

# Conformal invariance from non-conformal gravity

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## Abstract

We discuss the conditions under which classically conformally invariant models in four dimensions can arise out of non-conformal (Einstein) gravity. As an ‘existence proof’ that this is indeed possible we show how to derive  $N = 4$  super Yang Mills theory with any compact gauge group  $G$  from non-conformal *gauged*  $N = 4$  supergravity as a special flat space limit. We stress the role that the anticipated UV finiteness of the (so far unknown) underlying theory of quantum gravity would have to play in such a scheme, as well as the fact that the masses of elementary particles would have to arise via quantum gravitational effects which mimic the conformal anomalies of standard (flat space) UV divergent quantum field theory.

## 1 Introduction

That conformal symmetry<sup>1</sup> might play a key role in resolving the hierarchy problem remains a distinct possibility, especially in view of the remarkable fact that the standard model (=SM) of elementary particle physics [4, 5] *is* classically conformally invariant, except for the explicit mass term in the scalar potential that is commonly introduced for electroweak symmetry breaking. In such a ‘conformal scenario’ the observed mass scales of particle physics and their smallness *vis-à-vis* the Planck scale might be explained solely via the quantum mechanical breaking of conformal symmetry, in accordance with the naturalness criterion of [6] (see also [7]). Simultaneously with conformal symmetry, the electroweak  $SU(2)_w \times U(1)_Y$  symmetry

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<sup>1</sup>Classic references are [1, 2]; see also [3] for a comprehensive review of conformal invariance in field theory.

of the SM would have to be broken via radiative corrections by means of the Coleman-Weinberg (=CW) mechanism [8, 9].

While it is still not clear whether this idea can be made to work in the realistic context of the SM, a main objection that has been raised against it is that the SM couples to gravity and must eventually merge into a theory of quantum gravity at the Planck scale. Einstein’s theory (with SM-like matter couplings) is certainly *not* conformally invariant due to the presence of the dimensionful coupling  $\kappa = M_P^{-1}$ , and it is therefore far from evident how a classically conformal Lagrangian might arise out of such a theory at low energies. For this reason, recent and not so recent attempts to incorporate scale invariance have proceeded from the assumption that the ‘true’ theory of gravity might be conformal (Weyl) invariant gravity, that is, a theory invariant under *local* rescalings of the metric and the matter fields, out of which Einstein’s theory might emerge only after spontaneous breaking of scale invariance (see [10, 11] for an early proposal along these lines and for an *entrée* into the literature, and [12, 13] for more recent work). Such a theory would ultimately also be expected to involve terms quadratic in the Weyl tensor. However, apart from the known difficulties with  $(Weyl)^2$  theories of gravity, the known ansätze at unification in general do not give rise to effectively Weyl invariant low energy theories<sup>2</sup>, despite the ubiquity of dilaton-like fields in supergravity and superstring theory. For this reason we here suggest a different route by exploring whether and under what circumstances it may be possible to get a classically conformal theory out of *non-conformal* Einstein gravity or some of its supersymmetric extensions.

The specific example we focus on is the conformal theory *par excellence*, maximally supersymmetric  $N = 4$  Yang-Mills theory in four dimensions with compact gauge group  $G$  [15, 16]. We will demonstrate explicitly how this theory can be obtained as a conformal limit of a non-conformal theory, namely gauged  $N = 4$  supergravity [17, 18, 19] as  $\kappa \rightarrow 0$ . Our construction is inspired by a recent re-derivation from gauged supergravities in three dimensions [20, 21] of the conformally invariant and globally supersymmetric ( $N \leq 8$ ) models thought to describe multiple M2 branes. To be sure, the present construction only furnishes an ‘existence proof’: the example of  $N = 4$  super Yang Mills theory ‘overshoots’ in that this theory is UV finite to all orders [22, 23, 24], hence exactly, *i.e.* quantum mechanically confor-

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<sup>2</sup>A possible exception is maximally extended conformal  $N = 4$  supergravity in four dimensions [14], but the status of this theory remains very unclear.

mal. Furthermore, the unbroken supersymmetry entails that there can be no radiative symmetry breaking [25, 26]. Nevertheless, the derivation of this model from a non-conformal theory of gravity does illustrate our main point.

The present article is thus complementary to our recent proposal [27, 28] to implement the CW mechanism in the SM <sup>3</sup>. That work rests on two basic assumptions, namely (i) the absence of intermediate mass scales between the weak scale and the Planck scale  $M_P$ ; and (ii) the requirement that the RG evolved couplings exhibit neither Landau poles nor instabilities over this whole range of energies. The first assumption (which is obviously subject to experimental falsification) is necessary because any large intermediate scale appearing in a quantum field theoretic context (such as a GUT scale at  $10^{16}$  GeV) is evidently at odds with classically unbroken conformal invariance. The second hypothesis is to ensure the ‘survival’ of the SM up to the Planck scale. As shown in [27] this requirement implies strong restrictions on the SM parameters. A further important feature is that it may make electroweak symmetry breaking mandatory due to the indirect coupling of the scalars to the strong interactions [28].

The present ansatz based on classically unbroken conformal symmetry thus pursues the same goal as low energy supersymmetry models, namely to explain the emergence and stability of small scales in particle physics. In both scenarios we must *assume* that the Planck scale theory of quantum gravity is sufficiently benign so as not to affect low energy physics in too drastic a manner. In supersymmetric models this is achieved in part via the cancellation of quadratic divergences. However, in addition to having to introduce a multitude of new (and so far unobserved) particles and couplings one faces the notorious problem that supersymmetry is impossible to break spontaneously in a way that would fully conform with low energy physics: in all ‘realistic’ scenarios, it must be broken explicitly *by hand*, but there is no explanation why the soft breaking terms are not Planck scale. By contrast, (classical) conformal symmetry is not as averse to breaking by quantum effects, and allows for greater economy in low energy model building. There, it is the (postulated) structure of the anomalous Ward identity [7] (see eqn. (38) below) which ensures the absence of quadratic divergences, hence of Planck scale mass terms and a Planck scale cosmological constant in the low energy effective action. Consequently, mass terms and symmetry breaking would arise solely from the logarithmic terms in the effective potential induced by

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<sup>3</sup>There is a large literature on this subject, see e.g. [29] for references to earlier work.

quantum corrections à la Coleman–Weinberg. In this perspective the main issue is not only to explain the embedding of the SM into an UV complete theory of quantum gravity and quantum space-time, but also to understand how a UV finite Planck scale theory can produce conformally anomalous corrections to a classically conformal low energy effective action. Even though we have so far no working model, we would thus expect that the CW mechanism in a UV finite theory must undergo a metamorphosis and be replaced a gravitational analog, in such a way that  $\kappa^{-1}$  acts as the effective cutoff. Conformal symmetry would then be broken not because of the need to regulate UV divergences, but because the quantum gravity theory into which the SM is embedded is itself *not* conformal, but leaves its footprint in the low energy effective action only in the form of logarithmic corrections.

This paper is organized as follows. In section 2 we review the basics of gauged  $N = 4$  supergravity, and in section 3 we explain in detail how to take the flat space limit of this theory in such a way that a classically conformal theory emerges; these two sections contain our main technical results. In the final section, we restate our main conjecture and discuss possible avenues towards its solution.

## 2 $N=4$ gauged supergravities

In this section we review  $N = 4, d = 4$  supergravities and their most general gaugings [17, 18, 19]. For details we refer readers to [17] whose conventions and notations we follow almost without exception.<sup>4</sup> The most general  $N = 4$  theory couples the gravitational  $N = 4$  multiplet

$$1 \times [2] \oplus 4 \times \left[\frac{3}{2}\right] \oplus 6 \times [1] \oplus 4 \times \left[\frac{1}{2}\right] \oplus 2 \times [0] \quad (1)$$

to  $n$  vector multiplets of  $N = 4$  supersymmetry

$$n \times \left( 1 \times [1] \oplus 4 \times \left[\frac{1}{2}\right] \oplus 6 \times [0] \right) \quad (2)$$

where the spin is indicated in square brackets as  $[s]$ . The scalar sectors describing the self-interactions of the two gravitational scalars and the  $6n$  scalars from the vector multiplets are governed by non-linear  $\sigma$ -models over the coset spaces  $SU(1, 1)/U(1)$  and  $SO(6, n)/SO(6) \times SO(n)$ , respectively.

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<sup>4</sup>We use  $\alpha, \beta, \dots$  rather than  $m, n, \dots$  for flat (Lorentz) indices in four dimensions.

Here  $SO(6) \cong SU(4)$  is part of the R symmetry  $U(4)$  that rotates the four supercharges [30], while  $SO(n)$  acts as an outer automorphism that rotates the  $n$  vector multiplets. The gravitational coset  $SU(1,1)/U(1)$  is parametrized by one complex field  $Z(x)$  and its complex conjugate  $Z^*(x)$  subject to the constraint  $ZZ^* < 1$ , while the  $6n$  matter scalars coordinatize the second factor  $SO(6,n)/SO(6) \times SO(n)$ . More concretely, the latter is described by a matrix  $L_I^A(x)$  with curved indices  $I, J, \dots$  and flat indices  $A, B, \dots$  assuming the values  $1, \dots, 6+n$ . This matrix transforms in the usual way as

$$L(x) \rightarrow gL(x)h(x) \quad \text{with } g \in SO(6,n) \text{ and } h(x) \in SO(6) \times SO(n) \quad (3)$$

under rigid  $SO(6,n)$  and under local  $SO(6) \times SO(n)$ . For the further analysis we need to split the (flat) matrix indices as

$$L_I^A = (L_I^{ij}, L_I^a) \quad i, j = 1, \dots, 4, \quad a = 1, \dots, n \quad (4)$$

with the  $SO(n)$  indices  $a, b, \dots$ . For the first six components, we have exploited the local isomorphism  $SO(6) \cong SU(4)$  to replace the  $SO(6)$  vector indices by an antisymmetric pair  $[ij]$  of  $SU(4)$  indices with the self-duality constraint

$$L_I^{ij} = (L_I^{ij})^* = \frac{1}{2} \varepsilon^{ijkl} L_I^{kl} \quad (5)$$

(We are here using the standard convention that complex conjugation of the  $SU(4)$  tensors corresponds to raising or lowering indices, while the position of the  $SO(n)$  indices does not matter.) The inverse matrix  $L^J_A$  then obeys  $L_I^A L_A^J = \delta_I^J$ , or

$$-L_I^{ij} L_{Jij} + L_I^a L_{Ja} = \eta_{IJ} \quad , \quad L_I^{ij} L^I_{kl} = \delta_{kl}^{ij} \quad , \quad L_I^a L^I_{ij} = 0 \quad (6)$$

where

$$\eta_{IJ} = \text{diag}(-1, -1, -1, -1, -1, -1; +1, \dots, +1) \quad (7)$$

is the Cartan-Killing metric on  $SO(6,n)$ . The bosonic fields of the theory thus consist of the vierbein  $e_\mu^\alpha$  (with the metric  $g_{\mu\nu} = e_\mu^\alpha e_{\nu\alpha}$ ), the  $(6+n)$  vector fields  $A_\mu^I$  and the scalars  $(Z, Z^*)$  and  $L_I^A$ . The fermionic sector contains four gravitinos  $\psi_\mu^i$ , four ‘dilatinos’  $\chi^i$  and  $4n$  spin- $\frac{1}{2}$  matter fermions  $\lambda^{ai}$ ; these fermionic fields are subject to

$$\gamma^5 \psi_\mu^i = +\psi_\mu^i \quad , \quad \gamma^5 \chi^i = -\chi^i \quad , \quad \gamma^5 \lambda^{ai} = +\lambda^{ai} \quad (8)$$

With the gravitational coupling (=inverse Planck mass)  $\kappa$  of (length) dimension  $cm$ , the canonical dimensions of these fields and the supersymmetry parameters  $\varepsilon^i$  are as follows:

$$\begin{aligned} [g_{\mu\nu}] &= [e_\mu^\alpha] = [L_I^A] = [Z] = 0 \quad , \quad [A_\mu^I] = -1 \\ [\psi_\mu^i] &= [\chi^i] = [\lambda^{ai}] = -3/2 \quad , \quad [\varepsilon^i] = +1/2 \end{aligned} \quad (9)$$

In passing we note that the torus reduction of pure half maximal  $D = 10$  supergravity from ten to four dimensions would give rise to a theory with six vector multiplets and thus 12 vectors  $A_\mu^I$ , with the first six vectors corresponding to the Kaluza-Klein vectors arising from the  $D = 10$  metric  $G_{MN}$ , and the remaining six from the 2-form field  $B_{MN}$ . The resulting theory would thus yield the coset space  $SO(6,6)/SO(6) \times SO(6)$ .

The Lagrangian of the theory is obtained in the usual way by means of the Noether procedure [17]. We will here give it right away for the *gauged* theory. This means that one promotes a subgroup of the global (rigid) symmetry group  $SO(6,n)$  to a local group using the vector fields  $A_\mu^I$  as Yang-Mills fields. In order to preserve the local  $N = 4$  supersymmetry of the original (ungauged) Lagrangian certain consistency conditions must be obeyed.

Let us first present the Maurer-Cartan forms. For the coset  $SU(1,1)/U(1)$  they are given by

$$Q_\mu = -\frac{1}{2}(\Phi\partial_\mu\Psi^* + \Psi\partial_\mu\Phi^*) \quad , \quad P_\mu = \frac{1}{2}(\Phi\partial_\mu\Psi - \Psi\partial_\mu\Phi) \quad (10)$$

with

$$\Phi := \frac{1 - Z^*}{\sqrt{1 - ZZ^*}} \quad , \quad \Psi := \frac{1 + Z^*}{\sqrt{1 - ZZ^*}} \quad (11)$$

Here the (imaginary) vector  $Q_\mu$  is the  $U(1)$  connection while the vector  $P_\mu$  corresponding to the two coset degrees of freedom is complex. For the matter coset  $SO(6,n)/SO(6) \times SO(n)$  the Maurer-Cartan forms are

$$\begin{aligned} P_{\mu a}{}^{ij} &= L^I{}_a \left( \partial_\mu \delta_I^K + f_{IJ}{}^K A_\mu^J \right) L_K{}^{ij} \\ Q_{\mu ab} &= L^I{}_a \left( \partial_\mu \delta_I^K + f_{IJ}{}^K A_\mu^J \right) L_K b \\ Q_{\mu j}{}^i &= L^{I ik} \left( \partial_\mu \delta_I^K + f_{IJ}{}^K A_\mu^J \right) L_K{}^{kj} \end{aligned} \quad (12)$$

Here we have already included the gauge couplings via the structure constants  $f_{IJK}$  of the gauge group (we absorb the gauge coupling constants into  $f_{IJK}$ ).

The integrability (Maurer-Cartan) relations that follow from this definition are given in eq. (10) of [17]. The structure constants must be completely antisymmetric after lowering the index  $K$  [17]

$$f_{IJ}{}^L \eta_{KL} = f_{[IJ}{}^L \eta_{K]L} \quad (13)$$

The quantities  $(Q_{\mu ab}, Q_{\mu j}{}^i)$  are the ‘composite’ gauge connections for the local  $SO(6) \times SO(n)$ . In the ungauged theory ( $f_{IJK} = 0$ ) the fermions couple to the scalar fields *only* via the  $Q$ ’s and  $P$ ’s. The full covariant derivatives are

$$\begin{aligned} D_\mu \psi_\nu^i &= \partial_\mu \psi_\nu^i + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta} \psi_\nu^i - \frac{1}{2} Q_\mu \psi_\nu^i \\ D_\mu \chi^i &= \partial_\mu \chi^i + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta} \chi^i + Q_{\mu j}{}^i \chi^j + \frac{3}{2} Q_\mu \chi^i \\ D_\mu \lambda_a^i &= \partial_\mu \lambda_a^i + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta} \lambda_a^i + Q_{\mu a}{}^b \lambda_b^i + Q_{\mu j}{}^i \lambda_a^j + \frac{1}{2} Q_\mu \lambda_a^i \end{aligned} \quad (14)$$

with the usual spin connection  $\omega_\mu{}^{\alpha\beta}(e)$ . The quantities  $P_\mu$  belonging to the coset, on the other hand, appear in the kinetic terms of the scalar fields and in the Noether couplings to the fermions.

While the ungauged theory has only derivative couplings to the scalar fields, in the gauged theory there appear Yukawa-like couplings of the fermions as well as a (non-linear) potential for the scalar fields. These new couplings (which vanish when the gauge couplings are set to zero) involve the dimensionless quantities

$$\begin{aligned} C_{ij} &:= f_{IJK} L^I{}_{ik} L^J{}_{jl} L^{Kkl} = C_{ji} \\ C_{aj}{}^i &:= f_{IJK} L^I{}^{ik} L^J{}_{kj} L_{Ka} \\ C_{ab}{}^{ij} &:= f_{IJK} L^I{}_{a} L^J{}_{b} L^{Kij} = -C_{ba}{}^{ij} \end{aligned} \quad (15)$$

These are just a variant of the so-called ‘T tensor’ introduced in [31]. The preservation of local supersymmetry then requires that the following identities must be satisfied

$$\begin{aligned} C^{ai}{}_{k} C_{aj}{}^k - \frac{4}{9} C^{ik} C_{kj} - \text{trace} &= 0 \\ C^{ab}{}_{k(i} C_{bj)}{}^k - \frac{2}{3} C^{ak}{}_{(i} C_{j)k} &= 0 \end{aligned} \quad (16)$$

In addition, these tensors must satisfy the differential identities given in eq. (12) of [17]. Together these identities restrict the possible choices of consistent gauge groups  $\subset SO(6, n)$ . While only specific examples were studied

in [17, 18], more recent work [19] based on embedding tensor techniques (see [32] and references therein) has led to a more systematic classification of gauge groups (which can be compact, non-compact or non-semisimple). As a special case one also recovers the theories obtained by torus reduction of  $D = 10$  supergravity coupled to  $k$  vector multiplets with a Yang Mills gauge group  $G$  of dimension  $= k$ . This would give global  $SO(6, 6 + k)$  in four dimensions such that  $G \subset U(1)^6 \times SO(k)$ .

Modulo terms quartic in the fermions the full Lagrangian has been derived in [17, 18] and we here simply quote the result from [17]. Because we are here interested in the flat space limit we re-instate the dimensionful coupling  $\kappa$  in all formulas. We split the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{gauge} \quad (17)$$

with

$$\begin{aligned} e^{-1}\mathcal{L}_0 &= \frac{1}{2\kappa^2}R - \bar{\psi}_\mu^i \gamma^{\mu\nu\rho} D_\nu \psi_{\rho i} - \frac{1}{2} a_{IJ} F_{\mu\nu}^+{}^I F_{\mu\nu}^+{}^J \\ &\quad - \frac{1}{2} \bar{\chi}^i \gamma^\mu D_\mu \chi_i - \frac{1}{\kappa^2} P_\mu^* P^\mu - \frac{1}{2} \bar{\lambda}^{ai} \gamma^\mu D_\mu \lambda_{ai} - \frac{1}{\kappa^2} P_\mu^{aij} P_{aij}^\mu \\ &\quad + \bar{\chi}^i \gamma^\mu \gamma^\nu \psi_{\mu i} P_\nu^* - 2i \bar{\lambda}^{ai} \gamma^\mu \gamma^\nu \psi_\mu^j P_{\nu aij} \\ &\quad + \kappa \Phi^{-1} F_{\mu\nu}^+{}^I \left[ \frac{i}{2} \bar{\psi}_i^\lambda \gamma_{[\lambda} \gamma^{\mu\nu} \gamma_{\tau]} \psi_j^\tau L_I^{ij} - \frac{i}{2} \bar{\psi}^{\lambda i} \gamma^{\mu\nu} \gamma_\lambda \chi^j L_{I ij} \right. \\ &\quad \left. + \frac{1}{2} \bar{\psi}_{\lambda i} \gamma^{\mu\nu} \gamma^\lambda \lambda^{ai} L_{Ia} + \frac{1}{2} \bar{\lambda}_i^a \gamma^{\mu\nu} \chi^i L_I^a + \frac{i}{2} \bar{\lambda}^{ai} \gamma^{\mu\nu} \lambda^{aj} L_{I ij} \right] \\ &\quad + h.c. \end{aligned} \quad (18)$$

with

$$\begin{aligned} F_{\mu\nu}^I &:= \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + f^I{}_{JK} A_\mu^J A_\nu^K \\ F_{\mu\nu}^\pm &:= \frac{1}{2} \left( F_{\mu\nu} \pm \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \right) \end{aligned} \quad (19)$$

and

$$a_{IJ} := \frac{1+Z}{1-Z^*} L_I^{ij} L_{Jij} + \frac{1+Z^*}{1-Z} L_I^a L_{Ja} \quad (20)$$

All terms proportional to the gauge couplings are assembled into  $\mathcal{L}_{gauge}$  (and thus absent in the ungauged theory), *viz.*

$$e^{-1}\mathcal{L}_{gauge} = -\frac{2i}{3\kappa} \Phi C_{ij} \left( \bar{\psi}_\mu^i \gamma^{\mu\nu} \psi_\nu^j - \bar{\psi}_\mu^i \gamma^\mu \chi^j - \bar{\lambda}^{ai} \lambda_a^j \right)$$

$$\begin{aligned}
& + \frac{2i}{\kappa} \Phi C_{aj}{}^i (\bar{\psi}_\mu^j \gamma^\mu \lambda_i^a + \bar{\chi}^j \lambda_i^a) - \frac{2i}{\kappa} \Phi C_{ab}{}^{ij} \bar{\lambda}_i^a \lambda_j^b \\
& - \frac{1}{2\kappa^4} \Phi^* \Phi \left( C_{aj}{}^i C^{aj}{}_i - \frac{4}{9} C^{ij} C_{ij} \right) + h.c. \tag{21}
\end{aligned}$$

The last term in (21) is the *scalar potential*: as is obvious from the expression given it is unbounded from below — a standard feature of gauged supergravity potentials.

The local supersymmetry variations are given by (modulo cubic terms)

$$\begin{aligned}
\delta e_\mu{}^\alpha &= \kappa \bar{\varepsilon}^i \gamma^\alpha \psi_{\mu i} + h.c. \\
\delta \psi_\mu^i &= \frac{1}{\kappa} D_\mu \varepsilon^i + \frac{i}{2\Phi} \gamma^{\mu\nu} F_{\mu\nu}{}^I L_I{}^{ij} \varepsilon_j + \frac{2i}{3\kappa^2} \Phi^* \gamma_\mu C^{ij} \varepsilon_j \\
\delta A_\mu^I &= -2i L^I{}_{ij} \left( \bar{\varepsilon}^i \psi_\mu^j - \frac{1}{2} \bar{\varepsilon}^i \gamma_\mu \chi^j \right) + \Phi L^I{}_a \bar{\varepsilon}^i \gamma_\mu \lambda_i^a + h.c. \\
\delta L^{Ia} &= 2i \kappa L^I{}_{ij} \bar{\varepsilon}^i \lambda^{aj} + h.c. \\
\delta L^I{}_{ij} &= -2i \kappa \bar{\varepsilon}_{[i} \lambda_{j]}^a L^I{}_a - \text{dual} \\
\delta \chi^i &= \frac{i}{\Phi^*} \gamma^{\mu\nu} F_{\mu\nu}{}^I L_I{}^{ij} \varepsilon_j + \frac{2}{\kappa} \gamma^\mu P_\mu \varepsilon^i + \frac{4}{3\kappa^2} \Phi C^{ij} \varepsilon_j \\
\delta \lambda^{ai} &= -\frac{1}{2\Phi^*} \gamma^{\mu\nu} F_{\mu\nu}{}^I L_I{}^a \varepsilon^i + \frac{2i}{\kappa} \gamma^\mu P_\mu{}^{aij} \varepsilon_j - \frac{2i}{\kappa^2} \Phi C^{ai}{}_j \varepsilon^j \\
\delta \Phi &= -\kappa \Phi^* \bar{\varepsilon}_i \chi^i \tag{22}
\end{aligned}$$

### 3 The conformal limit

For the flat space limit we take  $\kappa \rightarrow 0$ . With the usual metric ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{23}$$

the curved space-time is flattened out in this limit, and gravity decouples from the matter fields. Next, we must make sure that not only gravity, but the full gravitational supermultiplet decouples from the  $n$  vector multiplets in this limit. The gravitino variation contains the derivative on  $\varepsilon^i$ , which comes with a factor of  $\kappa^{-1}$ ; to keep this finite in the limit we must demand

$$D_\mu \varepsilon^i = 0 \quad \Rightarrow \quad \varepsilon^i(x) = \text{const} \tag{24}$$

As expected, the limiting theory, if it exists, can only be globally (rigidly) supersymmetric. For the scalar fields, we have a formula analogous to (23),

$$Z = \kappa z \quad , \quad L_I^A = (\exp(\kappa\phi))_I{}^A = \delta_I^A + \kappa \phi_I^A + \dots \tag{25}$$

where the redefined scalar fields  $z$  and  $\phi_I^A$  are now of dimension  $(cm)^{-1}$  (like the vector fields) and the range of  $z$  becomes  $|z| < \kappa^{-1}$ . In the limit  $\kappa \rightarrow 0$  both cosets are thus flattened; in particular, from (11) we see that the fields  $Z(x)$  decouple and  $\Phi, \Psi \rightarrow 1$ . Furthermore,

$$SO(6, n)/SO(6) \times SO(n) \longrightarrow \mathbb{R}^{6n} \quad (26)$$

In the triangular gauge (where the ‘diagonal’ components of  $\phi_I^A$  vanish), the  $6n$  scalar fields  $\phi_a^{ij}$  from the  $n$  vector multiplets thus become coordinates of  $\mathbb{R}^{6n}$ . For the matrix  $L_I^A$  we obtain

$$L_{ij}^{kl} = \delta_{ij}^{kl} + \mathcal{O}(\kappa^2) \quad , \quad L_a^b = \delta_a^b + \mathcal{O}(\kappa^2) \quad , \quad L_a^{ij} = \kappa \phi_a^{ij} + \mathcal{O}(\kappa^3) \quad (27)$$

(for notational simplicity, we do not distinguish anymore between flat and curved indices here).

To obtain a *conformal theory*, however, (23) is not enough. To see that extra requirements are needed we note that non-compact and non-semisimple gaugings (the latter were not considered in [17, 18], but see [19]) cannot yield a conformal limit: both of these gaugings would involve Lie algebra generators in the non-compact part of  $SO(6, n)$ , and thus mix the six gravitational with the  $n$  matter multiplet vectors in such a way that the gravitational vectors cannot decouple. This can be directly seen by expanding the Maurer-Cartan connections, cf. (12), making use of (27)

$$\begin{aligned} Q_{\mu ab} &= f_{abc} A_{\mu}^c + f_{abij} A_{\mu}^{ij} + \mathcal{O}(\kappa^2) \\ Q_{\mu j}^i &= f_{jklm}^{ik} A_{\mu}^{lm} + f_{jkc}^{ik} A_{\mu}^c + \mathcal{O}(\kappa^2) \end{aligned} \quad (28)$$

In order to decouple the six vectors from the gravitational multiplet (1) we must therefore demand that all components of  $f_{IJK}$  vanish except for  $f_{abc}$ . In other words the gauge group must be entirely contained in the  $SO(n)$  symmetry group that rotates the  $n$  vector multiplets (2) and commutes with the local  $N = 4$  supersymmetry; that is, we have <sup>5</sup>

$$G \subset SO(n) \subset SO(6, n) \quad , \quad \dim G = n \quad (29)$$

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<sup>5</sup>An analogous restriction is found in the construction of [20, 21], and explains why only free field theories (which are trivially conformal) can arise when this limit is applied to  $D = 3$  gauged supergravities with  $N > 8$ , that is, why globally supersymmetric interacting conformal theories in three dimensions exist only for  $N \leq 8$ . Here it follows similarly that no non-trivial flat space limit with residual rigid supersymmetry exists for  $N > 4$  supergravities in four dimensions.

Let us mention that the exclusion of non-compact and non-semisimple gauge groups also ensures the positive definiteness of the kinetic terms of the vector fields (whereas otherwise the flat space limit would suffer from indefinite kinetic terms, unlike the original supergravity theory where non-compact or non-semisimple gauge groups are compatible with positive kinetic terms for the gauge fields). According to [17] one possible set of consistent gaugings is obtained with gauge groups  $SU(2) \times SU(2) \times G$  where  $SU(2) \times SU(2) \subset SO(6)$ , and  $G$  is of dimension  $n$ ; the relevant gaugings here are thus obtained by setting the  $SU(2)$  gauge couplings equal to zero. For the theories obtained by dimensional reduction of  $D = 10$  type I supergravity coupled to  $k$  vector multiplets to four dimensions we have  $G = U(1)^6 \times H \subset SO(6+k)$  where  $H$  is the (compact) gauge group in ten dimensions.

Let us now return to the Lagrangian and supersymmetry variations. First, we note that with (29),  $L_I^A = \delta_I^A$  (where  $\phi_a^{ij} = 0$ ) is a fully supersymmetric stationary point with vanishing cosmological constant. Namely, from (15) it follows immediately that all  $C$ -tensors vanish at this point, hence both the extremization and the supersymmetry condition eqs. (20) and (21) of [17] are trivially satisfied (this would not be so for other choices of  $G$ ). Let us then discuss the terms that decouple ‘trivially’, namely all those terms in (18) and (22) that do not involve the gauge couplings, and hence are independent of the choice of gauge group. With (23) the vierbein variation obviously reduces to that of a free spin-2 field. In the gravitino variation, the first term is absent due to (24) while the second term starts at  $\mathcal{O}(1)$ . Splitting the index  $I$ , we see that the field strengths  $F_{\mu\nu}^{ij}$  of  $\mathcal{O}(1)$  belong to the gravitational multiplet. The coupling to the matter vectors  $A_\mu^a$ , on the other hand, occurs via the matrix element  $L_a^{ij}$ ; by (27) it is therefore suppressed by an extra factor of  $\kappa$ , hence vanishes in the limit, as required. The same mechanism is at work in the variation  $\delta A_\mu^I$ : the Kaluza Klein vectors pair up with  $(\psi_\mu^i, \chi^i)$  while the  $6n$  matter vector terms have at least one extra factor of  $\kappa$ . The remaining variations are dealt with similarly. Idem for the gauge-independent terms collected in the Lagrangian (18); for instance, with (29) the coset part of the Maurer Cartan form becomes

$$P_{\mu a}^{ij} = \kappa \mathcal{D}_\mu \phi_a^{ij} + \mathcal{O}(\kappa^3) \equiv \kappa (\partial_\mu \phi_a^{ij} + f_{abc} A_\mu^b \phi^{cij}) + \mathcal{O}(\kappa^3) \quad (30)$$

hence

$$\frac{1}{\kappa^2} P_\mu^{a ij} P_{a ij}^\mu = \mathcal{D}_\mu \phi_a^{ij} \mathcal{D}^\mu \phi^a_{ij} + \mathcal{O}(\kappa^2) \quad (31)$$

and the scalar kinetic term acquires the requisite form for  $\kappa \rightarrow 0$ . Again

it is straightforward to see that all the other matter terms with derivative scalar couplings decouple from the supergravitational degrees of freedom in this limit.

The ‘non-trivial’ terms are the ones involving the gauge couplings and the non-derivative scalar couplings. In order to analyze the relevant contributions we expand the  $C$ -tensors of (15) in terms of the  $6n$  scalars  $\phi_a^{ij}$  in (27). From inspection of the scalar potential and the Yukawa terms in (21) we see that the potentially dangerous contributions are the ones involving inverse powers of  $\kappa$ , and we must therefore ensure that these terms are absent before taking  $\kappa \rightarrow 0$ . With the choice of gauge group (29) we find

$$\begin{aligned} C_{ij} &= \kappa^3 f_{abc} \phi_a^{ik} \phi_b^{jl} \phi_c^{kl} + \mathcal{O}(\kappa^4) \\ C_{aj}{}^i &= \kappa^2 f_{abc} \phi_b^{ik} \phi_c^{jk} + \mathcal{O}(\kappa^3) \\ C_{ab}{}^{ij} &= \kappa f_{abc} \phi_c^{ij} + \mathcal{O}(\kappa^2) \end{aligned} \quad (32)$$

Again we see that the choice of gauge group is crucial in order to ensure consistency of the decoupling limit: if there was a non-vanishing structure constant  $f_{abij}$  mixing the  $SO(6)$  and  $SO(n)$  subgroups  $C_{ij}$  would start at  $\mathcal{O}(\kappa^2)$  instead, and for instance the scalars would not decouple in the gravitino variation and other terms. We next substitute these expansions into the Lagrangian and the supersymmetry variations. For the potential we get

$$\frac{1}{2\kappa^4} \Phi^* \Phi \left( C_{aj}{}^i C^{aj}{}_i - \frac{4}{9} C^{ij} C_{ij} \right) = \frac{1}{2} f_{abe} f_{cde} \phi^{aik} \phi^b{}_{jk} \phi^{cjl} \phi^d{}_{il} + \mathcal{O}(\kappa) \quad (33)$$

To see that this is just the usual potential of  $N = 4$  Yang Mills theory in terms of six real Lie algebra valued scalar fields  $X^{ma}$  (with  $m, n = 1, \dots, 6$ ) we make use of the six real antisymmetric matrices  $(\alpha_{ij}^r, \beta_{ij}^r)$  for  $r = 1, 2, 3$  (explicit expressions are given, for instance, in [15, 16])

$$\phi^a{}_{ij} = \sum_{r=1,2,3} (\alpha_{ij}^r X^{ra} + i\beta_{ij}^r X^{(r+3)a}) \quad (34)$$

whence the  $\mathcal{O}(1)$  term in (33) reduces to the standard expression

$$f_{abe} f_{cde} \phi^{aik} \phi^b{}_{jk} \phi^{cjl} \phi^d{}_{il} \propto \text{Tr} ([X^m, X^n])^2. \quad (35)$$

Conveniently, the conformal limit also disposes of the term  $\propto C^{ij} C_{ij}$  which makes the potential unbounded from below. Again it is easy to see that

the unboundedness of the potential would persist with the ‘wrong’ choice of gauge group.

In an analogous fashion we can show that in the Yukawa-like terms in the Lagrangian only the term containing  $C_{ab}{}^{ij}$  survives the limit, and we end up with

$$-\frac{2i}{\kappa}C_{ab}{}^{ij}\bar{\lambda}_i^a\lambda_j^b = -2if_{abc}\phi^{cij}\bar{\lambda}_i^a\lambda_j^b + \mathcal{O}(\kappa) \quad (36)$$

As before it follows that

$$f_{abc}\phi^{cij}\bar{\lambda}_i^a\lambda_j^b + h.c. \propto \text{Tr } \bar{\lambda}\Gamma^m[X^m, \lambda] \quad (37)$$

(where  $\Gamma^m$  are the  $SO(6)$   $\Gamma$ -matrices), and similarly for the remaining terms in the Lagrangian.

## 4 Metamorphosis of the CW mechanism?

As we already pointed out, the example of  $N = 4$  super Yang Mills falls short of what we want in several respects (apart from anyway not being a realistic model of particle physics). First, the flat space theory is exactly conformally invariant: its  $\beta$ -functions vanishes to all orders and there is no conformal anomaly (however, there are non-trivial anomalous dimensions for composite operators). Secondly, radiative breaking of symmetries cannot occur as a consequence of unbroken supersymmetry. Finally, the non-conformal  $N = 4$  supergravity into which this theory is embedded is not expected to be UV finite; hence, the UV divergent gravitational corrections will destroy whatever finiteness or conformal properties the flat space theory possesses.

Nevertheless there are some general conclusions that can be drawn from the above construction and that may apply to other and hopefully more realistic models.<sup>6</sup> To obtain a classically conformal theory as a flat space limit from a non-conformal UV completion of the SM, we must ensure that negative powers of  $\kappa$  are absent before taking the limit  $\kappa \rightarrow 0$ , thereby also excluding a ‘bare’ cosmological term  $\propto \mathcal{O}(\kappa^{-4})$ . As it happens, the  $N = 4$  theories discussed above do allow for a trivial stationary point that meets all these requirements, but more interesting extrema with non-vanishing expectation values of the scalar fields with these properties appear not to exist for (29)

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<sup>6</sup>See also [33] for an early attempt to construct a realistic model with broken supersymmetry by taking the  $\kappa \rightarrow 0$  limit and imposing restrictions similar to the ones discussed here.

or other (non-compact) gauge groups [17]. In addition, we would require the stationary points to break supersymmetry completely in order to allow for radiative breaking of symmetries. This is not an easy task to accomplish: for instance, while the potential of  $N = 8$  supergravity (presumably the only quantum field theoretic extension of Einstein's theory that may be UV finite) does admit non-trivial stationary points [34], all of these come with a non-vanishing cosmological constant  $\propto \mathcal{O}(\kappa^{-4})$ , and the ones that do break supersymmetry are all unstable. It is therefore clear that 'something extra' beyond quantum field theory is required to avoid this *impasse*, and that we must invoke an as yet unknown mechanism operating at the Planck scale to make this idea work.

Independently of what the UV completion of the SM is, let us therefore restate our main hypothesis: apart from the explicit  $\mathcal{O}(\kappa)$  terms (which can be neglected in the  $\kappa \rightarrow 0$  limit) the UV completion of the SM gives rise to finite logarithmic quantum corrections (depending on  $\log(\kappa\phi)$ ) which induce conformal symmetry breaking. These terms would constitute an 'observable' signature of quantum gravity, in the sense that electroweak symmetry breaking might be entirely due to finite (and in principle computable) quantum gravitational effects. This immediately raises the question how such anomalous terms might appear in the low energy effective action of a theory which is expected to be UV finite. Conventional wisdom would suggest that there is no need to regulate UV divergences in such a theory, hence there cannot exist anomalies in the usual sense. The CW breaking would thus have to come from *finite* logarithmic corrections induced by quantum gravity, in accordance with the anomalous Ward identity (for  $\kappa \rightarrow 0$ )

$$T^\mu{}_\mu = \beta(\hat{\lambda}(\kappa\phi))\phi^4 + Z(\hat{\lambda}(\kappa\phi))\partial_\mu\phi\partial^\mu\phi \quad (38)$$

where the gravitational coupling  $\kappa$  replaces the dimensionful scale that must be introduced in quantum field theory in order to parametrize the CW potential, and where the function  $Z$  (which starts only at higher orders in the couplings) and the effective running coupling  $\hat{\lambda}$  depend only logarithmically on their argument. While the effective potential must obey a renormalization group equation to account for the fact that this scale can be chosen arbitrarily, the gravitational coupling  $\kappa$  is a *fixed* parameter. The effective potential at low energies can then be determined in principle by 'integrating' the anomalous Ward identity. We note that the potential importance of (38) for the solution of the hierarchy problem was already emphasized in [7].

The existence of anomalous logarithmic terms in a UV finite theory is also suggested by fascinating recent advances in the computation of  $n$ -point amplitudes in  $N = 4$  super Yang Mills theory [35, 36, 37]. These have exposed (amongst other things) a new ‘dual’ conformal symmetry subtly intertwining IR and UV domains [38, 39, 40] as well as subtle ‘anomalous’ effects in  $n$ -point correlators via so-called cusp anomalies in the dual Wilson loops. Despite the exact conformal invariance one obtains a non-vanishing result (conjectured to be exact to all orders)

$$\Gamma_4(p_1, p_2, p_3, p_4) \propto \Gamma_{cusp}(g) \log^2 \left( \frac{p_1 \cdot p_2}{p_2 \cdot p_3} \right) \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \quad (39)$$

for the 4-point amplitude with  $p_1^2 = \dots = p_4^2 = 0$  (this expression is cyclically invariant because  $p_1 \cdot p_2 = p_3 \cdot p_4$  and  $p_2 \cdot p_3 = p_4 \cdot p_1$ ). This result is obtained by integrating the anomalous Ward identity for the generator of conformal boosts [36, 37]. Because the coupling  $g$  in  $N = 4$  super Yang Mills theory does not run, there can be no dynamically generated scale as in QCD, and therefore the anomaly cannot appear in the effective potential (which vanishes) but only in momentum dependent terms, that is, the non-local part of the effective action. By contrast, gravity does possess a dimensionful scale, so we can form the dimensionless quantity  $\kappa\phi$ , and anomalous contributions in principle can show up in the *local* part of the effective action. Let us also remark that it is much easier to arrange for hierarchical expectation values if there are only logarithmic corrections to the potential. This expectation is also supported by the numerical finding that the minima of the CW effective potential [27, 28] tend to be very shallow.

The main conjecture put forward in this paper can therefore be summarized as follows: the hierarchy problem can conceivably be solved via ‘anomalous’ logarithmic quantum corrections in a UV finite theory of quantum gravity, if the latter admits a flat space limit which is classically conformally invariant. The mass spectrum and pattern of couplings observed in elementary particle physics could then have their origin in quantum gravity.

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