of the shortest closed geodesic of $(M, g(t))$ is no less than ε on the smaller interval $[0, T_{s_2}]$. Using the H^s estimates of Proposition 2.2 this allows us to conclude that g is C^1 as a curve into $\mathcal{M}^{s_2}_{-1}$ on the closed interval $[0, T_{s_2}]$, compare with the proof of Lemma 3.2 below.

Using Lemma 2.8, we then write $g(T_{s_2}) \in \mathcal{M}^{s_2}_{-1}$ in the form $g(T_{s_2}) = f^*\bar{g}(T_{s_2})$ for an H^{s_2+1} diffeomorphism f and a smooth metric $\bar{g} \in \mathcal{M}_{-1}$. Restarting the flow with the pulled-back initial data $(\bar{u}(T_{s_2}), \bar{g}(T_{s_2})) = (u(T_{s_2}) \circ f^{-1}, (f^{-1})^* g(T_{s_2})) \in C^{2,\alpha} \times \mathcal{M}_{-1}$ we obtain a solution $(\bar{u}, \bar{g})$ of (1.1) in $C^{2,1,\alpha}(M \times I) \times C^1(I, \mathcal{M}^{s_2}_{-1})$ on a time interval $I = [T_{s_2}, T_{s_2} + \delta)$. But equation (1.1) is invariant under the pull-back by diffeomorphisms applied simultaneously to both the map and the metric component and solutions of (1.1) are unique. Thus the pull-back of $(\bar{u}, \bar{g})$ by f is nothing else than our original solution (u,g) so that g is in $C^1([0, T_{s_2} + \delta), \mathcal{M}^{s_2}_{-1})$, leading to a contradiction.

At this point we are now in a position to argue by a standard bootstrapping argument, using parabolic regularity theory to improve the regularity of u, as well as the explicit formula for P_g given in (2.9) to analyse higher order time derivatives of g. We obtain that (u,g) is indeed smooth. □

We remark that the results of section 2 allow us not only to establish short-time existence of solutions to (1.1) but already give the following characterisation of the behaviour of the metric component at a singular time

Lemma 3.2. *Let (u,g) be a smooth solution of (1.1) defined (and smooth) on a maximal interval $[0, T_1)$. Then one of the following three statements holds*

(i) $T_1 = \infty$, *or*
(ii) $T_1 < \infty$ *but as* $t \nearrow T_1$ *the metrics* $g(t)$ *converge smoothly to a limit* $g(T_1) \in \mathcal{M}_{-1}$*; indeed g can be extended to a Lipschitz continuous curve from the closed interval* $[0, T_1]$ *into each of the Banach manifold* $\mathcal{M}^s_{-1}$, $s > 3$, *or*
(iii) *the metrics degenerate in moduli space at a finite time* T_1, *i.e.* $\lim_{t \nearrow T_1} \ell(g(t)) = 0$.

Proof of Lemma 3.2. Assume that $T_1 < \infty$ and that the length of the shortest closed geodesics in $(M, g(t))$ does not converge to zero

$$\limsup_{t \nearrow T_1} \ell(g(t)) > \varepsilon > 0.$$

Then given any number $s > 3$ we let $\theta = \theta(s, \varepsilon) > 0$ be the constant of Proposition 2.2. We recall that according to the energy identity (1.2) the L^2-length of the curve g is finite on intervals of finite length. We may thus choose $t_0 < T_1$ with $\ell(g(t_0)) \geq \varepsilon$ and close enough to T_1 such that $L_{L^2}(g|_{[t_0,T_1)}) < \theta$. Proposition 2.2 then implies that $g(t)$ is a Cauchy sequence in $\mathcal{M}^s_{-1}$ and thus converges to a limit $g(T_1)$ in $\mathcal{M}^s_{-1}$ as $t \nearrow T_1$. Indeed, combining Proposition 2.2 with the energy identity (1.2) gives $C^{1/2}$-Hölder estimates in time for g considered as map into $\mathcal{M}^s_{-1}$. Moreover, thanks to the uniform bound on the energy of u and thus on the L^1 norm of the Hopf-differential

$$\|k(u,g)\|_{L^1} \leq C \cdot \|\nabla u\|^2_{L^2} \leq C \cdot E(u,g) \leq C \cdot E(u_0, g_0),$$

the improved estimates on P_g stated in Lemma 2.9 give uniform bounds on $\|\frac{\mathrm{d}}{\mathrm{d}t}g(t)\|_{H^s}$. Thus g is not only $C^{1/2}$ but indeed Lipschitz continuous with respect to each H^s metric on the *closed* interval $[0,T_1]$. □

We remark that the possibility of solutions degenerating in moduli space will be addressed in future work and that here we focus on the analysis of singularities of the second type, essentially due to the map component becoming singular.

So let (u,g) be a smooth solution of (1.1) on a maximal interval $[0,T)$. Assume that the metrics do not degenerate in moduli space as we approach the singular time and thus that $g(t) \to g(T_1) \in \mathcal{M}_{-1}$ smoothly as $t \nearrow T_1$. We remark that the evolution of the metric component is uniformly controlled,

$$\|\frac{\mathrm{d}}{\mathrm{d}t} g\|_{H^s} \leq C\|k(u,g)\|_{L^1} \leq C \cdot \|\nabla u\|^2_{L^2} \leq C \cdot E_0 \tag{3.2}$$

for times in an interval of length $\delta = \delta(g(T_1), s) > 0$ not just for the one solution (u,g) of (1.1) that becomes singular, but also for all solutions evolving from an initial data $(\bar{u}, g(T_1))$ with energy bounded by E_0. Thanks to this strong bound on the metric component we can carry out the analysis of the map component of solution to (1.1) near singular times using methods familiar from the work of Struwe [15] on the harmonic map flow. Since our analysis closely follows the ideas of [15] we shall omit some details and calculations in the following presentation. We also remark that a similar argument was briefly outlined in [1] in the special case of maps from a torus.

Notation: We let $g_1 \in \mathcal{M}_{-1}$ be a fixed metric that should be thought of as a limiting metric of a solution of (1.1) at a singular time. Then unless indicated otherwise all occurring objects such as norms, operators (like Δ), integrals, balls and so on are to be understood as the corresponding objects on the fixed Riemannian surface (M, g_1). Furthermore, we denote generic constants (allowed to change from line to line) by C in case they depend only on g_1 and E_0 and will indicate any dependence on additional quantities accordingly.

Based on (3.2) we henceforth restrict our attention to solutions of (1.1) satisfying

$$\|g_1 - g(t)\|_{H^s} \leq \varepsilon_1 \tag{3.3}$$

for some fixed number $s > 3$ and a small $\varepsilon_1 = \varepsilon_1(g_1, s) > 0$, chosen in particular such that Lemma 2.9 applies on this $\mathcal{M}^s_{-1}$ neighbourhood of g_1.

We first remark that the evolution of the local energy is controlled by

Lemma 3.3. *For solutions (u,g) of (1.1) satisfying (3.3) the following local energy bounds hold true for any point $x \in M$ and any radius $0 < r < r_{inj}$*

$$E(u(t), B_{r/2}(x)) \leq 2E(u(0), B_r(x)) + C\frac{t}{r^2}$$

and

$$E(u(t), B_r(x)) \geq \frac{1}{2}E(u(0), B_{r/2}(x)) - 4\int_0^t \int_M \varphi^2 \left|\partial_t u\right|^2 dvdt - C\frac{t}{r^2}.$$

Sketch of proof. Given $x \in (M, g_1)$ and $0 < r < r_{inj}(M, g_1)$ we let $\varphi \in C_0^\infty(B_r(x), [0,1])$ be a standard cut-off function, i.e. such that $\varphi \equiv 1$ on $B_{r/2}(x)$ and $|\nabla \varphi| \leq \frac{C}{r}$. A short calculation shows that for a solution (u, g) of (1.1)

$$\begin{aligned} 0 &= \int \varphi^2 |\partial_t u|^2 \, dv - \int \varphi^2 \partial_t u \cdot \Delta_{g(t)} u \, dv \\ &= \int \varphi^2 |\partial_t u|^2 \, dv + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \varphi^2 |\nabla u|^2_{g(t)} \, dv + R(u(t), g(t)) \end{aligned} \tag{3.4}$$

with an error term that is bounded by

$$|R(u, g)| \leq (\frac{C}{r^2} + C \|\frac{\mathrm{d}}{\mathrm{d}t} g\|_{C^0}) \cdot E(u, B_r(x)) + \frac{1}{8} \int_M \varphi^2 |\partial_t u|^2 \, dv.$$

Since $\|\frac{\mathrm{d}}{\mathrm{d}t} g\|_{C^0}$ is uniformly bounded, this estimate integrates to give an upper and a lower bound on $\int \varphi^2 |\nabla u|^2_{g(t)} - \int \varphi^2 |\nabla u(0)|_{g(0)}$. Combined with the fact that $\frac{1}{2} g \leq \tilde{g} \leq 2g$ for all $g, \tilde{g}$ satisfying (3.3), we obtain the claims of Lemma 3.3. □

An important consequence of the previous calculation is

Corollary 3.4. *Suppose (u, g) is a smooth solution of (1.1) defined on a maximal interval $[0, T_1)$ for which (3.3) is satisfied. Then for any $\varepsilon_0 > 0$ the set of points*

$$S := \{x \in M : \limsup_{t \nearrow T_1} E(u(t), B_R(x)) \geq \varepsilon_0 \text{ for all } R > 0\},$$

is finite.

In fact $\#S \leq E_0/\varepsilon_0$, since energy concentrates near points of S not just along a suitable sequence $t_j \nearrow T_1$ but indeed for *all* sequences $t \nearrow T_1$, compare (3.4).

Away from the finite set S we control the map component of the flow using the following lemma which should be seen as the analogue of Lemmas 3.10 and 3.10' of [15]

Lemma 3.5. *There exists a number $\varepsilon_0 > 0$ depending only on g_1 and E_0 such that the following statement holds true. Let (u, g) be a smooth solution of (1.1) on an open interval $(0, T)$ and assume that (3.3) is satisfied. Let $M' \subseteq M$ be an open set such that there exists a number $R > 0$ with*

$$E(u(t), B_R(x)) \leq \varepsilon_0 \quad \text{for all } (x, t) \in M' \times (0, T). \tag{3.5}$$

Then the parabolic Hölder-norms of u and its spatial derivatives (upto order $s - 2$) are bounded uniformly on the sets $[\tau, T] \times M'$, $\tau > 0$, with bounds depending only on τ, R, g_1, T, s, M' and the energy bound E_0.

Remark 3.6. If the initial map $u(0)$ is smooth on a neighbourhood of M' then the above result can be extended to give bounds on the Hölder norms of $u|_{M'}$ and its spatial derivatives on M' up to time $t = 0$, now with bounds depending additionally on $u(0)$, compare with remark 3.11 and 3.11' of [15].

Remark 3.7. Because of the non-local nature of the projection operator P_g these estimates on $u|_{M'}$ allow us to improve the regularity of $g|_{M'}$ from the a priori known $C^{0,1}$ dependence on time only in case $M = M'$. For $M' \neq M$ we can improve the bounds of Lemma 3.5 to give $C^{1,\alpha}$ bounds in time on $u|_{M'}$ and its spatial derivatives while for $M = M'$, i.e. away from singluar *times*, a bootstrapping argument gives estimates on any C^k norm (in space and time) of (u, g) in terms of the quantities specified in Lemma 3.5.

Proof of Lemma 3.5. For the proof of this lemma we follow largely the ideas of [15]. We make use of the well known interpolation estimate, see e.g. [2]

Lemma 3.8. *There are numbers $\varepsilon_0 > 0$ and $C < \infty$ (depending on (M, g_1) and the target manifold) such that for all maps $u \in H^2(M, N)$, a bound on the local energy of*

$$E(u, B_r(x)) \leq \varepsilon_0$$

implies an H^2-bound of the form

$$\int \varphi^2 \left|\nabla^2 u\right|^2 dv \leq \frac{C}{r^2} E(u, B_r(x)) + C \int \varphi^2 \left|\tau(u)\right|^2 dv, \tag{3.6}$$

as well as an estimate of

$$\int \varphi^2 \left|\nabla u\right|^4 dv \leq C E(u, B_r(x)) \cdot \left[\frac{1}{r^2} E(u, B_r(x)) + \int \varphi^2 \left|\tau(u)\right|^2 dv\right]. \tag{3.7}$$

Here $\varphi \in C_0^\infty(B_r(x))$ denotes a cut-off function.

Let now (u, g), M' and $R > 0$ be as in Lemma 3.5, let $x \in M'$ and choose a cut-off function $\varphi \in C_0^\infty(B_{R/2}(x))$. We first remark that for $\varepsilon_1 > 0$ sufficiently small, the pointwise bound $|\tau_g(u) - \tau(u)| \leq C\varepsilon_1(\left|\nabla^2 u\right| + |\nabla u|^2)$ implies that (3.6) and (3.7) remain valid with $\tau(u(t))$ replaced by $\tau_{g(t)}(u(t)) = \partial_t u(t)$.

As in [15] we now differentiate equation (1.1a) in time and multiply with $\varphi^2 \partial_t u$. After carefully analysing all occurring terms, in particular the terms due to the time-dependence of the metric, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi^2 \left|\partial_t u\right|^2 + \int \varphi^2 \left|\nabla \partial_t u\right|^2 &\leq (\frac{1}{4} + C\varepsilon_1) \cdot \int \varphi^2 \left|\nabla \partial_t u\right|^2 + C \int \varphi^2 \left|\partial_t u\right|^2 \left|\nabla u\right|^2 \\ &\quad + C(R)\big(1 + \|\frac{dg}{dt}\|_{C^2}^2 + \int \left|\partial_t u\right|^2\big).\end{aligned}$$

Since we know that the H^s norm, and thus also the C^2 norm, of $\frac{\mathrm{d}}{\mathrm{d}t} g$ is uniformly bounded by a multiple of the energy, we obtain that for $\varepsilon_1 = \varepsilon_1(g_1) > 0$ chosen small enough

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi^2 \left|\partial_t u\right|^2 + \int \varphi^2 \left|\nabla \partial_t u\right|^2 \leq C(R)(1 + \frac{dE}{dt}) + C \int \varphi^2 \left|\partial_t u\right|^2 \left|\nabla u\right|^2. \tag{3.8}$$

Using the Sobolev embedding $W^{1,1} \hookrightarrow L^2$ as well as Lemma 3.8, we find

$$\begin{aligned} C \int \varphi^2 \left|\partial_t u\right|^2 \left|\nabla u\right|^2 &\leq C \|\varphi \left|\partial_t u\right|^2\|_{L^2} \cdot \big(\int \varphi^2 \left|\nabla u\right|^4\big)^{1/2} \\ &\leq C(\|\nabla(\varphi \left|\partial_t u\right|^2)\|_{L^1} + \|\left|\partial_t u\right|^2\|_{L^1})\varepsilon_0^{1/2}\big[C(R) + \int \varphi^2 \left|\partial_t u\right|^2\big]^{1/2} \\ &\leq \int \varphi^2 \left|\nabla \partial_t u\right|^2 + C\varepsilon_0 \frac{dE}{dt} \int \varphi^2 \left|\partial_t u\right|^2 + C(R) \cdot (\frac{dE}{dt} + 1). \end{aligned}$$

Integrating the resulting estimate (3.8) over any interval $[t_1, t_2] \subset (0, T)$ we thus obtain

$$\int \varphi^2 \left|\partial_t u\right|^2 dv \Big|_{t_1}^{t_2} \leq C\varepsilon_0 \cdot \sup_{t \in [t_1, t_2]} \int \varphi^2 \left|\partial_t u\right|^2 dv + C(R, T).$$

After possibly reducing $\varepsilon_0 = \varepsilon_0(E_0, g_1) > 0$ so that the factor $C\varepsilon_0 \leq \frac{1}{2}$, we conclude that for any $\tau > 0$

$$\begin{aligned}\sup_{t \in [\tau, T)} \int_{B_{R/4}(x)} \left|\partial_t u\right|^2 dv &\leq 2 \inf_{t \in [0, \tau)} \int \varphi^2 \left|\partial_t u(t)\right|^2 dv + C(R, T) \\ &\leq 2\frac{E_0}{\tau} + C(R, T).\end{aligned}$$

Repeating the above argument for a finite cover of balls $B_{R/4}(x_i)$ of M' we obtain a uniform estimate of

$$\int_U |\partial_t u(t)|^2 \, dv \le C \qquad \text{for all } t \in [\tau, T)$$

on a small neighbourhood U of M'. According to Lemma 3.8 this implies a bound on $\int_{U'} \left|\nabla^2 u(t)\right|^2 dv$ on a slightly smaller neighbourhood of M'. Applying Sobolev's embedding theorem we then conclude that for any exponent $p < \infty$

$$\int_{U'} |\nabla u(t)|^p \, dv \le C_p \qquad t \in [\tau, T).$$

Then as in [15] we think of (1.1a) as an inhomogeneous heat equation

$$\partial_t u - \Delta_g u = A_g(u)(\nabla u, \nabla u) \in L^p(U' \times [\tau, T))$$

allowing us to apply standard regularity results for parabolic equations, see e.g. [8], chapter VII; we get bounds in the parabolic Sobolev-spaces $W_p^{2,1}$, and thus in the parabolic Hölder spaces C^α, on sets $M' \times [\tau', T]$, $\tau' > \tau$. We finally obtain estimates on the Hölder norms of spatial derivatives of u (up to order $s-2$) by a standard bootstrapping argument which relies on the strong bounds on the velocity of horizontal curves given in Lemma 2.9. □

Let now (u, g) be a smooth solution of (1.1) on $[0, T_1)$ whose metric component does not degenerate and thus smoothly converges $g(t) \to g(T_1) =: g_1 \in \mathcal{M}_{-1}$ as $t \nearrow T_1$. We first remark that the uniform bounds on the energies $E(u(t)) \le 2E(u(t), g(t)) \le 2E_0$ combined with the fact that $\partial_t u \in L^2([0, T_1) \times M)$ imply that the maps $u(t)$ converges weakly in $H^1(M)$ to a limit $u(T_1)$ as $t \nearrow T_1$. Additionally, Lemma 3.5 gives uniform Hölder bounds on u and its *spatial* derivatives away from the finite set S of concentration points so that $u(t)$ converges also in $C^\infty_{loc}(M \setminus S)$.

We now remark that any concentration of energy must be due to the so-called bubbling off of (at least) one harmonic sphere. Indeed, the analysis carried out in [15] (p. 578/9) remains unchanged as long as the local energy estimates and H^2 bounds used in [15] are replaced by Lemmas 3.3 and 3.8. We obtain the following: For any point $x_0 \in S$ there are sequences of times $t_i \nearrow T_1$, radii $r_i \to 0$ and points $x_i \to x_0$ with energies on balls around x_i of

$$E(u(t_i), B_{2r_i}(x_i)) \le \varepsilon_0 \quad \text{and} \quad E(u(t_i), B_{r_i}(x_i)) \ge c\varepsilon_0, \quad c = c(M, g_1) > 0$$

and with tension satisfying

$$r_i^2 \int_{B_{2r_i}(x_i)} |\tau(u(t_i))|^2 \to 0 \qquad \text{as } i \to \infty.$$

The rescaled maps $u_i(x) = u(\exp_{x_i}(r_i x), t_i)$, defined on larger and larger subsets of $\mathbb{R}^2$ are then bounded uniformly in H^2 and subconverge (weakly in H^2, strongly in $W^{1,p}$, $p < \infty$) to a non-constant harmonic map of finite energy that is defined on $\mathbb{R}^2 \cup \{\infty\} \cong S^2$, called a harmonic sphere or bubble. The amount of energy that concentrates near x_0, and that is consequently lost as we pass to the limit $t \nearrow T_1$, is no less than $\varepsilon^* = \varepsilon^*(N)$, the minimal energy of such a non-constant harmonic map from S^2 to the target.

Finally, as in [15], we (weakly) continue the flow past any such singular time by restarting from initial data $g(T_1) \in \mathcal{M}_{-1}$ and $u(T_1) \in H^1(M, N) \cap C^\infty_{loc}(M \setminus S)$ as follows: let $u_{j,0} \in C^\infty(M, N)$ be a sequence of maps that converge to $u(T_1)$ in $H^1(M)$ as well as in $C^\infty_{loc}(M \setminus S)$ and let (u_j, g_j) be the smooth solution of (1.1) corresponding to initial data $(u_{j,0}, g(T_1))$ that exists at least on some interval $[T_1, T_1 + \delta_j)$ according to Lemma 3.1. We remark that the metrics g_j are uniformly Lipschitz continuous and thus that the estimates derived above can be applied on $[T_1, T_1 + \min(\delta_j, \delta_0))$ for a number $\delta_0(g(T_1)) > 0$ independent of j.

We now choose $r > 0$ such that $\sup_{x\in M} E(u_{j,0}, B_r(x)) < \varepsilon_0/4$, which is possible due to the strong H^1-convergence of the initial maps. Then the local energy estimates of Lemma 3.3 imply that there is no concentration of energy and thus in particular no blow-up for any of the maps u_j, on a uniform interval $I = [T_1, T_1 + cr^2]$, $c = c(g_1) > 0$, in the sense that

$$E(u_j(t), B_{r/2}(x)) < \varepsilon_0 \text{ for all } j \in \mathbb{N},\ x \in M,\ t \in I.$$

According to Lemma 3.5 as well as remarks 3.6 and 3.7 we thus obtain uniform $C^{1,\alpha}$ estimates in time for the maps u_j (and their spatial derivatives) in every compact subset of $M\times[T_1, T_1+cr^2]\setminus(S\times\{T_1\})$. Away from the singular time, we furthermore get uniform bounds on all C^k norms of (u_j, g_j) in space-time. We conclude that a subsequence of (u_j, g_j) converges smoothly on $M \times (T_1, T_1 + cr^2]$ to a pair (u, g) which solves (1.1) classically on $(T_1, T_1+cr^2]$ and weakly on $[T_1, T_1+cr^2]$. This solution achieves the initial data $(u(T_1), g(T_1))$ in the sense that for $t \searrow T_1$ the maps $u(t)$ converges to $u(T_1)$ weakly in $H^1(M)$ and smoothly away from the set S while the metric component g is Lipschitz-continuous across the singular time. Since the energy of the approximating solutions (u_j, g_j) is no more than $E(u_{j,0}, g(T_1)) \to E(u(T_1), g(T_1))$, the extended weak solution (u, g) has non-increasing energy also across the singular time T_1. In particular the total number of all singular points $\cup_i S(T_i)\times\{T_i\}$ of such a solution is bounded by $\frac{E(u(0),g(0))}{\varepsilon^*}$. After possibly repeating the above argument to analyse any further singularities, we thus obtain a weak solution satisfying the properties (i)-(iii) of Theorem 1.1 and existing for as long as the metrics do not degenerate in moduli space.

4. Uniqueness of weak solutions

We finally discuss the issue of uniqueness of weak solutions. We prove that the solution (u, g) of (1.1) constructed in the previous section is uniquely determined by its initial data, not only among all solutions satisfying the properties of Theorem 1.1, but in the natural class of all weak solutions with non-increasing energy. We remark that a further argument as carried out in [10] actually gives uniqueness under the weaker assumption that the total energy does not instantaneously increase by more than a certain quantum at any time. We also remark that it is necessary to impose restrictions on the evolution of the total energy in view of the possibility of reverse bubbling, see [16].

So let $(u_i, g_i)_{i=1,2}$ be two weak solutions of (1.1) defined on an interval $[0, T)$ that evolve from the same intial data

$$(u_1, g_1)(0) = (u_0, g_0) = (u_2, g_2)(0) \in H^1(M, N) \times \mathcal{M}_{-1}$$

and assume that the total energies $t \mapsto E(u_{1,2}(t), g_{1,2}(t))$ are non-increasing. Since

$$I := \{t \in [0, T) : (u_1, g_1) \equiv (u_2, g_2) \text{ on } [0, t]\}$$

is trivially closed in $[0, T)$, we need to prove that I is also open.

Given any $t_0 \in I$ we recall that $g_i(t) \to g_i(t_0)$ in each $\mathcal{M}^s_{-1}$ and thus certainly uniformly as $t \searrow t_0$. Combined with the fact that $u_i(t) \to u_i(t_0)$ strongly in L^2 and weakly in H^1, we thus obtain

$$E(u_i(t_0), g_i(t_0)) \geq \lim_{t\searrow t_0} E(u_i(t), g_i(t)) = \lim_{t\searrow t_0} E(u_i(t), g_i(t_0)) \geq E(u_i(t_0), g_i(t_0)),$$

where we used the assumption on the evolution of the energy in the first step. Thus $u_i(t) \to u_i(t_0)$ indeed strongly in $H^1(M, g_0)$, which implies in particular that local energies, say on balls, converge as $t \searrow t_0$. Choosing a finite cover of balls $B_r(x_i)$, $i = 1 \ldots K$, of $(M, g_i(t_0))$ such that $E(u_i(t_0), g_i(t_0), B_{2r}(x_i)) \leq \varepsilon_0/2$, we may thus choose $\delta > 0$ so small that

$$E(u_i(t), g_i(t_0), B_{2r}(x_i)) \leq \varepsilon_0 \text{ for } t \in [t_0, t_0 + \delta],\ i = 1, 2.$$

Here we let $\varepsilon_0 > 0$ be the constant of Lemma 3.8.

It is now crucial to remark that on almost every time slice the functions $u_i(t)$ weakly solves an *almost harmonic map equation*, that is an equation of the form $\tau_g v = f$ for a function $f \in L^2$ and a metric g, here of course $g = g_i(t)$ and $f = \partial_t u_i(t)$. Since any weak solution of such an elliptic equation is contained in the Sobolev space H^2, see e.g. [10], Proposition 2.1, we may apply Lemma 3.8 on almost every time slice resulting in an estimate of

$$\int_M |\nabla u_i(t)|^4 + \left|\nabla^2 u_i(t)\right|^2 dv_{g_0} \le C(r)\cdot(1+\int_M |\partial_t u(t)|^2\, dv_{g(t)})$$

for $t \in [t_0, t_0+\delta)$ and $i = 1,2$. We can thus reduce the uniqueness statement in the general class of weak solutions with non-increasing energy to the following lemma whose analogue for the harmonic map flow was proven in [15]

Lemma 4.1. *Let (u_1, g_1) and (u_2, g_2) be weak solutions of (1.1) to the same initial data $(u_1, g_1)(0) = (u_2, g_2)(0)$ and suppose that*

$$\nabla u_i \in L^4(M\times[0,T)),\ \textit{and}\ \nabla^2 u_i \in L^2(M\times[0,T)) \quad i = 1,2. \tag{4.1}$$

Then $(u_1, g_1) \equiv (u_2, g_2)$.

Proof of Lemma 4.1. Using an open-closed argument as above it is enough to prove that the solutions agree on a possibly smaller interval $[0,\delta)$, which we can chose in particular such that the metrics $g_{1,2}$ are contained in an H^s neighbourhood of $g_0 = g_1(0) = g_2(0)$ for which Lemma 2.9 applies. Here s can be chosen to be any fixed number $s > 3$.

Notation: For the following computations we denote by $\|\cdot\|_{L^p}$ the $L^p(M, g_0)$ norm and by $d(\cdot,\cdot)$ the metric on $\mathcal{M}^s_{-1}$ respectively by $\|\cdot\|$ the H^s norm on $T\mathcal{M}^s_{-1}$. Furthermore, we use the short-hand notation of $|\nabla V| := \max\{|\nabla u_1|, |\nabla u_2|\}$ which, by assumption, is a function in $L^4(M\times[0,T])$ with L^2 norm on time-slices bounded by the energy, $\|\nabla V(t)\|^2_{L^2(M)} \le C\cdot(E(u_1,g_1)+E(u_2,g_2)) \le CE(u_0,g_0)$.

Subtracting the equations (1.1a) for the map components u_i we obtain that the difference $w = u_1 - u_2$ satisfies

$$\partial_t w - \Delta_{g_1} w = (\Delta_{g_1} - \Delta_{g_2})(u_2) + A_{g_1}(u_1)(\nabla u_1, \nabla u_1) - A_{g_2}(u_2)(\nabla u_2, \nabla u_2) \tag{4.2}$$

where A denotes the second fundamental form of the target $N \hookrightarrow \mathbb{R}^N$, $A_g(u)(\nabla u, \nabla u) := g^{ij}A(u)(\partial_i u, \partial_j u)$.

Following [15] we multiply equation (4.2) with w, integrate over the fixed surface (M, g_0) and estimate the resulting terms using Hölder's inequality. This leads to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2} \le\,& C\cdot d(g_1,g_0)\cdot\|\nabla w\|^2_{L^2}\\ &+ C\cdot d(g_1,g_2)\cdot\big(\|\nabla w\|_{L^2} + \|w\|_{L^2}\cdot(1+\|\nabla V\|^2_{L^4})\big)\\ &+ C\|\nabla w\|_{L^2}\cdot\|\nabla V\|_{L^4}\cdot\|w\|_{L^4} + C\|\nabla V\|^2_{L^4}\cdot\|w\|^2_{L^4}\\ \le\,& (\frac{1}{4} + C\cdot t)\|\nabla w\|^2_{L^2} + C\cdot d(g_1,g_2)^2\\ &+ C(1+\|\nabla V\|^2_{L^4})\cdot\|w\|^2_{L^4}.\end{aligned} \tag{4.3}$$

Using the Sobolev embedding $W^{1,1} \hookrightarrow L^2$, we may furthermore estimate

$$\|w\|^2_{L^4} = \|w^2\|_{L^2} \le C(\|\nabla(w^2)\|_{L^1} + \|w^2\|_{L^1}) \le C\cdot\|w\|_{L^2}\cdot(\|w\|_{L^2} + \|\nabla w\|_{L^2})$$

so that the last term on the right hand side of (4.3) is bounded by

$$C(1+\|\nabla V\|^2_{L^4})\cdot\|w\|^2_{L^4} \le \frac{1}{8}\|\nabla w\|^2_{L^2} + C\psi(t)\cdot\|w\|^2_{L^2},$$

for $\psi(t) = (\|\nabla V(t)\|_{L^4}^4 + 1) \in L^1([0,T])$.

In order to estimate the distance of the metric components g_1 and g_2 in terms of w we recall that the evolution of the tensor $g_1 - g_2$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_1 - g_2) = P_{g_1}(k(u_1, g_1)) - P_{g_2}(k(u_2, g_2)),$$

$k(u,g) = 2u^*G_N - 2e(u,g)g$. We have a pointwise estimate of the difference of the involved tensors of

$$|k(u_1, g_1) - k(u_2, g_2)| \leq C \cdot d(g_1, g_2) \cdot |\nabla V|^2 + C \cdot |w| \cdot |\nabla V|^2 + C \cdot |\nabla w| \cdot |\nabla V| .$$

Remark, that any L^2 estimate of this tensor would involve integrals of the form $\int |\nabla V|^4 |w|^2$ and $\int |\nabla V|^2 \cdot |\nabla w|^2$ which are *not* controlled by the quantities of the left hand side of (4.3). It thus crucial at this point that the improved bounds on P_g given in Lemma 2.9 only ask for L^1 bounds on the involved tensors, allowing us to estimate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} d(g_1, g_2) &\leq C \cdot d(g_1, g_2) \cdot \|k(u_1, g_1)\|_{L^1} + C \cdot \|k(u_1, g_1) - k(u_2, g_2)\|_{L^1} \\ &\leq C \cdot d(g_1, g_2) + C \cdot \|\nabla w\|_{L^2} + C \cdot \psi(t)^{1/2} \|w\|_{L^2}. \end{aligned} \tag{4.4}$$

Gronvall's lemma thus leads to an estimate of

$$\begin{aligned} d(g_1, g_2)(t)^2 &\leq C \cdot \left(\int_0^t \|\nabla w(s)\|_{L^2(M)}\, ds \right)^2 + C \left(\int_0^t \psi(s)^{1/2} \cdot \|w(s)\|_{L^2(M)}\, ds \right)^2 \\ &\leq t \cdot \int_0^t \int_M |\nabla w|^2 + C \int_0^t \psi(s) ds \cdot \int_0^t \int_M w^2, \end{aligned} \tag{4.5}$$

which we insert into (4.3). Integrating the resulting estimate over time, we find

$$\|w(t)\|_{L^2}^2 + \int_0^t \int |\nabla w|^2 \leq C \cdot \int_0^t \psi(s) ds \cdot \sup_{s \in [0,t]} \|w(s)\|_{L^2}^2 + Ct^2 \int_0^t \int |\nabla w|^2 .$$

Since ψ is integrable, we conclude that for all t sufficiently small, say $t \in (0, t_0)$,

$$\int |w(t)|^2 \leq \frac{1}{2} \sup_{s \in [0,t]} \int |w(s)|^2 .$$

Thus w must vanish identically on $(0, t_0)$ so $u_1 \equiv u_2$ and $g_1 \equiv g_2$ as desired. □

A. Appendix

A.1. **Solving the equation on a fixed Banach manifold.**

Let $(u_0, g_0) \in C^{2,\alpha}(M) \times \mathcal{M}_{-1}$, $\alpha > 0$ be given and let $s > 3$ be a fixed number. Here we outline an iteration argument that can be used to obtain a solution $(u, g) \in C^{2,1,\alpha}([0, \delta) \times M) \times C^1([0, \delta), \mathcal{M}_{-1}^s)$ of (1.1) for such initial data.

For $\delta_0 = \delta_0(u_0, g_0, s) > 0$ to be determined later, we extend u_0 to a constant in time map defined on $M \times [0, \delta_0^2)$ and define iteratively for $i = 1 \ldots$

- $g_i \in C^1([0, \delta_{i-1}^2], \mathcal{M}_{-1}^s)$ as the solution of $\frac{\mathrm{d}}{\mathrm{d}t} g_i = P_{g_i}(k(u_{i-1}, g_i))$ with $g_i(0) = g_0$;
- $u_i \in C^{2,1,\alpha}(M_{\delta_i})$ as the solution of $\partial_t u_i = \tau_{g_i}(u_i)$, $u_i(0) = u_0$, defined and smooth on a maximal domain $M_{\delta_i} := M \times [0, \delta_i^2)$, $\delta_i \leq \delta_{i-1}$.

Here we use the Lipschitz-continuity of the map P_g on the Banach manifold $\mathcal{M}_{-1}^s$ in the first step. We also remark that the equation for u_i is a semilinear parabolic equation so standard methods, see e.g. [9] Theorem 5.2.1, lead to the existence of a solution u_i of the above equation, defined on all of $[0, \delta_{i-1}^2)$ unless there is a blow-up in the gradient at some time δ_i^2, $0 < \delta_i < \delta_{i-1}$.

We claim that for δ_0 initially chosen small enough, the iterates are all defined on $[0,\delta_0^2)$ and satisfy

$$\begin{aligned}\|u_{i+1}-u_i\|^*_{C^{2,1,\alpha}(M_{\delta_0})} &\leq \frac{1}{2}\|u_i-u_{i-1}\|^*_{C^{2,1,\alpha}(M_{\delta_0})}\\ \|g_{i+1}-g_i\|^*_{C^1([0,\delta_0^2],\mathcal{M}^s_{-1})} &\leq C\cdot\delta_0\|u_{i+1}-u_i\|^*_{C^{2,1,\alpha}(M_{\delta_0})},\end{aligned} \tag{A.1}$$

thus converging to a classical solution $(u,g)\in C^{2,1,\alpha}([0,\delta_0^2)\times M)\times C^1([0,\delta_0^2),\mathcal{M}^s_{-1})$ in the limit $i\to\infty$. Here, we use scaling invariant versions of the standard parabolic Hölder norms, defined by $\|u\|^*_{C^{0,\alpha}(M_\delta)}=\|u\|_{C^0(M_\delta)}+\delta^\alpha[u]_{C^\alpha(M_\delta)}$ and more generally

$$\|u\|^*_{C^{a,b,\alpha}(M_\delta)}=\sum_{\substack{k+2j\leq a\\ j\leq b}}\delta^{2j+k}\|\partial_t^j\nabla^k u\|^*_{C^\alpha(M_\delta)},\qquad M_\delta=[0,\delta^2)\times M.$$

We remark that the second estimate of (A.1) immediately follows from Propostion 2.1 and the Gronvall lemma, compare with (4.4). To estimate $w_i=u_i-u_{i-1}$, we observe that

$$\partial_t w_i-L_iw_i=f_i \tag{A.2}$$

for the elliptic linear operator

$$L_iw:=\Delta_{g_i}w+A_{g_i}(u_{i-1})(\nabla u_{i-1},\nabla w)+\big(dA_{g_i}(u_{i-1})\big)(w)(\nabla u_{i-1},\nabla u_{i-1})$$

and a right hand side that is bounded in $C^{0,\alpha}(M_\delta)$ for any $\delta\leq\delta_i$ by

$$\delta^2\|f_i\|^*_{C^{0,\alpha}(M_\delta)}\leq C\|g_i-g_{i-1}\|^*_{C^1([0,\delta^2],\mathcal{M}^s_{-1})}+C(\|w_i\|^*_{C^{2,1,\alpha}(M_\delta)})^2$$

with a constant depending on a $C^{2,1,\alpha}$ bound on the previous iterate u_{i-1} but not on u_i. We then apply the following scaling invariant version of parabolic Schauder estimates

Proposition A.1. Let M be a closed manifold and let $\lambda>0$, $A<\infty$ be fixed. Then there exists a number $C<\infty$ such that the following holds true. Let L be any second order differential operator on M_δ, $\delta\in(0,1)$ any number, that is given in local coordinate charts as $Lu=\partial_{x_i}(a^{ij}\partial_{x_j}u)+b^i\partial_{x_i}u+cu$ with

$$a^{ij}(x,t)\xi_i\xi_j\geq\lambda|\xi|^2\quad\text{for all }\xi\in\mathbb{R}^m\text{ and }(x,t)\in M_\delta$$
$$\|a^{ij}\|^*_{C^{1,0,\alpha}(M_\delta)}+\delta\|b^i\|^*_{C^\alpha(M_\delta)}+\delta^2\|c\|^*_{C^\alpha(M_\delta)}\leq A.$$

Then the solution $u\in C^{2,1,\alpha}(M_\delta)$ of $\partial_t w-Lw=f\in C^\alpha(M_\delta), w(0)=0$ satisfies

$$\|w\|^*_{C^{2,1,\alpha}(M_\delta)}\leq C\delta^2\|f\|^*_{C^\alpha(M_\delta)}.$$

In terms of giving a proof of this result, we remark that standard Schauder estimates combined with a scaling argument give in a first step an estimate of the form

$$\|w\|^*_{C^{2,1,\alpha}(M_\delta)}\leq C\delta^2\|f\|^*_{C^\alpha(M_\delta)}+C\|w\|_{C^0(M_\delta)} \tag{A.3}$$

for constants C independent of δ. To retain this scaling invariance, we can then use the Ehrling-lemma and a further rescaling argument to estimate the second term of (A.3) by

$$\|w(t)\|_{C^0(M)}\leq\varepsilon\cdot\|w\|^*_{C^\alpha(M_\delta)}+C_\varepsilon\sup_{x\in M}\delta^{-1}\|w(t)\|_{L^2(B_\delta(x))}$$

on every time slice. Finally considering the evolution of local energy quantities of the form $\int\varphi^2\big[a^{ij}\partial_{x_i}w(t)\partial_{x_j}w(t)+\delta^{-2}w(t)^2\big]\,dx$, φ a cut-off function supported on balls of radius 2δ, gives that the last term in this estimate is bounded by a fixed (independent of δ) multiple of $\delta^2\|f\|_{L^\infty}$, completing the proof of Propostion A.1.

Turning back to the equation (A.2) satisfied by w_i, this Schauder-estimate allows us to conclude that for any $\delta<\delta_i$

$$\|w_i\|^*_{C^{2,1,\alpha}(M_\delta)}\leq C\delta\|w_{i-1}\|^*_{C^{2,1,\alpha}(M_\delta)}+C\big(\|w_i\|^*_{C^{2,1,\alpha}(M_\delta)}\big)^2.$$

Since $w_i(0) = 0$, the norm $\|w_i\|^*_{C^{2,1,\alpha}(M_\delta)}$ is small at least for δ small (a priori depending on i). We conclude that the first estimate of (A.3) holds true, initially for δ small and then, by a continuity argument, indeed for as long as the solution exists (provided $\delta_0 = \delta_0(u_0, g_0)$ was initially chosen small enough). But this very estimate prevents a blow-up before time δ_{i-1}, so that $\delta_i = \delta_{i-1} = .. = \delta_0$, completing the proof.

A.2. **Proof of Lemma 2.8.** We finally provide a possible proof of the fact that any metric in $\mathcal{M}^s_{-1}$ can be written in the form $f^*\bar{g}$ with $f \in \mathcal{D}^{s+1}$ and $\bar{g} \in \mathcal{M}_{-1}$.

Let $s > 3$ and let $\Omega \subset \mathcal{M}^s_{-1}$ be the subset of all metrics which can be written in the form $g = f^*\bar{g}$ for a smooth metric $\bar{g} \in \mathcal{M}_{-1}$ and a diffeomorphism f of class H^{s+1}.

We first prove that Ω is an open subset of $\mathcal{M}^s_{-1}$ using the slice theorem. Given any metric of the form $g_0 = f_0^*\tilde{g}_0$, $f_0 \in \mathcal{D}^{s+1}$ and $\tilde{g}_0 \in \mathcal{M}_{-1}$ we apply the slice-theorem 2.3 to the smooth metric $\tilde{g}_0$ resulting in an $\mathcal{M}^s_{-1}$-neighbourhood $\tilde{W}$ of $\tilde{g}_0$, consisting only of metrics of the form f^*g_S, g_S an element of a slice S around $\tilde{g}_0$ and thus in particular smooth. The pull-back $W = f_0^*\tilde{W}$ is then an $\mathcal{M}^s_{-1}$-neighbourhood of the original metric $g_0 \in \mathcal{M}^s_{-1}$, containing only metrics of the form $g = f_0^*(f^*g_S) = (f \circ f_0)^*g_S$, $g_S \in S \subset \mathcal{M}_{-1}$ and $f \circ f_0 \in \mathcal{D}^{s+1}$. So indeed $W \subset \Omega$ and Ω is open.

To see that Ω is also closed, we use a result due to Ebin and Palais which says that the action of $\mathcal{D}^{s+1}$ on $\mathcal{M}^s_{-1}$ is proper, see e.g. Theorem 2.3.1 in [17]; in practice this means that if we are given a sequence of diffeomorphisms $f_i \in \mathcal{D}^{s+1}$ and a convergent sequence of metrics $g_i \to g$ in $\mathcal{M}^s_{-1}$ then knowing that $f_i^*g_i \to \bar{g} \in \mathcal{M}^s_{-1}$ converges (in H^s topology) is enough to conclude that also (a subsequence of) the diffeomorphisms f_i converges, $f_i \to f$ in $\mathcal{D}^{s+1}$.

Let now $g \in \mathcal{M}^s_{-1}$ be such that there are diffeomorphisms $f_i \in \mathcal{D}^{s+1}$ and metrics $g_i \in \mathcal{M}_{-1}$ with $f_i^*g_i \to g$ (in $\mathcal{M}^s_{-1}$). This convergence implies in particular that the length $\ell(g_i) = \ell(f_i^*g_i)$ of the shortest closed geodesic of (M, g_i) is bounded away from zero. Thus the Mumford compactness theorem implies that after pulling back g_i by a *smooth* family of diffeomorphisms $\tilde{f}_i$, a subsequence of g_i converges smoothly

$$(\tilde{f}_i)^*g_i = (f_i^{-1} \circ \tilde{f}_i)^*(f_i^*g_i) \to \bar{g} \in \mathcal{M}_{-1}.$$

We conclude that the diffeomorphisms $f_i^{-1} \circ \tilde{f}_i$ converge to another diffeomorphism $f \in \mathcal{D}^{s+1}$ and thus that $g = (f^{-1})^*\bar{g} \in \Omega$.

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