Loop and Spin Foam Quantum Gravity:
A Brief Guide for Beginners

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Abstract:

We review aspects of loop quantum gravity and spin foam models at an introductory level, with special attention to questions frequently asked by non-specialists.

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1. Quantum Einstein gravity

The assumption that Einstein's classical theory of gravity can be quantised non-perturbatively is at the root of a wide variety of approaches to quantum gravity. The assumption constitutes the basis of several discrete methods [1], such as dynamical triangulations and Regge calculus, but it also implicitly underlies the older Euclidean path integral approach [2, 3] and the somewhat more indirect arguments which suggest that there may exist a non-trivial fixed point of the renormalisation group [4–6]. Finally, it is the key assumption which underlies loop and spin foam quantum gravity. Although the assumption is certainly far-reaching, there is to date no proof that Einstein gravity cannot be quantised non-perturbatively, either along the lines of one of the programs listed above or perhaps in an entirely different way.

In contrast to string theory, which posits that the Einstein-Hilbert action is only an effective low energy approximation to some other, more fundamental, underlying theory, loop and spin foam gravity take Einstein's theory in four spacetime dimensions as the basic starting point, either with the conventional or with a (constrained) 'BF-type' formulation.[1] These approaches are background independent in the sense that they do not presuppose the existence of a given background metric. In comparison to the older geometrodynamics approach (which is also formally background independent) they make use of many new conceptual and technical ingredients. A key role is played by the reformulation of gravity in terms of connections and holonomies. A related feature is the use of spin networks in three (for canonical formulations) and four (for spin foams) dimensions. These, in turn, require other mathematical ingredients, such as non-separable ('polymer') Hilbert spaces and representations of operators which are not weakly continuous. Undoubtedly, novel concepts and ingredients such as these will be necessary in order to circumvent the problems of perturbatively quantised gravity (that novel ingredients are necessary is, in any case, not just the point of view of LQG but also of most other approaches to quantum gravity). Nevertheless, it is important not to lose track of the physical questions that one is trying to answer.

Evidently, in view of our continuing ignorance about the 'true theory' of quantum gravity, the best strategy is surely to explore all possible avenues. LQG, just like the older geometrodynamics approach [7], addresses several aspects of the problem that are currently outside the main focus of string theory, in particular the question of background independence and the quantisation of geometry. Whereas there is a rather direct link between (perturbative) string theory and classical space-time concepts, and string theory can therefore rely on familiar notions and concepts, such as the notion of a particle and the S-matrix, the task is harder for LQG, as it must face up right away to the question of what an observable quantity is in the absence of a proper semi-classical space-time with fixed asymptotics.

The present text, which is based in part on the companion review [8], is intended as a brief introductory and critical survey of loop and spin foam quantum gravity [2], with special attention to some of the questions that are frequently asked by non-experts, but not always adequately emphasised (for our taste, at least) in the pertinent literature. For the canonical formulation of LQG, these concern in particular the definition and implementation of the Hamiltonian (scalar) constraint and its lack of uniqueness. An important question (which we will not even touch on here) concerns the consistent incorporation of matter couplings, and especially the question as to whether the consistent quantisation of gravity imposes any kind of restrictions on them. Establishing the existence of a semi-classical limit, in which classical space-time and the Einstein field equations are supposed to emerge, is widely regarded as the main open problem of the LQG approach. This is also a prerequisite for understanding the ultimate fate of the non-renormalisable UV divergences that arise in the conventional perturbative treatment. Finally, in any canonical approach there is the question whether one has succeeded in achieving (a quantum version of) full space-time covariance, rather than merely covariance under diffeomorphisms of the three-dimensional slices. In [8] we have argued (against a widely held view in the LQG community) that for this, it is not enough to check the closure of two Hamiltonian constraints on diffeomorphism invariant states, but that it is rather the *off-shell closure* of the constraint algebra that should be made the crucial requirement in establishing quantum space-time covariance.

Many of these questions have counterparts in the spin foam approach, which can be viewed as a 'space-time covariant version' of LQG, and at the same time as a modern variant of earlier attempts to define a discretised path integral in quantum gravity. For instance, the existence of a semi-classical limit is re-

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[1]In the remainder, we will often follow established (though perhaps misleading) custom and summarily refer to this framework of ideas simply as "Loop Quantum Gravity", or LQG for short.

[2]Whereas [8] is focused on the 'orthodox' approach to loop quantum gravity, to wit the Hamiltonian framework.

lated to the question whether the Einstein-Hilbert action can be shown to emerge in the infrared (long distance) limit, as is the case in (2+1) gravity in the Ponzano-Regge formulation, cf. eq. (38). Regarding the non-renormalisable UV divergences of perturbative quantum gravity, many spin foam practitioners seem to hold the view that there is no need to worry about short distance singularities and the like because the divergences are simply 'not there' in spin foam models, due to the existence of an intrinsic cutoff at the Planck scale. However, the same statement applies to any regulated quantum field theory (such as lattice gauge theory) before the regulator is removed, and on the basis of this more traditional understanding, one would therefore expect the 'correct' theory to require some kind of refinement (continuum) limit [3], or a sum 'over all spin foams' (corresponding to the 'sum over all metrics' in a formal path integral). If one accepts this point of view, a key question is whether it is possible to obtain results which do not depend on the specific way in which the discretisation and the continuum limit are performed (this is also a main question in other discrete approaches which work with reparametrisation invariant quantities, such as in Regge calculus). On the other hand, the very need to take such a limit is often called into question by LQG proponents, who claim that the discrete (regulated) model correctly describes physics at the Planck scale. However, it is then difficult to see (and, for gravity in (3+1) dimensions has not been demonstrated all the way in a single example) how a classical theory with all the requisite properties, and in particular full space-time covariance, can emerge at large distances. Furthermore, without considering such limits, and in the absence of some other unifying principle, one may well remain stuck with a multitude of possible models, whose lack of uniqueness simply mirrors the lack of uniqueness that comes with the need to fix infinitely many coupling parameters in the conventional perturbative approach to quantum gravity.

Obviously, a brief introductory text such as this cannot do justice to the numerous recent developments in a very active field of current research. For this reason, we would like to conclude this introduction by referring readers to several 'inside' reviews for recent advances and alternative points of view, namely [9–11] for the canonical formulation, [12–14] for spin foams, and [15] for both. A very similar point of view to ours has been put forward in [16, 17][4]. Readers are also invited to have a look at [18] for an update on the very latest developments in the subject.

2. The kinematical Hilbert space of LQG

There is a general expectation (not only in the LQG community) that at the very shortest distances, the smooth geometry of Einstein's theory will be replaced by some quantum space or spacetime, and hence the continuum will be replaced by some 'discretuum'. Canonical LQG does not do away with conventional spacetime concepts entirely, in that it still relies on a spatial continuum Σ as its 'substrate', on which holonomies and spin networks live (or 'float') — of course, with the idea of eventually 'forgetting about it' by considering abstract spin networks and only the combinatorial relations between them. On this substrate, it takes as the classical phase space variables the holonomies of the Ashtekar connection,

$$h_e[A] = \mathcal{P} \exp \int_e A_m^a \tau_a \mathrm{d}x^m \,,$$

$$\text{with} \quad A_m^a := -\tfrac{1}{2}\epsilon^{abc}\omega_{mbc} + \gamma\, K_m^a \,. \quad (1)$$

Here, τ_a are the standard generators of SU(2) (Pauli matrices), but one can also replace the basic representation by a representation of arbitrary spin, denoted by $\rho_j(h_e[A])$. The Ashtekar connection A is thus a particular linear combination of the spin connection ω_{mbc} and the extrinsic curvature K_m^a which appear in a standard (3+1) decomposition. The parameter γ is the so-called Barbero-Immirzi parameter. The variable conjugate to the Ashtekar connection turns out to be the inverse densitised dreibein $\tilde{E}_a{}^m := e\, e_a{}^m$. Using this conjugate variable, one can find the objects which are conjugate to the holonomies. These are given by integrals of the associated two-form over two-dimensional surfaces S embedded in Σ,

$$F_S[\tilde{E}, f] := \int_S \epsilon_{mnp}\tilde{E}_a^m\, f^a\, \mathrm{d}x^n \wedge \mathrm{d}x^p \,, \quad (2)$$

where $f^a(x)$ is a test function. This flux vector is indeed conjugate to the holonomy in the sense described in figure 1: if the edge associated to the holonomy intersects the surface associated to the flux, the Poisson bracket between the two is non-

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[3]Unless quantum gravity is ultimately a *topological* theory, in which case the sequence of refinements becomes stationary. Such speculations have also been entertained in the context of string and M theory.

[4]However, [16, 17] only addresses the so-called m-ambiguity, whereas we will argue that there are infinitely many other parameters which a microscopic theory of quantum gravity must fix.
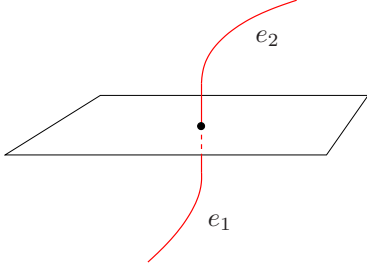
Figure 1: LQG employs holonomies and fluxes as elementary conjugate variables. When the edge of the holonomy and the two-surface element of the flux intersect, the canonical Poisson bracket of the associated operators is non-vanishing, and inserts a τ-matrix at the point of intersection, cf. (3).

zero,

$$\left\{ (h_e[A])_{\alpha\beta}, F_S[\tilde{E}, f] \right\}$$

$$= \pm\, \gamma\, f_a(P) \left(h_{e_1}[A]\, \tau^a\, h_{e_2}[A] \right)_{\alpha\beta}, \quad (3)$$

where $e = e_1 \cup e_2$ and the sign depends on the relative orientation of the edge and the two-surface. This Poisson structure is the one which gets promoted to a commutator algebra in the quantum theory.

Instead of building a Hilbert space as the space of functions over configurations of the Ashtekar connection, i.e. instead of constructing wave-functionals $\Psi[A_m(\mathbf{x})]$, LQG uses a Hilbert space of wave functionals which "probe" the geometry only on one-dimensional submanifolds, so-called *spin networks*. The latter are (not necessarily connected) graphs Γ embedded in Σ consisting of finitely many edges (links). The wave functionals are functionals over the space of holonomies. In order to make them $\mathbb{C}$-valued, the SU(2) indices of the holonomies have to be contracted using invariant tensors (i.e. Clebsch-Gordan coefficients). The wave function associated to the spin network in figure 2 is, for instance, given by

$$\Psi[\text{fig.2}] = \left(\rho_{j_1}(h_{e_1}[A]) \right)_{\alpha_1\beta_1} \left(\rho_{j_2}(h_{e_2}[A]) \right)_{\alpha_2\beta_2}$$

$$\times \left(\rho_{j_3}(h_{e_3}[A]) \right)_{\alpha_3\beta_3} C^{j_1 j_2 j_3}_{\beta_1\beta_2\beta_3} \dots, \quad (4)$$

where dots represent the remainder of the graph. The spin labels $j_1, \dots$ must obey the standard rules for the vector addition of angular momenta, but otherwise can be chosen arbitrarily. The spin network wave functions Ψ are thus labelled by Γ (the spin network graph), by the spins $\{j\}$ attached to the edges, and the intertwiners $\{C\}$ associated to the vertices.

At this point, we have merely defined a space of wave functions in terms of rather unusual variables, and it now remains to define a proper Hilbert space structure on them. The discrete kinematical structure which LQG imposes does, accordingly, *not* come from the description in terms of holonomies and fluxes. After all, this very language can also be used to describe ordinary Yang-Mills theory. The discrete structure which LQG imposes is also entirely different from the discreteness of a lattice or naive discretisation of space (*i.e.* of a finite or countable set). Namely, it arises by 'polymerising' the continuum via an unusual *scalar product*. For any two spin network states, one defines this scalar product to be

$$\left\langle \Psi_{\Gamma,\{j\},\{C\}} \,\middle|\, \Psi'_{\Gamma',\{j'\},\{C'\}} \right\rangle$$

$$= \begin{cases} 0 & \text{if } \Gamma \neq \Gamma', \\ \displaystyle\int \prod_{e_i \in \Gamma} \mathrm{d}h_{e_i}\, \bar{\psi}_{\Gamma,\{j\},\{C\}}\, \psi'_{\Gamma',\{j'\},\{C'\}} & \text{if } \Gamma = \Gamma', \end{cases}$$

$$(5)$$

where the integrals $\int \mathrm{d}h_e$ are to be performed with the SU(2) Haar measure. The spin network wave functions ψ depend on the Ashtekar connection only through the holonomies. The *kinematical Hilbert space* $\mathcal{H}_{\mathrm{kin}}$ is then defined as the completion of the space of spin network wave functions w.r.t. this scalar product (5). The topology induced by the latter is similar to the discrete topology ('pulverisation') of the real line with countable unions of points as the open sets. Because the only notion of 'closeness' between two points in this topology is whether or not they are coincident, whence *any* function is continuous in this topology, this raises the question as to how one can recover conventional notions of continuity in this scheme.

The very special choice of the scalar product (5) leads to representations of operators which need not be weakly continuous: this means that expectation values of operators depending on some parameter do not vary continuously as these parameters are varied. Consequently, the Hilbert space does not admit a countable basis, hence is *non-separable*, because the set of all spin network graphs in Σ is uncountable, and non-coincident spin networks are orthogonal w.r.t. (5). Therefore, any operation (such as a diffeomorphism) which moves around graphs continuously corresponds to an uncountable sequence of mutually orthogonal states in $\mathcal{H}_{\mathrm{kin}}$. That is, no matter how 'small' the deformation of the graph in Σ, the associated elements of $\mathcal{H}_{\mathrm{kin}}$ always remain a finite distance apart, and consequently, the continuous motion in 'real space' gets mapped to a highly discontinuous one in $\mathcal{H}_{\mathrm{kin}}$. Although unusual, and per-
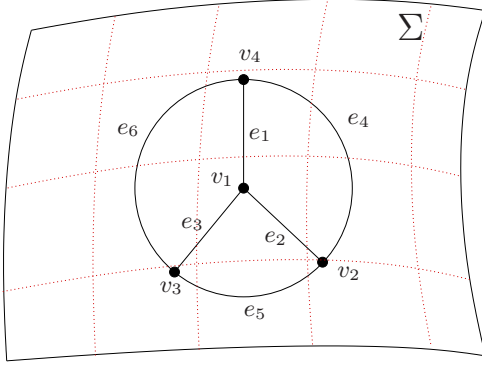
Figure 2: A simple spin network, embedded in the spatial hypersurface Σ. The hypersurface is only present in order to provide coordinates which label the positions of the vertices and edges. Spin network wave functions only probe the geometry along the one-dimensional edges and are insensitive to the geometry elsewhere on Σ.

haps counter-intuitive, as they are, these properties constitute a cornerstone for the hopes that LQG can overcome the seemingly unsurmountable problems of conventional geometrodynamics: if the representations used in LQG were equivalent to the ones of geometrodynamics, there would be no reason to expect LQG not to end up in the same quandary.

Because the space of quantum states used in LQG is very different from the one used in Fock space quantisation, it becomes non-trivial to see how semi-classical 'coherent' states can be constructed, and how a smooth classical spacetime might emerge. In simple toy examples, such as the harmonic oscillator, it has been shown that the LQG Hilbert space indeed admits states (complicated linear superpositions) whose properties are close to those of the usual Fock space coherent states [19]. In full (3+1)-dimensional LQG, the classical limit is, however, far from understood (so far only kinematical coherent states are known [20–25], i.e. states which do not satisfy the quantum constraints). In particular, it is not known how to describe or approximate classical spacetimes in this framework that 'look' like, say, Minkowski space, or how to properly derive the classical Einstein equations and their quantum corrections. A proper understanding of the semi-classical limit is also indispensable to clarify the connection (or lack thereof) between conventional perturbation theory in terms of Feynman diagrams, and the non-perturbative quantisation proposed by LQG.

However, the truly relevant question here concerns the structure (and definition!) of *physical* space and time. This, and not the kinematical 'discretuum' on which holonomies and spin networks 'float', is the

arena where one should try to recover familiar and well-established concepts like the Wilsonian renormalisation group, with its *continuous* 'flows'. Because the measurement of lengths and distances ultimately requires an operational definition in terms of appropriate matter fields and states obeying the physical state constraints, 'dynamical' discreteness is expected to manifest itself in the spectra of the relevant physical observables. Therefore, let us now turn to a discussion of the spectra of three important operators and to the discussion of physical states.

3. Area, volume and the Hamiltonian

In the current setup of LQG, an important role is played by two relatively simple operators: the 'area operator' measuring the area of a two-dimensional surface $S \subset \Sigma$, and the 'volume operator' measuring the volume of a three-dimensional subset $V \subset \Sigma$. The latter enters the definition of the Hamiltonian constraint in an essential way. Nevertheless, it must be emphasised that the area and volume operators are *not* observables in the Dirac sense, as they do not commute with the Hamiltonian. To construct *physical* operators corresponding to area and volume is more difficult and would require the inclusion of matter (in the form of 'measuring rod fields').

The area operator is most easily expressed as

$$A_S[g] = \int_S \sqrt{\mathrm{d}F^a \cdot \mathrm{d}F^a}\,,$$

$$\text{with} \quad \mathrm{d}F_a := \epsilon_{mnp}\tilde{E}_a^m \mathrm{d}x^n \wedge \mathrm{d}x^p \quad (6)$$

(the area element is here expressed in terms of the new 'flux variables' $\tilde{E}_a^m$, but is equal to the standard expression $\mathrm{d}F_a := \epsilon_{abc}e_m{}^b e_n{}^c \mathrm{d}x^m \wedge \mathrm{d}x^n$). The next step is to re-write this area element in terms of the spin network variables, in particular the momentum $\tilde{E}_a{}^m$ conjugate to the Ashtekar connection. In order to do so, we subdivide the surface into infinitesimally small surfaces S_I as in figure 3. Next, one approximates the area by a Riemann sum (which, of course, converges for well-behaved surfaces S), using

$$\int_{S_I} \sqrt{\mathrm{d}F^a \cdot \mathrm{d}F^a} \approx \sqrt{F_{S_I}^a[\tilde{E}]\, F_{S_I}^a[\tilde{E}]}\,. \quad (7)$$

This turns the operator into the expression

$$A_S[\tilde{E}_m^a] = \lim_{N\to\infty} \sum_{I=1}^N \sqrt{F_{S_I}^a[\tilde{E}]\, F_{S_I}^a[\tilde{E}]}\,. \quad (8)$$

If one applies the operator (8) to a wave function associated with a fixed graph Γ and refines it in such
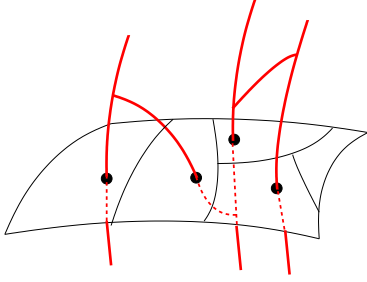
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Figure 3: The computation of the spectrum of the area operator involves the division of the surface into cells, such that at most one edge of the spin network intersects each given cell.

a way that each elementary surface S_I is pierced by only *one* edge of the network, one obtains, making use of (3) twice,

$$\hat{A}_S \Psi = 8\pi l_p^2 \gamma \sum_{p=1}^{\#\text{edges}} \sqrt{j_p(j_p+1)}\, \Psi\,.\qquad(9)$$

These spin network states are thus eigenstates of the area operator. The situation becomes considerably more complicated for wave functions which contain a spin network vertex which lies in the surface S; in this case the area operator does not necessarily act diagonally anymore (see figure 4). Expression (9) lies at the core of the statement that areas are quantised in LQG.

The construction of the volume operator follows similar logic, although it is substantially more involved. One starts with the classical expression for the volume of a three-dimensional region $\Omega \subset \Sigma$,

$$V(\Omega) = \int_{\Omega} \mathrm{d}^3x \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\epsilon^{mnp}\tilde{E}_m^a \tilde{E}_n^b \tilde{E}_p^c\right|}\,.\qquad(10)$$

Just as with the area operator, one partitions Ω into small cells $\Omega = \cup_I \Omega_I$, so that the integral can be replaced with a Riemann sum. In order to express the volume element in terms of the canonical quantities introduced before, one then again approximates the area elements $\mathrm{d}F^a$ by the small but finite area operators $F_S^a[\tilde{E}]$, such that the volume is obtained as the limit of a Riemann sum

$$V(\Omega) = \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{\left|\frac{1}{3!}\epsilon_{abc}F_{S_I^1}^a[\tilde{E}]\,F_{S_I^2}^b[\tilde{E}]\,F_{S_I^3}^c[\tilde{E}]\right|}\,.\qquad(11)$$

The main problem is now to choose appropriate surfaces $S_{1,2,3}$ in each cell. This should be done in such a way that the r.h.s. of (11) reproduces the correct

classical value. For instance, one can choose a point inside each cube Ω_I, then connect these points by lines and 'fill in' the faces. In each cell Ω_I one then has three lines labelled by $a = 1, 2, 3$; the surface S_I^a is then the one that is traversed by the a-th line. With this choice it can be shown that the result is insensitive to small 'wigglings' of the surfaces, hence independent of the shape of S_I^a, and the above expression converges to the desired result. See [26, 27] for some recent results on the spectrum of the volume operator.

The key problem in canonical gravity is the definition and implementation of the Hamiltonian (scalar) constraint operator, and the verification that this operator possesses all the requisite properties. The latter include (quantum) space-time covariance as well as the existence of a proper semi-classical limit, in which the classical Einstein equations are supposed to emerge. It is this operator which replaces the Hamiltonian evolution operator of ordinary quantum mechanics, and encodes all the important dynamical information of the theory (whereas the Gauss and diffeomorphism constraints are merely 'kinematical'). More specifically, together with the kinematical constraints, it defines the *physical states* of the theory, and thereby the physical Hilbert space $\mathcal{H}_{\text{phys}}$ (which may be separable [28], even is $\mathcal{H}_{\text{kin}}$ is not).

To motivate the form of the *quantum Hamiltonian* one starts with the classical expression, written in loop variables. To this aim one re-writes the Hamiltonian in terms of Ashtekar variables, with the result

$$H[N] = \int_{\Sigma} \mathrm{d}^3x\, N \frac{\tilde{E}_a^m \tilde{E}_b^n}{\sqrt{\det \tilde{E}}}\Big(\epsilon^{abc}F_{mnc}$$
$$-\frac{1}{2}(1+\gamma^2)K_{[m}{}^a K_{n]}{}^b\Big)\,.\quad(12)$$

For the special values $\gamma = \pm i$, the last term drops out, and the Hamiltonian simplifies considerably. This was indeed the value originally proposed by Ashtekar, and it would also appear to be the natural one required by local Lorentz invariance (as the Ashtekar variable is, in this case, just the pullback of the four-dimensional spin connection). However, imaginary γ obviously implies that the phase space of general relativity in terms of these variables would have to be complexified, such that the original phase space could be recovered only after imposing a reality constraint. In order to avoid the difficulties related to quantising this reality constraint, γ is now usually taken to be real. With this choice, it becomes much more involved to rewrite (12) in terms of loop and flux variables.
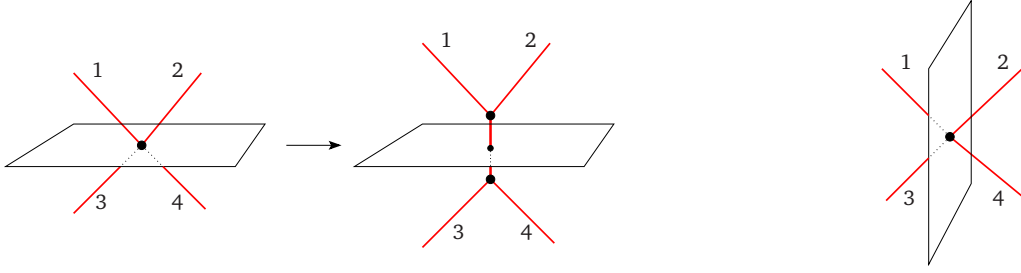
Figure 4: The action of the area operator on a node with intertwiner $C^{j_1 j_2 k}_{\alpha_1 \alpha_2 \beta} C^{j_3 j_4 k}_{\alpha_3 \alpha_4 \beta}$. Whether or not this action is diagonal depends on the orientation of the surface associated to the area operator. In the figure on the left, the location of the edges with respect to the surface is such that the invariance of the Clebsch-Gordan coefficients can be used to evaluate the action of the area operator. The result can be written in terms of a "virtual" edge. In the figure on the right, however, this is not the case, a recoupling relation is needed, and the spin network state is not an eigenstate of the corresponding area operator.

4. Implementation of the constraints

In canonical gravity, the simplest constraint is the Gauss constraint. In the setting of LQG, it simply requires that the SU(2) representation indices entering a given vertex of a spin network enter in an SU(2) invariant manner. More complicated are the diffeomorphism and Hamiltonian constraint. In LQG these are implemented in two entirely different ways. Moreover, the implementation of the Hamiltonian constraint is not completely independent, as its very definition relies on the existence of a subspace of diffeomorphism invariant states.

Let us start with the diffeomorphism constraint. Unlike in geometrodynamics, one cannot immediately write down formal states which are manifestly diffeomorphism invariant, because the spin network functions are not supported on all of Σ, but only on one-dimensional links, which 'move around' under the action of a diffeomorphism. A formally diffeomorphism invariant state is obtained by 'averaging' over the diffeomorphism group, and more specifically by considering the formal sum

$$\eta(\Psi)[A] := \sum_{\phi \in \mathrm{Diff}(\Sigma|\Gamma)} \Psi_\Gamma[A \circ \phi]. \qquad (13)$$

Here $\mathrm{Diff}(\Sigma|\Gamma)$ is obtained by dividing out the diffeomorphisms leaving invariant the graph Γ. Although this is a continuous sum which might seem to be ill-defined, it can be given a mathematically precise meaning because the unusual scalar product (5) ensures that the inner product between a state and a diffeomorphism-averaged state,

$$\langle \eta(\Psi_{\Gamma'}) | \Psi_\Gamma \rangle = \sum_{\phi \in \mathrm{Diff}(\Sigma|\Gamma')} \langle \phi^* \circ \Psi_{\Gamma'} | \Psi_\Gamma \rangle \qquad (14)$$

consists at most of a *finite* number of terms. It is this fact which ensures that $\langle \eta(\Psi_\Gamma)|$ is indeed well-

defined as an element of the space dual to the space of spin networks (which is dense in $\mathcal{H}_{\mathrm{kin}}$). In other words, although $\eta(\Psi)$ is certainly outside of $\mathcal{H}_{\mathrm{kin}}$, it does make sense *as a distribution*. On the space of diffeomorphism averaged spin network states (regarded as a subspace of a distribution space) one can now again introduce a Hilbert space structure 'by dividing out' spatial diffeomorphisms, namely

$$\langle\!\langle \eta(\Psi)|\eta(\Psi')\rangle\!\rangle := \langle \eta(\Psi)|\Psi'\rangle. \qquad (15)$$

The completion by means of this scalar product defines the space $\mathcal{H}_{\mathrm{diff}}$; but note that $\mathcal{H}_{\mathrm{diff}}$ is *not* a subspace of $\mathcal{H}_{\mathrm{kin}}$!

As we said above, however, it is the Hamiltonian constraint which plays the key role in canonical gravity, as it this operator which encodes the dynamics. Implementing this constraint on $\mathcal{H}_{\mathrm{diff}}$ or some other space is fraught with numerous choices and ambiguities, inherent in the construction of the quantum Hamiltonian as well as the extraordinary complexity of the resulting expression for the constraint operator [29]. The number of ambiguities can be reduced by invoking independence of the spatial background [10], and indeed, without making such choices, one would not even obtain sensible expressions. In other words, the formalism is partly 'on-shell' in that the very existence of the (unregulated) Hamiltonian constraint operator depends very delicately on its 'diffeomorphism covariance', and the choice of a proper 'habitat', on which it is supposed to act in a well defined manner. A further source of ambiguities, which, for all we know, has not been considered in the literature so far, consists in possible $\hbar$-dependent 'higher order' modifications of the Hamiltonian, which might still be compatible with all consistency requirements of LQG.

In order to write the constraint in terms of only holonomies and fluxes, one has to eliminate the in-

verse square root $\tilde{E}^{-1/2}$ in (12) as well as the extrinsic curvature factors. This can be done through a number of tricks found by Thiemann [30]. The vielbein determinant is eliminated using

$$\epsilon_{mnp}\epsilon^{abc}\tilde{E}^{-1/2}\tilde{E}_b{}^n\tilde{E}_c{}^p = \frac{1}{4\gamma}\left\{A_m{}^a(\mathbf{x}), V\right\}. \quad (16)$$

where $V \equiv V(\Sigma)$ is the total volume, cf. (10). The extrinsic curvature is eliminated by writing it as

$$K_m{}^a(\mathbf{x}) = \frac{1}{\gamma}\left\{A_m{}^a(\mathbf{x}), \bar{K}\right\}$$

$$\text{where} \qquad \bar{K} := \int_\Sigma \mathrm{d}^3x\, K_m{}^a\tilde{E}_a{}^m, \quad (17)$$

and then eliminating the integrand of $\bar{K}$ using

$$\begin{aligned}
\bar{K}(\mathbf{x}) &= \frac{1}{\gamma^{3/2}}\left\{\frac{\tilde{E}_a{}^m\tilde{E}_b{}^n}{\sqrt{\tilde{E}}}\epsilon^{abc}F_{mnc}(\mathbf{x}), V\right\} \\
&= \frac{1}{4\gamma^{5/2}}\epsilon^{mnp}\left\{\{A_m{}^a, V\}F_{npa}, V\right\},
\end{aligned} \quad (18)$$

that is, writing it as a nested Poisson bracket. Inserting these tricks into the Hamiltonian constraint, one replaces (12) with the expression

$$\begin{aligned}
H[N] = \int_\Sigma \mathrm{d}^3x\, N\epsilon^{mnp}\, \mathrm{Tr}\,\Big(& F_{mn}\{A_p, V\} \\
&- \frac{1}{2}(1+\gamma^2)\{A_m, \bar{K}\}\{A_n, \bar{K}\}\{A_p, V\}\Big), \quad (19)
\end{aligned}$$

with $\bar{K}$ understood to be eliminated using (18). This expression, which now contains only the connection A and the volume V, is the starting point for the construction of the quantum constraint operator.

In order to quantise the classical Hamiltonian (19), one next elevates all classical objects to quantum operators as described in the foregoing sections, and replaces the Poisson brackets in (19) by quantum commutators. The resulting *regulated Hamiltonian* then reduces to a sum over the vertices v_α of the spin network with lapses $N(v_\alpha)$

$$\begin{aligned}
\hat{H}[N, \epsilon] = \sum_\alpha N(v_\alpha)\, \epsilon^{mnp} \\
\times \mathrm{Tr}\,\Big\{\left(h_{\partial P_{mn}(\epsilon)} - h^{-1}_{\partial P_{mn}(\epsilon)}\right)h_p^{-1}\left[h_p, \hat{V}\right] \\
-\tfrac{1}{2}(1+\gamma^2)h_m^{-1}\left[h_m, \bar{K}\right]h_n^{-1}\left[h_n, \bar{K}\right]h_p^{-1}\left[h_p, \hat{V}\right]\Big\}, \\
(20)
\end{aligned}$$

where $\partial P_{mn}(\epsilon)$ is a small loop attached to the vertex v_α that must eventually be shrunk to zero. In writing
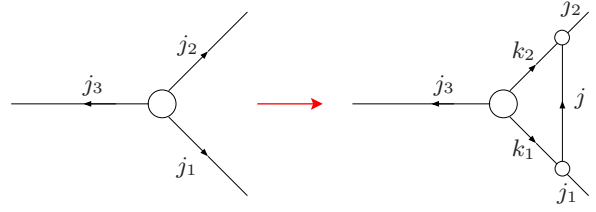

Figure 5: Schematic depiction of the action of the Hamiltonian constraint on a vertex of a spin network wave function. Two new vertices are introduced, and the original vertex is modified. Note that in order for this to be true, particular choices have been made in the quantisation prescription.

the above expression, we have furthermore assumed a specific (but, at this point, not specially preferred) ordering of the operators.

Working out the action of (20) on a given spin network wave function is rather non-trivial, and we are not aware of any concrete calculations in this regard, other than for very simple special configurations (see e.g. [31]); to get an idea of the complications, readers may have a look at a recent analysis of the volume operator and its spectrum in [32]. In particular, the available calculations focus almost exclusively on the action of the first term in (20), whereas the second term (consisting of multiply nested commutators, cf. (18)) is usually not discussed in any detail. At any rate, this calculation involves a number of choices in order to fix various ambiguities, such as e.g. the ordering ambiguities in both terms in (20). An essential ingredient is the action of the operator $h_{\partial P_{mn}(\epsilon)} - h^{-1}_{\partial P_{mn}(\epsilon)}$, which is responsible for the addition of a plaquette to the spin network. The way in which this works is depicted (schematically) in figure 5. The plaquette is added in a certain SU(2) representation, corresponding to the representation of the trace in (20). This representation label j is arbitrary, and constitutes a quantisation ambiguity (often called 'm-ambiguity').

Having defined the action of the regulated Hamiltonian, the task is not finished, however, because one must still take the limit $\epsilon \to 0$, in which the attached loops are shrunk to zero. As it turns out, this limit cannot be taken straightforwardly: due to the scalar product (5) and the non-separability of $\mathcal{H}_{\mathrm{kin}}$ the limiting procedure runs through a sequence of mutually orthogonal states, and therefore does not converge in $\mathcal{H}_{\mathrm{kin}}$. For this reason, LQG must resort to a weaker notion of limit, either by defining the limit as a weak limit on a (subspace of the) algebraic dual of a dense subspace of $\mathcal{H}_{\mathrm{kin}}$ [11, 33], or by taking the limit in the weak * operator topology [10]. In the first case the relevant space (sometimes referred to
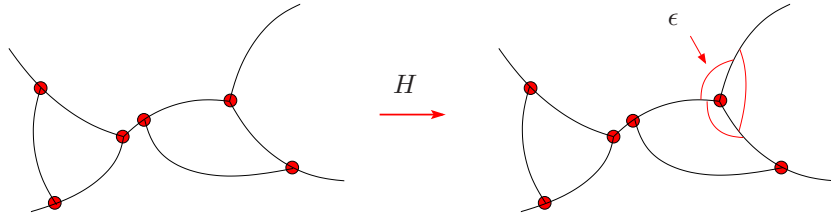
Figure 6: The action of the Hamiltonian constraint is "ultra-local", in the sense that it acts only in a neighbourhood of "size" ϵ around a spin network vertex.

as the 'habitat') is a distribution space which contains the space $\mathcal{H}_{\mathrm{diff}}$ of formally diffeomorphism invariant states as a subspace, but its precise nature and definition is still a matter of debate. In the second case, the limit is implemented (in a very weak sense) on the original kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$, but that space will not contain any diffeomorphism invariant states other than the 'vacuum' $\Psi = \mathbf{1}$. The question of the proper 'habitat' on which to implement the action of the Hamiltonian constraint is thus by no means conclusively settled.

From a more general point of view, it should be noted that the action of the Hamiltonian constraint is always 'ultralocal': all changes to the spin network are made in an $\epsilon \to 0$ neighbourhood of a given vertex, while the spin network graph is kept fixed [34–36]. Pictorially speaking, the only action of the (regulated) Hamiltonian is to dress up the vertices with 'spiderwebs', see figure 6. More specifically, it has been argued [33] that the Hamiltonian acts at a particular vertex only by changing the intertwiners at that vertex. This is in stark contrast to what happens in lattice field theories. There the action of the Hamiltonian always links two different existing nodes, the plaquettes are by construction always spanned between existing nodes, and the continuum limit involves the lattice as a whole, not only certain sub-plaquettes that shrink to a vertex. This is also what one would expect on physical grounds for a theory with non-trivial dynamics.

The attitude often expressed with regard to the ambiguities in the construction of the Hamiltonian is that they correspond to *different* physics, and therefore the choice of the correct Hamiltonian is ultimately a matter of physics (experiment?), and not mathematics. However, it appears unlikely to us that Nature will allow such a great degree of arbitrariness at its most fundamental level: in fact, our main point here is that the infinitely many ambiguities which killed perturbative quantum gravity, are also a problem that other (to wit, non-perturbative) approaches must address and solve. [5]

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[5]The abundance of 'consistent' Hamiltonians and spin foam

5. Quantum space-time covariance?

Spacetime covariance is a central property of Einstein's theory. Although the Hamiltonian formulation is not manifestly covariant, full covariance is still present in the classical theory, albeit in a hidden form, via the classical (Poisson or Dirac) algebra of constraints acting on phase space. However, this is not necessarily so for the quantised theory. As we explained, LQG treats the diffeomorphism constraint and the Hamiltonian constraint in a very different manner. Why and how then should one expect such a theory to recover full spacetime (as opposed to purely spatial) covariance? The crucial issue here is clearly what LQG has to say about the quantum algebra of constraints. Unfortunately, to the best of our knowledge, the 'off-shell' calculation of the commutator of two Hamiltonian constraints in LQG – with an explicit operatorial expression as the final result – has never been fully carried out. Instead, a survey of the possible terms arising in this computation has led to the conclusion that the commutator vanishes on a certain restricted 'habitat' of states [33, 37, 38], and that therefore the LQG constraint algebra closes without anomalies. By contrast, we have argued in [8] that this 'on shell closure' is not sufficient for a full proof of *quantum spacetime covariance*, but that a proper theory of quantum gravity requires a constraint algebra that closes 'off shell', *i.e.* without prior imposition of a subset of the constraints. The fallacies that may ensue if one does not insist on off-shell closure can be illustrated with simple examples. In our opinion, this requirement may well provide the acid test on which any proposed theory of canonical quantum gravity will stand or fail.

While there is general agreement as to what one means when one speaks of 'closure of the constraint algebra' in classical gravity (or any other classical constrained system [39]), this notion is more subtle

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models (see below) is sometimes compared to the vacuum degeneracy problem of string theory, but the latter concerns *different solutions* of the *same* theory, as there is no dispute as to what (perturbative) string theory *is*. However, the concomitant lack of predictivity is obviously a problem for both approaches.

in the quantised theory. [6] Let us therefore clarify first the various notions of closure that can arise: we see at least three different possibilities. The strongest notion is 'off-shell closure' (or 'strong closure'), where one seeks to calculate the commutator of two Hamiltonians

$$\left[\hat{H}[N_1],\, \hat{H}[N_2]\right] = \hat{O}(N_1; N_2)\,. \qquad (21)$$

Here we assume that the quantum Hamiltonian constraint operator

$$\hat{H}[N] := \lim_{\epsilon \to 0} \hat{H}[N, \epsilon]\,, \qquad (22)$$

has been rigorously defined as a suitably weak limit, and without further restrictions on the states on which (21) is supposed to hold. In writing the above equations, we have thus been (and will be) cavalier about habitat questions and the precise definition of the Hamiltonian; see, however [8, 33, 38] for further details and critical comments.

Unfortunately, it appears that the goal of determining $\hat{O}(N_1; N_2)$ as a *bona fide* 'off-shell' operator on a suitable 'habitat' of states, and prior to the imposition of any constraints, is unattainable within the current framework of LQG. For this reason, LQG must resort to weaker notions of closure, by making partial use of the constraints. More specifically, equation (21) can be relaxed substantially by demanding only

$$\left[\hat{H}[N_1],\, \hat{H}[N_2]\right]|\mathcal{X}\rangle = 0\,, \qquad (23)$$

but still with the unregulated Hamiltonian constraint $\hat{H}[N]$. This 'weak closure' should hold for all states $|\mathcal{X}\rangle$ in a restricted habitat of states that are 'naturally' expected to be annihilated by the r.h.s. of (21), and that are subject to the further requirement that the Hamiltonian can be applied twice without leaving the 'habitat'. The latter condition is, for instance, met by the 'vertex smooth' states of [33]. As shown in [33, 38], the commutator of two Hamiltonians indeed vanishes on this 'habitat', and one is therefore led to conclude that the full constraint algebra closes 'without anomalies'.

The same conclusion was already arrived at in an earlier computation of the constraint algebra in [30, 37], which was done from a different perspective (no 'habitats'), and makes essential use of the space of diffeomorphism invariant states $\mathcal{H}_{\text{diff}}$, the 'natural' kernel of the r.h.s. of (21). Here the idea is to verify that [30, 37]

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \langle \mathcal{X} \,|\, \left[\hat{H}[N_1, \epsilon_1],\, \hat{H}[N_2, \epsilon_2]\right] \Psi \rangle = 0\,, \qquad (24)$$

for all $|\mathcal{X}\rangle \in \mathcal{H}_{\text{diff}}$, and for all $|\Psi\rangle$ in the space of finite linear combinations of spin network states. As for the Hamiltonian itself, letting $\epsilon_{1,2} \to 0$ in this expression produces an uncountable sequence of mutually orthogonal states w.r.t. the scalar product (5). Consequently, the limit again does not exist in the usual sense, but only as a weak* limit. The 'diffeomorphism covariance' of the Hamiltonian is essential for this result. Let us stress that (23) and (24) are by no means the same: in (23) one uses the unregulated Hamiltonian (where the limit $\epsilon \to 0$ has already been taken), whereas the calculation of the commutator in (24) takes place inside $\mathcal{H}_{\text{kin}}$, and the limit $\epsilon \to 0$ is taken only *after* computing the commutator of two regulated Hamiltonians. These two operations (taking the limit $\epsilon \to 0$, and calculating the commutator) need not commute. Because with both (23) and (24) one forgoes the aim of finding an operatorial expression for the commutator $\left[\hat{H}[N_1], \hat{H}[N_2]\right]$, making partial use of the constraints, we say (in a partly supergravity inspired terminology) that the algebra closes 'on-shell'.

Although on-shell closure may perhaps look like a sufficient condition on the quantum Hamiltonian constraint, it is easy to see, at the level of simple examples, that this is not true. Consider, for instance, the Hamiltonian constraint of bosonic string theory, and consider modifying it by multiplying it with an operator which commutes with all Virasoro generators. There are many such operators in string theory, for instance the mass-squared operator (minus an arbitrary integer). In this way, we arrive at a realisation of the constraint operators which is very similar to the one used in LQG: the algebra of spatial diffeomorphisms is realised via a (projective) unitary representation, and the Hamiltonian constraint transforms covariantly (the extra factor does not matter, because it commutes with all constraints). In a first step, one can restrict attention to the subspace of states annihilated by the diffeomorphism constraint, the analog of the space $\mathcal{H}_{\text{diff}}$. Imposing now the new Hamiltonian constraint (the one multiplied with the Casimir) on this subspace would produce a 'non-standard' spectrum by allowing extra diffeomorphism invariant states of a certain prescribed mass. The algebra would also still close on-shell, i.e. on the 'habitat' of states annihilated by the diffeomorphism constraint. The point here is not so much whether this new spectrum is 'right' or 'wrong', but rather that in allowing such modifications which are compatible with on-shell closure of the constraint algebra, we introduce an infinite ambiguity and arbitrariness into the definition of the physical states. In other words, if we only demand on-shell closure as in LQG, there is no way of telling whether or not the
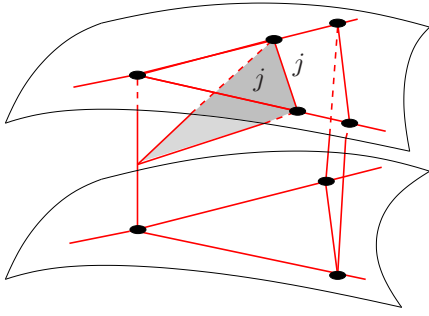
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[6]For reasons of space, we here restrict attention to the bracket between two Hamiltonian constraints, because the remainder of the algebra involving the kinematical constraints is relatively straightforward to implement.

Figure 7: From spin networks to spin foams, in (2+1) dimensions. The Hamiltonian constraint has created one new edge and two new vertices. The associated surface inherits the label j of the edge which is located on the initial or (in this case) final space-like surface.

vanishing of a commutator is merely accidental, that is, not really due to the diffeomorphism invariance of the state, but caused by some other circumstance.

By weakening the requirements on the constraint algebra and by no longer insisting on off-shell closure, crucial information gets lost. This loss of information is reflected in the ambiguities inherent in the construction of the LQG Hamiltonian. It is quite possible that the LQG Hamiltonian admits many further modifications on top of the ones we have already discussed, for which the commutator continues to vanish on a suitably restricted habitat of states — in which case neither (23) nor (24) would amount to much of a consistency test.

6. Canonical gravity and spin foams

Attempts to overcome the difficulties with the Hamiltonian constraint have led to another development, *spin foam models* [40–42]. These were originally proposed as space-time versions of spin networks, to wit, evolutions of spin networks in 'time', but have since developed into a class of models of their own, disconnected from the canonical formalism. Mathematically, spin foam models represent a generalisation of spin networks, in the sense that group theoretical objects (holonomies, representations, intertwiners, etc.) are attached not only to vertices and edges (links), but also to higher dimensional faces in a simplicial decomposition of space-time.

The relation between spin foam models and the canonical formalism is based on a few general features of the action of the Hamiltonian constraint operator on a spin network (for a review on the connection, see [43]). As we have discussed above, the Hamiltonian constraint acts, schematically, by adding a small plaquette close to an existing vertex of the spin network (as in figure 5). In terms of a

space-time picture, we see that the edges of the spin network sweep out surfaces, and the Hamiltonian constraint generates new surfaces, as in figure 7; but note that this graphical representation does not capture the details of how the action of the Hamiltonian affects the intertwiners at the vertices. Instead of associating spin labels to the edges of the spin network, one now associates the spin labels to the surfaces, in such a way that the label of the surface is determined by the label of the edge which lies in either the initial or final surface.

In analogy with proper-time transition amplitudes for a relativistic particle, it is tempting to define the transition amplitude between an initial spin network state and a final one as

$$
Z_T := \langle \psi_f | \exp \left(i \int_0^T \mathrm{d}t \, H \right) | \psi_i \rangle
$$

$$
= \sum_{n=0}^{\infty} \frac{(i\,T)^n}{n!} \int \mathrm{d}\psi_1 \ldots \mathrm{d}\psi_n \, \langle \psi_f | H | \psi_1 \rangle
$$

$$
\times \langle \psi_1 | H | \psi_2 \rangle \cdots \langle \psi_n | H | \psi_i \rangle , \quad (25)
$$

where we have repeatedly inserted resolutions of unity. A (somewhat heuristic) derivation of the above formula can be given by starting from a formal path integral [41], which, after gauge fixing and choice of a global time coordinate T, and with appropriate boundary conditions, can be argued to reduce to the above expression. There are many questions one could ask about the physical meaning of this expression, but one important property is that (just as with the relativistic particle), the transition amplitude will project onto physical states (formally, this projection is effected in the original path integral by integrating over the lapse function multiplying the Hamiltonian density). One might thus consider (25) as a way of defining a physical inner product.

Because path integrals with oscillatory measures are notoriously difficult to handle, one might wonder at this point whether to apply a formal Wick rotation to (25), replacing the Feynman weight with a Boltzmann weight, as is usually done in Euclidean quantum field theory. This is also what is suggested by the explicit formulae in [41], where i in (25) is replaced by (-1). However, this issue is much more subtle here than in ordinary (flat space) quantum field theory. First of all, the distinction between a Euclidean (Riemannian) and a Lorentzian (pseudo-Riemannian) manifold obviously requires the introduction of a metric of appropriate signature. However, spin foam models, having their roots in (background independent) LQG, do not come with a metric, and thus the terminology is to some extent up to the beholder. To avoid confusion, let us state

clearly that our use of the words 'Euclidean' and 'Lorentzian' here always refers to the use of oscillatory weights e^{iS_E} and e^{iS_L}, respectively, where the actions S_E and S_L are the respective actions for Riemannian resp. pseudo-Riemannian metrics. The term 'Wick rotated', on the other hand, refers to the replacement of the oscillatory weight e^{iS} by the exponential weight e^{-S}, with either $S = S_E$ or $S = S_L$. However, in making use of this terminology, one should always remember that there is no Osterwalder-Schrader type reconstruction theorem in quantum gravity, and therefore any procedure (or 'derivation') remains formal. Unlike the standard Euclidean path integral [2, 3], the spin foam models to be discussed below are generally interpreted to correspond to path integrals *with oscillatory weights e^{iS}*, but come in both Euclidean and Lorentzian variants (corresponding to the groups SO(4) and SO(1,3), respectively). This is true even if the state sums involve only *real* quantities (nj-symbols, edge amplitudes, etc.), cf. the discussion after (38).

The building blocks $\langle\psi_k|H|\psi_l\rangle$ in the transition amplitude (25) correspond to elementary spin network transition amplitudes, as in figure 7. For a given value of n, i.e. a given number of time slices, we should thus consider objects of the type

$$Z_{\psi_1,\ldots,\psi_n} = \langle\psi_f|H|\psi_1\rangle\langle\psi_1|H|\psi_2\rangle\cdots\langle\psi_n|H|\psi_i\rangle.$$
(26)

Each of the building blocks depends only on the values of the spins at the spin network edges and the various intertwiners in the spin network state. The points where the Hamiltonian constraint acts nontrivially get associated to spin foam vertices; see figure 8. Instead of working out (26) directly from the action of the Hamiltonian constraint, one could therefore also define the amplitude directly in terms of sums over expressions which depend on the various spins meeting at the spin foam nodes. In this way, one arrives at so-called *state sum* models, which we will describe in the following section.

A problematic issue in the relation between spin foams and the canonical formalism comes from covariance requirements. While tetrahedral symmetry (or the generalisation thereof in four dimensions) is natural in the spin foam picture, the action of the Hamiltonian constraint, depicted in figure 7, does not reflect this symmetry. The Hamiltonian constraint only leads to so-called $1\rightarrow 3$ moves, in which a single vertex in the initial spin network is mapped to three vertices in the final spin network. In the spin foam picture, the restriction to only these moves seems to be in conflict with the idea that the slicing of space-time into a space+time decomposition can be chosen arbitrarily. For space-time covariance, one ex-
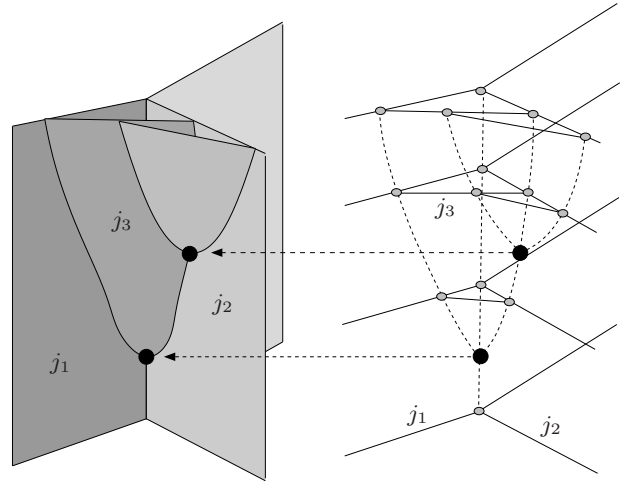

Figure 8: A spin foam (left) together with its spin network evolution (right) in (2+1) dimensions. Spin foam nodes correspond to the places where the Hamiltonian constraint in the spin network acts non-trivially (black dots). Spin foam edges correspond to evolved spin network nodes (grey dots), and spin foam faces correspond to spin network edges. The spin labels of the faces are inherited from the spin labels of spin network edges. If all spin network nodes are three-valent, the spin foam nodes sit at the intersection of six faces, and the dual triangulation consists of tetrahedrons.

pects $2\rightarrow 2$ and $0\rightarrow 4$ moves (and their time-reversed partners) as well, see figure 9. These considerations show that there is no unique path from canonical gravity to spin foam models, and thus no unique model either (even if there was a unique canonical Hamiltonian).

It has been argued [41] that these missing moves can be obtained from the Hamiltonian formalism by a suitable choice of operator ordering. In section 4 we have used an ordering, symbolically denoted by FEE, in which the Hamiltonian first opens up a spin network and subsequently glues in a plaquette. If one chooses the ordering to be EEF, then the inverse densitised vielbeine can open the plaquette, thereby potentially inducing a $2\rightarrow 2$ or $0\rightarrow 4$ move. However, ref. [30] has argued strongly against this operator ordering, claiming that in such a form the Hamiltonian operator cannot even be densely defined. In addition, the derivation sketched here is rather symbolic and hampered by the complexity of the Hamiltonian constraint [44]. Hence, to summarise: for (3+1) gravity a decisive proof of the connection between spin foam models and the full Einstein theory and its canonical formulation appears to be lacking, and it is by no means excluded that such a link does not even exist.
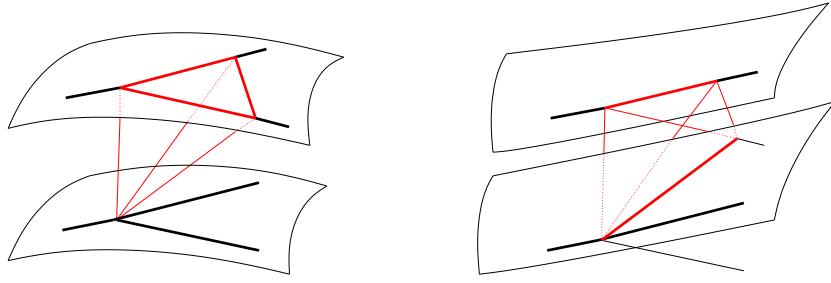
Figure 9: The Hamiltonian constraint induces a $1 \to 3$ move in the spin foam formalism (figure on the left). However, by slicing space-time in a different way, one can equivalently interpret this part of the spin foam as containing a $2 \to 2$ move (figure on the right). This argument suggests that the ultra-local Hamiltonian may not be sufficient to achieve space-time covariance. For clarity, the network edges which lie in one of the spatial slices have been drawn as thick lines.

7. Spin foam models: some basic features

In view of the discussion above, it is thus perhaps best to view spin foam models as models in their own right, and, in fact, as a novel way of defining a (regularised) path integral in quantum gravity. Even without a clear-cut link to the canonical spin network quantisation programme, it is conceivable that spin foam models can be constructed which possess a proper semi-classical limit in which the relation to classical gravitational physics becomes clear. For this reason, it has even been suggested that spin foam models may provide a possible 'way out' if the difficulties with the conventional Hamiltonian approach should really prove insurmountable.

The simplest context in which to study state sum models is (2+1) gravity, because it is a topological ('BF-type') theory, that is, without local degrees of freedom, which can be solved exactly (see e.g. [45–47] and [48] for a more recent analysis of the model within the spin foam picture). The most general expression for a state sum in (2+1) dimensions takes, for a given spin foam ϕ, the form

$$Z_\phi = \sum_{\text{spins } \{j\}} \prod_{f,e,v} A_f(\{j\}) \, A_e(\{j\}) \, A_v(\{j\}) \,, \quad (27)$$

where f, e, v denote the faces, edges and vertices respectively. The amplitudes depend on all these subsimplices, and are denoted by A_f, A_e and A_v respectively. There are many choices which one can make for these amplitudes. In three Euclidean dimensions, space-time covariance demands that the contribution to the partition sum has tetrahedral symmetry in the six spins of the faces which meet at a node (here we assume a 'minimal' spin foam; models with more faces intersecting at an edge are of course possible).

Now, a model of this type has been known for a long time: it is the Ponzano-Regge model for 3d gravity, which implements the above principles by defining the partition sum

$$Z_\phi^{\text{PR}} = \sum_{\text{spins } \{j_i\}} \prod_{\text{faces } f} (2\,j_f + 1) \prod_{\text{vertices } v} \quad (28)$$

The graphical notation denotes the Wigner $6j$ symbol, defined in terms of four contracted Clebsch-Gordan coefficients as

$$\{6j\} \sim \sum_{m_1,\ldots,m_6} C^{j_1 \ j_2 \ j_3}_{m_1 m_2 m_3} C^{j_5 \ j_6 \ j_1}_{m_5 m_6 m_1} C^{j_6 \ j_4 \ j_2}_{m_6 m_4 m_2} C^{j_4 \ j_5 \ j_3}_{m_4 m_5 m_3} \,.$$

$$(29)$$

For SU(2) representations, the sum over spins in the Ponzano-Regge state sum (28) requires that one divides by an infinite factor in order to ensure convergence (more on finiteness properties below) and independence of the triangulation. The tetrahedron appearing in (28) in fact has a direct geometrical interpretation as the object dual to the spin foam vertex. The dual tetrahedron can then also be seen as an elementary simplex in the triangulation of the manifold. Three-dimensional state sums with boundaries, appropriate for the calculation of transition amplitudes between two-dimensional spin networks, have been studied in [49].
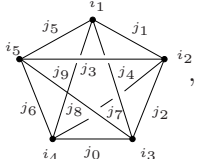
When one tries to formulate spin foam models in four dimensions, the first issue one has to deal with is the choice of the representation labels on the spin foam faces. From the point of view of the canonical formalism it would seem natural to again use SU(2) representations, as these are used to label the edges of a spin network in three spatial dimensions, whose evolution produces the faces (2-simplices) of the spin foam. However, this is not what is usually done. Instead, the faces of the spin foam are supposed to carry representations of SO(4) $\approx$ SO(3) $\times$ SO(3) [or SO(1,3) $\approx$ SL(2,$\mathbb{C}$) for Lorentzian space-times]. The corresponding models in four dimensions are purely topological theories, the so-called

"BF models", where $F(A)$ is a field strength, and B the Lagrange multiplier two-form field whose variation enforces $F(A) = 0$. Up to this point, the model is analogous to gravity in (2+1) dimensions, except that the relevant gauge group is now SO(4) [or SO(1,3)]. However, in order to recover general relativity and to re-introduce local (propagating) degrees of freedom into the theory, one must impose a constraint on B.

Classically, this constraint says that B is a 'bi-vector', that is $B^{ab} = e^a \wedge e^b$. The quantum mechanical analog of this constraint amounts to a restriction to a particular set of representations of SO(4) = SU(2) $\otimes$ SU(2), namely those where the spins of the two factors are equal, the so-called *balanced representations*, denoted by (j,j) (for $j = \frac{1}{2}, 1, \frac{3}{2}, \dots$). Imposing this restriction on the state sum leads to a class of models first proposed by Barrett & Crane [50, 51]. In these models the vertex amplitudes are given by combining the 10 spins of the faces which meet at a vertex, as well as possibly further 'virtual' spins associated to the vertices themselves, using an expression built from contracted Clebsch-Gordan coefficients. For instance, by introducing an extra 'virtual' spin i_k associated to each edge where four faces meet, one can construct an intertwiner between the four spins by means of the following expression

$$I^{j_1 \cdots j_4 \; ;i_k}_{m_1 \cdots m_4} = \sum_{m_k} C^{j_1 \; j_2 \; i_k}_{m_1 m_2 m_k} C^{j_3 \; j_4 \; i_k}_{m_3 m_4 m_k}. \quad (30)$$

However, this prescription is not unique as we can choose between three different 'channels' (here taken to be $12 \leftrightarrow 34$); this ambiguity can be fixed by imposing symmetry, see below. Evidently, the number of channels and virtual spins increases rapidly with the valence of the vertex. For the above four-vertex, this prescription results in a state sum[7]

$$Z_\phi^{\{i_k\}} = \sum_{\text{spins } \{j_i\}} \prod_{\text{faces } f} \prod_{\text{edges } e} A_f(\{j\}) A_e(\{j\})$$
$$\times \prod_{\text{vertices } v} \quad , \quad (31)$$

where the spins j denote spin labels of balanced representations (j,j) (as we already mentioned, without this restriction, the model above corresponds to the topological BF model [52–54]). The precise factor

corresponding to the pentagon (or "$15j$" symbol) in this formula is explicitly obtained by multiplying the factors (30) (actually, one for each SO(3) factor in SO(4)), and contracting (summing) over the labels m_i,

$$\{15j\} = \sum_{m_i} I^{j_1 \; j_4 \; j_9 \; j_5 \; ;i_1}_{m_1 m_4 m_9 m_5} I^{j_1 \; j_2 \; j_7 \; j_3 \; ;i_2}_{m_1 m_2 m_7 m_3}$$
$$\times I^{j_4 \; j_2 \; j_8 \; j_0 \; ;i_3}_{m_4 m_2 m_8 m_0} I^{j_9 \; j_7 \; j_0 \; j_6 \; ;i_4}_{m_9 m_7 m_0 m_6} I^{j_5 \; j_3 \; j_8 \; j_6 \; ;i_5}_{m_5 m_3 m_8 m_6}. \quad (32)$$

There are various ways in which one can make (31) independent of the spins i_k associated to the edges. One way is to simply sum over these spins. This leads to the so-called "$15j$ BC model",

$$Z_\phi^{15j} = \sum_{\text{spins } \{j_i, i_k\}} \prod_{\text{faces } f} \prod_{\text{edges } e} A_f(\{j\}) A_e(\{j\})$$
$$\times \prod_{\text{vertices } v} \{15j\}. \quad (33)$$

An alternative way to achieve independence of the edge intertwiner spins is to include a sum over the i_k in the definition of the vertex amplitude. These models are known as "$10j$ BC models",

$$Z_\phi^{10j} = \sum_{\text{spins } \{j_i\}} \prod_{\text{faces } f} \prod_{\text{edges } e} A_f(\{j\}) A_e(\{j\})$$
$$\times \prod_{\text{vertices } v} \sum_{\text{spins } \{i_k\}} f(\{i_k\}) \{15j\}, \quad (34)$$

labelled by an arbitrary function $f(\{i_k\})$ of the intertwiner spins. Only for the special choice [50]

$$f(\{i_k\}) = \prod_{k=1}^{5} (2i_k + 1) \quad (35)$$

does the vertex amplitude have simplicial symmetry [55], i.e. is invariant under the symmetries of the pentagon (31) (where the pentagon really represents a 4-simplex).[8]

While the choice (34,35) for the vertex amplitude $A_v(\{j\})$ is thus preferred from the point of view of covariance, there are still potentially many different choices for the face and edge amplitudes $A_f(\{j\})$

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[7]There is now no longer such a clear relation of the graphical object in (31) to the dual of the spin foam vertex: faces and edges of the spin foam map to faces and tetrahedrons of the dual in four dimensions, respectively, but these are nevertheless represented with edges and vertices in the figure in (31).

[8]There is an interesting way to express combinatorial objects such as the $10j$ symbol in terms of integrals over group manifolds, which goes under the name of 'group field theory' (see e.g. [56]), and which also allows an interpretation in terms of 'Feynman diagrams'. The relation between spin foams and group field theory is potentially useful to evaluate state sums because the corresponding integrals can be evaluated using stationary phase methods. We will, however, not comment on this development any further since there is (under certain assumptions) a one-to-one map between spin foam models and group field theory models.

and $A_e(\{j\})$. Different choices may lead to state sums with widely varying properties. The dependence of state sums on the face and edge amplitudes is clearly illustrated by e.g. the numerical comparison of various models presented in [57]. A natural and obvious restriction on the possible amplitudes is that the models should yield the correct classical limit – to wit, Einstein's equations – in the large j limit, corresponding to the infrared (see also the discussion in the following section). Therefore, any function of the face spins which satisfies the pentagon symmetries and is such that the state sum has appropriate behaviour in the large j limit is a priori allowed. Furthermore, the number of possible amplitudes, and thus of possible models, grows rapidly if one allows for more general valences of the vertices. In the literature, the neglect of higher-valence vertices is often justified by invoking the fact that the valence ≤ 4 spin network wave functions in the Hamiltonian formulation constitute a superselection sector in $\mathcal{H}_{\text{kin}}$ (because the 'spiderwebs' in figure 6 do not introduce higher valences). However, we find this argument unconvincing because (i) the precise relation between the Hamiltonian and the spin foam formulation remains unclear, and (ii) physical arguments based on ultralocality (cf. our discussion at the end of section 6) suggest that more general moves (hence, valences) should be allowed.

Let us also mention that, as an alternative to the Euclidean spin foam models, one can try to set up *Lorentzian spin foam models*, as has been done in [58, 59]. In this case, the (compact) group SO(4) is replaced by the *non-compact* Lorentz group SO(1,3) [or SL$(2,\mathbb{C})$]. Recall that in both cases we deal with oscillatory weights, not with a weight appropriate for a Wick rotated model. It appears unlikely that there is any relation between the Lorentzian models and the Euclidean ones. Furthermore, the analysis of the corresponding Lorentzian state sums is much more complicated due to the fact that the relevant (i.e. unitary) representations are now infinite-dimensional.

8. Spin foams and discrete gravity

To clarify the relation between spin foam models and earlier attempts to define a discretised path integral in quantum gravity, we recall that the latter can be roughly divided into two classes, namely:

- *Quantum Regge Calculus* (see e.g. [60]), where one approximates space-time by a triangulation consisting of a fixed number of simplices, and integrates over all edge lengths, keeping the 'shape' of the triangulation fixed;

- *Dynamical Triangulations* (see e.g. [61–63]), where the simplices are assigned fixed edge lengths, and one sums instead over different triangulations, but keeping the number of simplices fixed (thus changing only the 'shape', but not the 'volume' of the triangulation).

Both approaches are usually based on a positive signature (Euclidean) metric, where the Boltzmann factor is derived from, or at least motivated by, some discrete approximation to the Einstein-Hilbert action, possibly with a cosmological constant (but see [64, 65] for some recent progress with a Wick-rotated 'Lorentzian' dynamical triangulation approach which introduces and exploits a notion of causality on the space-time lattice). In both approaches, the ultimate aim is then to recover continuum space-time via a refinement limit in which the number of simplices is sent to infinity. Establishing the existence of such a limit is a notoriously difficult problem that has not been solved for four-dimensional gravity. In fact, for quantum Regge models in two dimensions such a continuum limit does not seem to agree with known continuum results [66–69] (see however [70]).

From the point of view of the above classification, spin foam models belong to the first, 'quantum Regge', type, as one sums over all spins for a given spin foam, but does not add, remove or replace edges, faces or vertices, at least not in the first step. Indeed, for the spin foams discussed in the foregoing section, we have so far focused on the partition sum for a *single* given spin foam. An obvious question then concerns the next step, or more specifically the question how spin foam models can recover (or even only define) a continuum limit. The canonical setup, where one sums over all spin network states in expressions like (25), would suggest that one should sum over all foams,

$$Z^{\text{total}} = \sum_{\text{foams } \phi} w_\phi \, Z_\phi \,, \tag{36}$$

where Z_ϕ denotes the partition function for a given spin foam ϕ, and where we have allowed for the possibility of a non-trivial weight w_ϕ depending only on the topological structure ('shape') of the foam. The reason for this sum would be to achieve formal independence of the triangulations. In a certain sense this would mimic the dynamical triangulation approach (except that one now would also sum over foams with a different number of simplices and different edge lengths), and thus turn the model into a hybrid version of the above approaches. However, this prescription is far from universally accepted, and several other ideas on how to extract classical, con-

tinuum physics from the partition sum Z_ϕ have been proposed.

One obvious alternative is to *not* sum over all foams, but instead look for a refinement with an increasing number of cells, [9]

$$Z^\infty = \lim_{\#\,\text{cells}\to\infty} Z_\phi\,. \qquad (37)$$

The key issue is then to ensure that the final result does not depend on the way in which the triangulations are performed and refined (this is a crucial step which one should understand in order to interpret results on single-simplex spin foams like those of [71, 72]). The refinement limit is motivated by the fact that it does appear to work in three space-time dimensions, where (allowing for some 'renormalisation') one can establish triangulation independence [82]. Furthermore, for large spins, the $6j$ symbol which appears in the Ponzano-Regge model approximates the Feynman weight for Regge gravity [73, 74]. More precisely, when all six spins are simultaneously taken large,

$$\{6j\} \sim \left(e^{iS_\text{Regge}(\{j\})+\frac{i\pi}{4}} + e^{-iS_\text{Regge}(\{j\})-\frac{i\pi}{4}}\right). \qquad (38)$$

Here $S_\text{Regge}(\{j\})$ is the Regge action of a tetrahedron, given by

$$S_\text{Regge}(\{j\}) = \sum_{i=1}^{6} j_i\,\theta_i\,, \qquad (39)$$

where θ_i is the dihedral angle between the two surfaces meeting at the ith edge. Related results in four dimensions are discussed in [75] and, using group field theory methods, in [76]. We emphasise once more that this by no means singles out the $6j$ symbol as the unique vertex amplitude: we can still multiply it by any function of the six spins which asymptotes to one for large spins.

The $6j$ symbol is of course real, which explains the presence of a cosine instead of a complex oscillatory weight on the right-hand side of (38). Indeed, it seems rather curious that, while the left-hand side of (38) arises from an expression resembling a Boltzmann sum, the right-hand side contains oscillatory factors which suggest a path integral with oscillatory weights. In view of our remarks in section 6, and in order to make the relation to Regge gravity somewhat more precise, one must therefore argue either that a proper path integral in gravity produces both terms, or otherwise that one can get rid of one of the terms by some other mechanisms. The first possibility appears to be realised in (2+1) gravity, because

one can cast the gravitational action into Chern Simons form $S = \int R \wedge e$, in which case a sum over orientations of the dreibein would lead to terms with both signs in the exponent. Unfortunately, this argument does not extend to four dimensions, where the gravitational action $S = \int R \wedge e \wedge e$ depends quadratically on the vierbein. For this reason, it has instead been suggested that one of the two oscillatory terms disappears for all physical correlation functions [71].

The vertex amplitudes represented by the $6j$ or $10j$ symbols only form part of the state sum (27). The known four-dimensional models depend rather strongly on the choice of the face and edge amplitudes: while some versions of the Barrett-Crane $10j$ model have diverging partition sums, others are dominated by configurations in which almost all spins are zero, i.e. configurations which correspond to zero-area faces [57]. Once more, it is important to remember that even in 'old' Regge models in two dimensions, where a comparison with exact computations in the continuum is possible [77–79], the continuum limit does not seem to agree with these exact results [66–69] (the expectation values of edge lengths do not scale as a power of the volume when a diffeomorphism invariant measure is used, in contrast to the exact results). Therefore, it is far from clear that (37) will lead to a proper continuum limit.

A third proposal is to take a fixed spin foam and to simply define the model as the sum over all spins [56, 80, 81]; this proposal differs considerably from both the Regge and dynamical triangulation approaches. Considering a fixed foam clearly only makes sense provided the partition sum is actually independent of the triangulation of the manifold (or more correctly, one would require that physical correlators are independent of the triangulation). Such a situation arises in the three-dimensional Ponzano-Regge model, but three-dimensional gravity does not contain any local degrees of freedom. For higher dimensions, the only triangulation independent models known so far are topological theories, i.e. theories for which the local degrees of freedom of the metric do not matter. If one insists on triangulation independence also for gravity, then one is forced to add new degrees of freedom to the spin foam models (presumably living on the edges). In this picture, a change from a fine triangulation to a coarse one is then compensated by more information stored at the edges of the coarse triangulation. This then also requires (presumably complicated) rules which say how these new degrees of freedom behave under a move from one triangulation to another. Note that even when the partition sum is independent of the refinement of the triangulation, one would probably still want to deal with complicated cross-sections of

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[9]But note that, formally, the sum over all foams can also be thought of as a refinement limit if one includes zero spin representations (meaning no edge) in the refinement limit.

foams to describe "in" and "out" coherent states. At present, there is little evidence that triangulation independence can be realised in non-topological theories, or that the problems related to the continuum limit will not reappear in a different guise.

9. Predictive (finite) quantum gravity?

Let us now return to the question as to what can be said about finiteness properties of spin foam models, and how they relate to finiteness properties (or rather, lack thereof!) of the standard perturbative approach – after all, one of the main claims of this approach is that it altogether avoids the difficulties of the standard approach. So far, investigations of finiteness have focused on the partition sum itself. Namely, it has been shown that for a variety of spin foam models, the partition sum for a *fixed* spin foam is finite,

$$\sum_{\text{spins } \{j\}} Z_\phi(\{j\}) = \text{finite}. \qquad (40)$$

Even though a given spin foam consists of a finite number of links, faces, ..., divergences could arise in principle because the range of each spin j is infinite. One way to circumvent infinite sums is to replace the group SU(2) by the quantum group SU(2)$_q$ (which has a finite number of irreps), or equivalently, by introducing an infinite positive cosmological constant [82]; in all these cases the state sum becomes finite.[10] A similar logic holds true in four dimensions and for Lorentzian models, although in the latter case the analysis becomes more complicated due to the non-compactness of the Lorentz group, and the fact that the unitary representations are all infinite dimensional [84]. Perhaps unsurprisingly, there exist choices for edge and surface amplitudes in four dimensions which look perfectly reasonable from the point of view of covariance, but which are nevertheless not finite [57].

It should, however, be emphasised that the finiteness of (40) is a statement about *infrared* finiteness. Roughly speaking, this is because the spin j corresponds to the 'length' of the link, whence the limit of large j should be associated with the *infinite volume limit*. In statistical mechanics, partition functions generically diverge in this limit, but in such a way that physical correlators possess a well-defined

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[10]The division by the infinite factor which is required to make the Ponzano-Regge state sum finite can be understood as dividing out the volume of the group of residual invariances of Regge models [83]. These invariances correspond to changes of the triangulation which leave the curvature fixed. However, dividing out by the volume of this group does not eliminate the formation of 'spikes' in Regge gravity.

limit (as quotients of two quantities which diverge). From this point of view, the finiteness properties established so far say nothing about the UV properties of quantum gravity, which should instead follow from some kind of refinement limit, or from an averaging procedure where one sums over all foams, as discussed above. The question of convergence or non-convergence of such limits has so far not received a great deal of attention in the literature.

This then, in a sense, brings us back to square one, namely the true problem of quantum gravity, which lies in the ambiguities associated with an infinite number of non-renormalisable UV divergences. As is well known this problem was originally revealed in a perturbative expansion of Einstein gravity around a fixed background, which requires an infinite series of counterterms, starting with the famous two-loop result [85–87]

$$\Gamma_{\text{div}}^{(2)} = \frac{1}{\epsilon} \frac{209}{2880} \frac{1}{(16\pi^2)^2} \int \mathrm{d}^4 x \, \sqrt{g} \, C_{\mu\nu\rho\sigma} C^{\rho\sigma\lambda\tau} C_{\lambda\tau}{}^{\mu\nu} . \qquad (41)$$

The need to fix an infinite number of couplings in order to make the theory predictive renders perturbatively quantised Einstein gravity useless as a physical theory. What we would like to emphasise here is that *any* approach to quantum gravity must confront this problem, and that the need to fix infinitely many couplings in the perturbative approach, and the appearance of infinitely many ambiguities in non-perturbative approaches are really just different sides of the same coin.

At least in its present incarnation, the canonical formulation of LQG does not encounter any UV divergences, but the problem reappears through the lack of uniqueness of the canonical Hamiltonian. For spin foams (or, more generally, discrete quantum gravity) the problem is no less virulent. The known finiteness proofs all deal with the behaviour of a single foam, but, as we argued, these proofs concern the infrared rather than the ultraviolet. Just like canonical LQG, spin foams thus show no signs of ultraviolet divergences so far, but, as we saw, there is an *embarras de richesse* of physically distinct models, again reflecting the non-uniqueness that manifests itself in the infinite number of couplings associated with the perturbative counterterms. Indeed, fixing the ambiguities of the non-perturbative models by *ad hoc*, albeit well motivated, assumptions is not much different from defining the perturbatively quantised theory by fixing infinitely many coupling constants 'by hand' (and thereby remove all divergences). Furthermore, even if they do not 'see' any UV divergences, non-perturbative approaches cannot be relieved of the duty to explain *in all detail* how the 2-loop divergence (41) and its higher loop analogues

'disappear', be it through cancellations or some other mechanism.

Finally, let us remark that in lattice gauge theories, the classical limit and the UV limit can be considered and treated as separate issues. As for quantum gravity, this also appears to be the prevailing view in the LQG community. However, the continuing failure to construct viable *physical* semi-classical states, solving the constraints even in only an approximate fashion, seems to suggest (at least to us) that in gravity the two problems cannot be solved separately, but are inextricably linked — also in view of the fact that the question as to the precise fate of the two-loop divergence (41) can then no longer be avoided.

Acknowledgements

References

[1] R. Loll, "Discrete approaches to quantum gravity in four dimensions", *Living Rev. Rel.* **1** (1998) 13, `gr-qc/9805049`.

[2] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity", *Phys. Rev.* **D15** (1977) 2752–2756.

[3] S. W. Hawking, "The path-integral approach to quantum gravity", in "An Einstein centenary survey", S. Hawking and W. Israel, eds., pp. 746–789. Cambridge University Press, 1979.

[4] S. Weinberg, "Ultraviolet divergences in quantum gravity", in "An Einstein centenary survey", S. Hawking and W. Israel, eds., pp. 790–832. Cambridge University Press, 1979.

[5] S. Weinberg, "What is quantum field theory, and what did we think it was?", `hep-th/9702027`.

[6] O. Lauscher and M. Reuter, "Is quantum Einstein gravity nonperturbatively renormalizable?", *Class. Quant. Grav.* **19** (2002) 483–492, `hep-th/0110021`.

[7] C. Kiefer, "Quantum Gravity", Clarendon Press, 2004.

[8] H. Nicolai, K. Peeters, and M. Zamaklar, "Loop quantum gravity: an outside view", *Class. Quant. Grav.* **22** (2005) R193–R247, `hep-th/0501114`.

[9] R. Gambini and J. Pullin, "Loops, knots, gauge theories and quantum gravity", Cambridge University Press, 1996.

[10] T. Thiemann, "Introduction to modern canonical quantum general relativity", `gr-qc/0110034`.

[11] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A status report", *Class. Quant. Grav.* **21** (2004) R53, `gr-qc/0404018`.

[12] A. Perez, "Introduction to loop quantum gravity and spin foams", `gr-qc/0409061`.

[13] J. C. Baez, "An introduction to spin foam models of BF theory and quantum gravity", *Lect. Notes Phys.* **543** (2000) 25–94, `gr-qc/9905087`.

[14] A. Perez, "Spin foam models for quantum gravity", *Class. Quant. Grav.* **20** (2003) R43, `gr-qc/0301113`.

[15] C. Rovelli, "Quantum gravity", Cambridge University Press, 2004.

[16] A. Perez, "On the regularization ambiguities in loop quantum gravity", `gr-qc/0509118`.

[17] A. Perez, "The spin-foam-representation of loop quantum gravity", `gr-qc/0601095`.

[18] "Loops '05", `http://loops05.aei.mpg.de/`.

[19] A. Ashtekar, S. Fairhurst, and J. L. Willis, "Quantum gravity, shadow states, and quantum mechanics", *Class. Quant. Grav.* **20** (2003) 1031–1062, `gr-qc/0207106`.

[20] T. Thiemann, "Gauge field theory coherent states (GCS). I: General properties", *Class. Quant. Grav.* **18** (2001) 2025–2064, `hep-th/0005233`.

[21] T. Thiemann and O. Winkler, "Gauge field theory coherent states (GCS). II: Peakedness properties", *Class. Quant. Grav.* **18** (2001) 2561–2636, `hep-th/0005237`.

[22] T. Thiemann and O. Winkler, "Gauge field theory coherent states (GCS) III: Ehrenfest theorems", *Class. Quant. Grav.* **18** (2001) 4629–4682, `hep-th/0005234`.

[23] T. Thiemann and O. Winkler, "Gauge field theory coherent states (GCS). IV: Infinite tensor product and thermodynamical limit", *Class. Quant. Grav.* **18** (2001) 4997–5054, `hep-th/0005235`.

[24] T. Thiemann, "Complexifier coherent states for quantum general relativity", `gr-qc/0206037`.

[25] H. Sahlmann, T. Thiemann, and O. Winkler, "Coherent states for canonical quantum general relativity and the infinite tensor product extension", *Nucl. Phys.* **B606** (2001) 401–440, `gr-qc/0102038`.

[26] J. Brunnemann and T. Thiemann, "On (cosmological) singularity avoidance in loop quantum gravity", `gr-qc/0505032`.

[27] K. A. Meissner, "Eigenvalues of the volume operator in loop quantum gravity", `gr-qc/0509049`.

[28] W. Fairbairn and C. Rovelli, "Separable Hilbert space in loop quantum gravity", *J. Math. Phys.* **45** (2004) 2802–2814, `gr-qc/0403047`.

[29] R. Borissov, R. De Pietri, and C. Rovelli, "Matrix elements of Thiemann's Hamiltonian constraint in loop quantum gravity", *Class. Quant. Grav.* **14**

(1997) 2793–2823, gr-qc/9703090.

[30] T. Thiemann, "Quantum spin dynamics (QSD)", *Class. Quant. Grav.* **15** (1998) 839–873, gr-qc/9606089.

[31] R. De Pietri and C. Rovelli, "Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity", *Phys. Rev.* **D54** (1996) 2664–2690, gr-qc/9602023.

[32] J. Brunnemann and T. Thiemann, "Simplification of the spectral analysis of the volume operator in loop quantum gravity", gr-qc/0405060.

[33] J. Lewandowski and D. Marolf, "Loop constraints: A habitat and their algebra", *Int. J. Mod. Phys.* **D7** (1998) 299–330, gr-qc/9710016.

[34] L. Smolin, "The classical limit and the form of the Hamiltonian constraint in non-perturbative quantum general relativity", gr-qc/9609034.

[35] D. E. Neville, "Long range correlations in quantum gravity", *Phys. Rev.* **D59** (1999) 044032, gr-qc/9803066.

[36] R. Loll, "On the diffeomorphism-commutators of lattice quantum gravity", *Class. Quant. Grav.* **15** (1998) 799–809, gr-qc/9708025.

[37] T. Thiemann, "Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity", *Phys. Lett.* **B380** (1996) 257–264, gr-qc/9606088.

[38] R. Gambini, J. Lewandowski, D. Marolf, and J. Pullin, "On the consistency of the constraint algebra in spin network quantum gravity", *Int. J. Mod. Phys.* **D7** (1998) 97–109, gr-qc/9710018.

[39] M. Henneaux and C. Teitelboim, "Quantization of gauge systems", Princeton University Press, 1992.

[40] M. P. Reisenberger, "World sheet formulations of gauge theories and gravity", gr-qc/9412035.

[41] M. P. Reisenberger and C. Rovelli, "'Sum over surfaces' form of loop quantum gravity", *Phys. Rev.* **D56** (1997) 3490–3508, gr-qc/9612035.

[42] J. C. Baez, "Spin foam models", *Class. Quant. Grav.* **15** (1998) 1827–1858, gr-qc/9709052.

[43] R. De Pietri, "Canonical "loop" quantum gravity and spin foam models", gr-qc/9903076.

[44] C. Rovelli, "The projector on physical states in loop quantum gravity", *Phys. Rev.* **D59** (1999) 104015, gr-qc/9806121.

[45] S. Deser, R. Jackiw, and G. 't Hooft, "Three-dimensional Einstein gravity: dynamics of flat space", *Ann. Phys.* **152** (1984) 220.

[46] E. Witten, "(2+1)-Dimensional gravity as an exactly soluble system", *Nucl. Phys.* **B311** (1988) 46.

[47] A. Ashtekar, V. Husain, C. Rovelli, J. Samuel, and L. Smolin, "(2+1)-Quantum gravity as a toy model for the (3+1) theory", *Class. Quant. Grav.* **6** (1989) L185.

[48] K. Noui and A. Perez, "Three dimensional loop quantum gravity: Physical scalar product and spin foam models", *Class. Quant. Grav.* **22** (2005) 1739–1762, gr-qc/0402110.

[49] M. Karowski, W. Muller, and R. Schrader, "State sum invariants of compact three manifolds with boundary and $6j$ symbols", *J. Phys.* **A25** (1992) 4847–4860.

[50] J. W. Barrett and L. Crane, "Relativistic spin networks and quantum gravity", *J. Math. Phys.* **39** (1998) 3296–3302, gr-qc/9709028.

[51] L. Freidel and K. Krasnov, "Spin foam models and the classical action principle", *Adv. Theor. Math. Phys.* **2** (1999) 1183–1247, hep-th/9807092.

[52] H. Ooguri, "Topological lattice models in four dimensions", *Mod. Phys. Lett.* **A7** (1992) 2799–2810, hep-th/9205090.

[53] L. Crane and D. Yetter, "A categorical construction of 4-D topological quantum field theories", 1993.

[54] L. Crane, L. H. Kauffman, and D. N. Yetter, "State sum invariants of four-manifolds I", *J. Knot Theor Ramifications* **6** (1997) 177–234, hep-th/9409167.

[55] M. P. Reisenberger, "On relativistic spin network vertices", *J. Math. Phys.* **40** (1999) 2046–2054, gr-qc/9809067.

[56] L. Freidel, "Group field theory: An overview", *Int. J. Theor. Phys.* **44** (2005) 1769–1783, hep-th/0505016.

[57] J. C. Baez, J. D. Christensen, T. R. Halford, and D. C. Tsang, "Spin foam models of Riemannian quantum gravity", *Class. Quant. Grav.* **19** (2002) 4627–4648, gr-qc/0202017.

[58] J. W. Barrett and L. Crane, "A Lorentzian signature model for quantum general relativity", *Class. Quant. Grav.* **17** (2000) 3101–3118, gr-qc/9904025.

[59] A. Perez and C. Rovelli, "Spin foam model for Lorentzian general relativity", *Phys. Rev.* **D63** (2001) 041501, gr-qc/0009021.

[60] R. M. Williams and P. A. Tuckey, "Regge calculus: A bibliography and brief review", *Class. Quant. Grav.* **9** (1992) 1409–1422.

[61] D. V. Boulatov, V. A. Kazakov, I. K. Kostov, and A. A. Migdal, "Analytical and numerical study of the model of dynamically triangulated random surfaces", *Nucl. Phys.* **B275** (1986) 641.

[62] A. Billoire and F. David, "Scaling properties of randomly triangulated planar random surfaces: a numerical study", *Nucl. Phys.* **B275** (1986) 617.

[63] J. Ambjørn, B. Durhuus, J. Frohlich, and P. Orland, "The appearance of critical dimensions in regulated string theories", *Nucl. Phys.* **B270** (1986) 457.

[64] J. Ambjørn, J. Jurkiewicz, and R. Loll, "Dynamically triangulating Lorentzian quantum gravity", *Nucl. Phys.* **B610** (2001) 347–382, hep-th/0105267.

[65] J. Ambjorn, J. Jurkiewicz, and R. Loll, "Emergence of a 4D world from causal quantum gravity", *Phys. Rev. Lett.* **93** (2004) 131301, hep-th/0404156.

[66] W. Bock and J. C. Vink, "Failure of the Regge approach in two-dimensional quantum gravity", *Nucl. Phys.* **B438** (1995) 320–346, `hep-lat/9406018`.

[67] C. Holm and W. Janke, "Measure dependence of 2D simplicial quantum gravity", *Nucl. Phys. Proc. Suppl.* **42** (1995) 722–724, `hep-lat/9501005`.

[68] J. Ambjorn, J. L. Nielsen, J. Rolf, and G. K. Savvidy, "Spikes in quantum Regge calculus", *Class. Quant. Grav.* **14** (1997) 3225–3241, `gr-qc/9704079`.

[69] J. Rolf, "Two-dimensional quantum gravity", PhD thesis, University of Copenhagen, 1998. `hep-th/9810027`.

[70] H. W. Hamber and R. M. Williams, "On the measure in simplicial gravity", *Phys. Rev.* **D59** (1999) 064014, `hep-th/9708019`.

[71] C. Rovelli, "Graviton propagator from background-independent quantum gravity", `gr-qc/0508124`.

[72] S. Speziale, "Towards the graviton from spinfoams: The 3d toy model", `gr-qc/0512102`.

[73] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients", in "Spectroscopic and group theoretical methods in physics". North-Holland, 1968.

[74] J. Roberts, "Classical $6j$-symbols and the tetrahedron", *Geom. Topol.* **3** (1999) 21–66, `math-ph/9812013`.

[75] J. W. Barrett and R. M. Williams, "The asymptotics of an amplitude for the 4-simplex", *Adv. Theor. Math. Phys.* **3** (1999) 209–215, `gr-qc/9809032`.

[76] L. Freidel and K. Krasnov, "Simple spin networks as Feynman graphs", *J. Math. Phys.* **41** (2000) 1681–1690, `hep-th/9903192`.

[77] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, "Fractal structure of 2d-quantum gravity", *Mod. Phys. Lett.* **A3** (1988) 819.

[78] F. David, "Conformal field theories coupled to 2-d gravity in the conformal gauge", *Mod. Phys. Lett.* **A3** (1988) 1651.

[79] J. Distler and H. Kawai, "Conformal field theory and 2-d quantum gravity or who's afraid of Joseph Liouville?", *Nucl. Phys.* **B321** (1989) 509.

[80] J. W. Barrett, "State sum models for quantum gravity", `gr-qc/0010050`.

[81] H. Pfeiffer, "Diffeomorphisms from finite triangulations and absence of 'local' degrees of freedom", *Phys. Lett.* **B591** (2004) 197–201, `gr-qc/0312060`.

[82] H. Ooguri, "Partition functions and topology changing amplitudes in the 3D lattice gravity of Ponzano and Regge", *Nucl. Phys.* **B382** (1992) 276–304, `hep-th/9112072`.

[83] L. Freidel and D. Louapre, "Diffeomorphisms and spin foam models", *Nucl. Phys.* **B662** (2003) 279–298, `gr-qc/0212001`.

[84] L. Crane, A. Perez, and C. Rovelli, "A finiteness proof for the Lorentzian state sum spinfoam model for quantum general relativity", `gr-qc/0104057`.

[85] M. H. Goroff and A. Sagnotti, "Quantum gravity at two loops", *Phys. Lett.* **B160** (1985) 81.

[86] M. H. Goroff and A. Sagnotti, "The ultraviolet behavior of Einstein gravity", *Nucl. Phys.* **B266** (1986) 709.

[87] A. E. M. van de Ven, "Two loop quantum gravity", *Nucl. Phys.* **B378** (1992) 309–366.