Exacting $\mathcal{N} = 4$ superconformal symmetry

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ABSTRACT: Tree level scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory are almost, but not exactly invariant under the free action of the $\mathcal{N} = 4$ superconformal algebra. What causes the non-invariance is the holomorphic anomaly at poles where external particles become collinear. In this paper we propose a deformation of the free superconformal representation by contributions which change the number of external legs. This modified classical representation not only makes tree amplitudes fully invariant, but it also leads to additional constraints from symmetry alone mediating between hitherto unrelated amplitudes. Moreover, in a constructive approach it appears to fully constrain all tree amplitudes when combined with dual superconformal alias Yangian symmetry.

KEYWORDS: Anomalies in Field and String Theories, Global Symmetries, AdS-CFT Correspondence, Supersymmetric gauge theory

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1   Introduction and overview

Maximally supersymmetric gauge theory in four spacetime dimensions — $\mathcal{N} = 4$ super Yang-Mills (SYM) — is an interacting quantum field theory with a host of useful features: It has a unique massless action with only a few adjustable parameters. Perturbative calculations typically show many cancellations such that, e.g., the model's classical conformal symmetry is preserved at the quantum level due to the absence of running couplings. Furthermore, a lot of evidence has accumulated in favour of the AdS/CFT correspondence [1] claiming that the model is exactly dual to a string theory on an AdS background.

On top of these features, calculations in the planar alias the large-N_c limit for a U(N_c) gauge group have turned out to produce surprising final results in many cases. Simplifications are certainly related to the absence of string interactions in the dual string theory, yet it takes more to explain most of the observed mysteries. Once fully understood and exploited, we hope that calculations at high perturbative orders and even at finite coupling become tractable. For instance, the spectrum of anomalous dimensions of local operators appears to be governed by a certain integrable model [2–4] which makes calculations very efficient, see e.g. the reviews [5]. Integrability is usually synonymous with the existence of an infinite dimensional algebra which enlarges the manifest symmetries of the model and which (almost) completely constrains the dynamics. In this case superconformal symmetry apparently extends to its loop algebra whose quantisation is a Yangian algebra [4, 6].

A different field of investigation in $\mathcal{N} = 4$ SYM which has advanced substantially in recent years is the study of on-shell scattering amplitudes. These are particularly important because of their relations to scattering amplitudes in QCD (for phenomenological purposes) and in $\mathcal{N} = 8$ supergravity through the KLT relations (for demonstrating finiteness of a particular theory of quantum gravity). In particular, the twistor space approach [7, 8] (see [9–11] and references therein for further accounts) following from the ideas of Penrose [12] has sparked many new investigations leading to a much better understanding. Subsequently, recursion relations for all tree-level amplitudes have been set up [13] and their on-shell superspace version [14–17] solved explicitly [18]. Moreover, amplitudes at loop level can be computed efficiently and reliably through the methods of generalised unitarity whose basic framework was introduced in [19, 20] and further developed in [21]; see [22] for a useful review. Among others, these enabled the computation of the planar amplitudes with four legs up to four loops and beyond [23–27] as well as amplitudes with six or more legs at two loops [28].

It is well known that scattering amplitudes for massless particles are problematic because asymptotic states cannot be defined properly: a single massless particle can decay into an unbounded number of massless particles with collinear momenta. This manifests itself in the appearance of infra-red divergences at loop level when integrating over collinear momentum configurations. The divergences call for the introduction of some regulator, most commonly a minimal subtraction scheme in dimensional regularisation or reduction to $d = 4 - 2\epsilon$ spacetime dimensions. The resulting amplitudes will then have singularities as $\epsilon \to 0$, typically two factors of $1/\epsilon$ per loop level. The structure of IR divergences is understood reasonably well: they combine into an exponent which can be factored out from the amplitude leaving a finite part behind [29]. The form of the exponent is constrained by field theory and symmetry considerations. The same would be true for the finite remainder function, however, some symmetries, such as special conformal transformations, may have been deformed or broken by the introduction of the regulator.

Some structural simplifications come about in the planar limit: There the IR divergences are determined through a single function of the coupling, the so-called cusp anomalous dimension [30], see also [24].[1] Interestingly, this very same cusp anomalous di-

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[1]The subleading collinear anomalous dimension is scheme dependent and not a good observable on its

mension can also be computed from anomalous dimensions of local operators which in turn are governed by the above mentioned integrable model, see in particular [32]. One might therefore wonder if there are further connections between planar scattering amplitudes and the integrable structures for planar anomalous dimensions.

Indeed, the unitarity construction of higher-loop planar amplitudes shows some surprises: Many of the integrals that could in principle contribute to the unitarity construction do not appear in practice. Only such integrals with certain conformal weights appear to have non-zero prefactors [25, 27, 33]. It is however not the standard conformal symmetry which leads to these restrictions, but rather a conformal symmetry acting on momentum space. Curiously, Wilson loops in this dual momentum space were seen to be equivalent to certain scattering amplitudes [34–37], see also the reviews [38]. Later the dual conformal symmetry was extended to superconformal symmetry and shown to apply to all tree level scattering amplitudes [15, 39]. In string theory the appearance of such dual superconformal symmetries can be explained by a supersymmetric T-duality transformation which turns out to map the string model to itself [34, 40]. The superconformal symmetries of the dual model become the dual superconformal symmetries of the original model. Moreover, the two sets of superconformal symmetries form two inequivalent superconformal subalgebras of the loop algebra representing classical string integrability [40–42]. Alternatively one can say that the loop algebra alias integrability results as the closure of the two sets of superconformal symmetries. On the gauge theory side, the realisation of integrability alias Yangian symmetry for tree-level scattering amplitudes was derived in [43] and shown to be self-consistent.

All of these developments together point towards integrability of planar scattering amplitudes in $\mathcal{N} = 4$ SYM, not only at tree level, but at all loops and even non-perturbatively. This suggests that one might be able to compute all planar scattering amplitudes very efficiently and without the need for lengthy field theory or generalised unitarity calculations. Could there be some differential or integral equation determining the finite part of scattering amplitudes?

Before such an equation can be established, several problems have to be overcome: The regulator for the IR divergences breaks the special conformal symmetries. E.g. in dimensional regularisation the dimensionality of spacetime is $d = 4 - 2\epsilon$ while conformal symmetry requires exactly $d = 4$. Consequently conformal symmetry for scattering amplitudes is either broken beyond repair or it is at least obscured at loop level. For dual conformal symmetry at loop level the second option seems to apply; its breakdown can be formulated as an anomaly originating from UV divergences for the dual Wilson loops [36]. One may expect the same to be true for the original conformal symmetry. The $\mathcal{N} = 4$ model is known to be exactly conformal at the quantum level. Conformal symmetry persists even in the presence of the UV divergences accompanying the anomalous dimensions of local operators. The main difficulty for scattering amplitudes rests in the IR nature of the divergences whose structure is less clear than for UV divergences.

The above discussion hides two important points at tree level which appear to paint a pessimistic picture. Firstly, conformal symmetry is subtle and even at tree level it does

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own. For a recent discussion of these subleading singularities see [31].

not strictly hold: Amplitudes were shown to be conformal when the external momenta are in a general position. Whenever two momenta become collinear, however, conformal symmetry becomes anomalous. A related anomaly is made obvious by going to the twistor space representation of the amplitudes [9, 10].[2] On a second thought this subtlety is not very surprising because it is precisely the collinear momenta which cause the IR divergences which in turn lead to the conformal anomaly. Only at tree level can collinearities be avoided through a choice of external momenta while at loop level internal momenta are integrated over and collinearities become inevitable. Secondly, conformal and dual conformal symmetry together are not even sufficient to fix tree level amplitudes completely. The basis of tree amplitudes introduced in [18] or similarly in the twistor space picture is (almost, see above) invariant under both symmetries. Consequently all linear combinations are invariant as well and symmetry alone does not determine the correct linear combination for the physical scattering amplitude.[3] Only additional physical input, such as a correct set of singularities, appears to fix the right coefficients, see also the very recent work [11] as well as [44] which appeared after an earlier version of the present work.

In this paper we propose a resolution to the problems of conformal symmetry at tree level discussed above: The naive action of infinitesimal conformal transformations on scattering amplitudes is not complete. It needs to be supplemented by correction terms which cure the collinear anomaly at tree level. We also believe that similar corrections can remove the anomalies at loop level and thus render scattering amplitudes exactly conformal, albeit using a deformed representation. The proposed corrections act in similar fashion as the symmetry generators of the integrable spin chain for anomalous dimensions. Most importantly, the corrections have the ability to change the number of legs of scattering amplitudes. Such generators cannot act on individual scattering amplitudes, but rather they must act on the generating functional of all amplitudes.

Altogether this paints a consistent picture in view of the problems introduced by massless asymptotic states:[4] The number of massless asymptotic particles is not a well-defined quantity. Hence it is natural to consider the generating functional of scattering amplitudes (which can be viewed as the scattering operator) rather than individual scattering amplitudes with a fixed number of legs. The purpose of the correction terms is to take into account the overcounting of states in the Fock space where momenta become collinear.
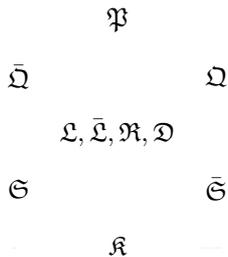
The paper is organised as follows: We start in section 2 by presenting how free superconformal symmetry acts on scattering amplitudes and compare it to the quantum action on local operators. We conclude that the action on amplitudes may require corrections whose qualitative form is derived by analogy with local operators. In section 3 we determine these corrections by demanding exact superconformal invariance of MHV amplitudes. We then show the closure of the superconformal algebra modulo gauge transformations in section 4. Finally, in section 5 we show invariance of all tree amplitudes under the deformed superconformal representation. We summarise our results in section 6 and give an outlook.

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[2]There are two subtleties here: a) the twistor space formulation requires the signature of spacetime to be $(2,2)$ and not $(3,1)$; b) the twistor transformation itself is singular at collinearities.

[3]We thank James Drummond, Johannes Henn and Emery Sokatchev for explanations.

[4]We thank David Skinner for pointing this out and for discussions.

$$\mathfrak{P}$$

$$\bar{\mathfrak{Q}} \qquad\qquad \mathfrak{Q}$$

$$\mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{R}, \mathfrak{D}$$

$$\mathfrak{S} \qquad\qquad \bar{\mathfrak{S}}$$

$$\mathfrak{K}$$

Figure 1. Structure of the superconformal algebra $\mathfrak{psu}(2,2|4)$. The generators are plotted according to their scaling dimensions (vertical) and their helicities (horizontal).

2 Representation of superconformal symmetry

In this section we review and discuss the representation of superconformal symmetry on scattering amplitudes. By means of analogy to local operators we propose how to qualitatively deform the free representation to an interacting one.

2.1 Free representation

Scattering amplitudes in $\mathcal{N} = 4$ SYM are most conveniently expressed in the spinor helicity superspace [45]: The light-like momentum p of each external particle is first converted to a bi-spinor $p^{a\dot{a}}$ which can consequently be written as a product $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$. Here λ^a and $\bar{\lambda}^{\dot{a}}$ are mutually conjugate bosonic spinors of the Lorentz algebra with $a, b, \ldots = 1, 2$ and $\dot{a}, \dot{b}, \ldots = 1, 2$. The decomposition is unique up to a complex phase $\lambda^a \to e^{i\varphi}\lambda^a$ and $\bar{\lambda}^{\dot{a}} \to e^{-i\varphi}\bar{\lambda}^{\dot{a}}$. Furthermore, it is advantageous to compute scattering amplitudes for the superfield [46]

$$\Phi(\lambda, \bar{\lambda}, \eta) = G^+(\lambda, \bar{\lambda}) + \eta^A \Gamma_A(\lambda, \bar{\lambda}) + \frac{1}{2}\eta^A\eta^B S_{AB}(\lambda, \bar{\lambda})$$
$$+ \frac{1}{6}\varepsilon_{ABCD}\eta^A\eta^B\eta^C \bar{\Gamma}^D(\lambda, \bar{\lambda}) + \frac{1}{24}\varepsilon_{ABCD}\eta^A\eta^B\eta^C\eta^D G^-(\lambda, \bar{\lambda}), \quad (2.1)$$

where $G^\pm$, $\Gamma/\bar{\Gamma}$, S are the on-shell gluons, fermions and scalars with definite helicity. By picking a suitable component in the expansion of fermionic spinors η^A, $A, B, \ldots = 1, 2, 3, 4$, of $\mathfrak{su}(4)$ one can select the desired type of external particle for each leg. The scattering amplitude for n external particles is thus a superspace function

$$A_n(\lambda_1, \bar{\lambda}_1, \eta_1, \ldots, \lambda_n, \bar{\lambda}_n, \eta_n). \quad (2.2)$$

The superconformal algebra $\mathfrak{psu}(2,2|4)$ can be represented in a simple fashion on such scattering amplitudes. We shall denote the superconformal generators through Gothic letters $\mathfrak{J}_\alpha$. More concretely, it is generated by Lorentz rotations $\mathfrak{L}, \bar{\mathfrak{L}}$, internal rotations $\mathfrak{R}$, momentum generators $\mathfrak{P}$, special conformal generators $\mathfrak{K}$, the dilatation generator $\mathfrak{D}$ as well as supercharges $\mathfrak{Q}, \bar{\mathfrak{Q}}$ and special conformal supercharges $\mathfrak{S}, \bar{\mathfrak{S}}$, see figure 1. Using the spinor helicity superspace coordinates the representation of the superconformal algebra

can be written in a very compact fashion [7] (cf. section 4)

$$\begin{aligned}
\mathfrak{L}^a{}_b &= \lambda^a \partial_b - \frac{1}{2}\delta^a_b \lambda^c \partial_c, & \bar{\mathfrak{L}}^{\dot{a}}{}_{\dot{b}} &= \bar{\lambda}^{\dot{a}} \bar{\partial}_{\dot{b}} - \frac{1}{2}\delta^{\dot{a}}_{\dot{b}} \bar{\lambda}^{\dot{c}} \bar{\partial}_{\dot{c}}, \\
\mathfrak{D} &= \frac{1}{2}\partial_c \lambda^c + \frac{1}{2}\bar{\lambda}^{\dot{c}} \bar{\partial}_{\dot{c}}, & \mathfrak{R}^A{}_B &= \eta^A \partial_B - \frac{1}{4}\delta^A_B \eta^C \partial_C, \\
\mathfrak{Q}^{aB} &= \lambda^a \eta^B, & \mathfrak{S}_{aB} &= \partial_a \partial_B, \\
\bar{\mathfrak{Q}}^{\dot{a}}{}_B &= \bar{\lambda}^{\dot{a}} \partial_B, & \bar{\mathfrak{S}}^B_{\dot{a}} &= \eta^B \bar{\partial}_{\dot{a}}, \\
\mathfrak{P}^{a\dot{b}} &= \lambda^a \bar{\lambda}^{\dot{b}}, & \mathfrak{K}_{a\dot{b}} &= \partial_a \bar{\partial}_{\dot{b}},
\end{aligned} \tag{2.3}$$

where we abbreviate $\partial_a = \partial/\partial\lambda^a$, $\bar{\partial}_{\dot{a}} = \partial/\partial\bar{\lambda}_{\dot{a}}$ and $\partial_A = \partial/\partial\eta^A$. Furthermore, let us introduce a central charge $\mathfrak{C}$ and the helicity charge $\mathfrak{B}$ which would extend the algebra to $\mathfrak{u}(2,2|4)$. Their representation reads

$$\mathfrak{C} = \partial_a \lambda^a - \bar{\lambda}^{\dot{c}} \bar{\partial}_{\dot{c}} - \eta^C \partial_C = 2 + \lambda^a \partial_a - \bar{\lambda}^{\dot{c}} \bar{\partial}_{\dot{c}} - \eta^C \partial_C, \qquad \mathfrak{B} = \eta^C \partial_C. \tag{2.4}$$

In fact, this is only one half of the story: The energy component in $p^{a\dot{b}} = \lambda^a \bar{\lambda}^{\dot{b}}$ is manifestly positive. However, reasonable scattering amplitudes require at least two particles with negative energy. For such particles we must set $p^{a\dot{b}} = -\lambda^a \bar{\lambda}^{\dot{b}}$. The negative energy representation is the same as the above (2.3), where the sign of all instances of $\bar{\lambda}$ is flipped. In most places this replacement is sufficient and can be done mechanically. We shall thus treat all particles as though their energy is positive and point out whenever negative energy particles make an essential difference (cf. section 3.3).

The representation on tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM takes the standard tensor product form

$$\mathfrak{J}_\alpha = \sum_{k=1}^n \mathfrak{J}_{k,\alpha}. \tag{2.5}$$

Here $\mathfrak{J}_{k,\alpha}$ is the representation of the conformal symmetry generator $\mathfrak{J}_\alpha$ on the k-th leg $(\lambda_k, \bar{\lambda}_k, \eta_k)$ of A_n as specified in (2.3). Invariance of A_n is the statement

$$\mathfrak{J}_\alpha A_n = 0. \tag{2.6}$$

In [43] a Yangian representation on tree-level scattering amplitudes in $\mathcal{N} = 4$ SYM was proposed. The action of the level-one Yangian generators $\widehat{\mathfrak{J}}_\alpha$ follows the standard Yangian coproduct rule for evaluation representations with homogeneous evaluation parameters

$$\widehat{\mathfrak{J}}_\alpha = \frac{1}{2} f_\alpha^{\beta\gamma} \sum_{1 \le k < \ell \le n} \mathfrak{J}_{k,\beta} \mathfrak{J}_{\ell,\gamma}. \tag{2.7}$$

This representation was shown to be compatible with cyclicity provided that the amplitude is invariant under superconformal symmetry. Making use of dual superconformal covariance [15, 39] and the Serre relations one can further deduce that the tree-level amplitudes are invariant under the complete Yangian algebra, $\widehat{\mathfrak{J}}_\alpha A_n = 0$ [43].
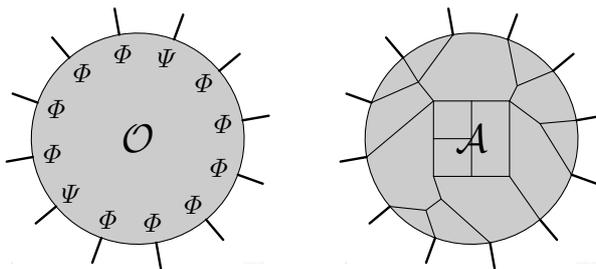
Figure 2. Comparison of single-trace local operators and colour-ordered scattering amplitudes

2.2 Higher-loop representation on local operators

This representation is the direct analog of the leading-order representation on local operators when $\mathfrak{J}_{k,\alpha}$ is the representation on the k-th site of the spin chain. The main difference is that $\mathfrak{J}_{k,\alpha}$ is a differential operator for scattering amplitudes while it is a spin operator for local operators.[5] In fact, the structures of single-trace local operators and colour-ordered scattering amplitudes are very much alike as illustrated in figure 2.

It is well-known that the representation of the superconformal algebra on local operators is deformed at loop level. This is required to incorporate the effects of anomalous dimensions; after all the dilatation generator measures conformal dimensions. Alternatively one can say that the deformation is due to regularisation of UV divergences. While the tree-level generators $\mathfrak{J}_{k,\alpha}$ act on a single site of the local operator and map it back to itself, the structure of the loop corrections is qualitatively different: They can act on several sites at the same time and map them back to themselves. Moreover, they are *dynamic* in the sense that they can change the number of sites, e.g. map a single site to two sites or vice versa [49]. This implies that local operators with well-defined scaling dimension do not have a well-defined number of component fields, but they are rather linear combinations of spin chains with different lengths. Note that some of these length-changing effects are known as "non-linear" or interacting realisations of the symmetry. For example, it is well-known that a supercharge $\mathfrak{Q}$ acting on a fermion can produce the commutator of two scalars [49] even in the classical theory.

The generic structure of the perturbative representation $\mathfrak{J}_\alpha(g)$ for some generator $\mathfrak{J}_\alpha$ around the free representation $(\mathfrak{J}_0)_\alpha = (\mathfrak{J}_{1,1}^{(0)})_\alpha$ reads

$$\mathfrak{J}_\alpha(g) = \sum_{m,n=1}^{\infty} \sum_{\ell=0}^{\infty} g^{2\ell+m+n-2} (\mathfrak{J}_{m,n}^{(\ell)})_\alpha. \tag{2.8}$$

This structure follows from the structure of planar Feynman graphs [3, 49], and it is depicted in figure 3. An ℓ-loop contribution $\mathfrak{J}_{m,n}^{(\ell)}$ which acts on m adjacent sites of the chain and which replaces them by n adjacent sites is of order $g^{2\ell+m+n-2}$. This is because an elementary interaction of $\mathcal{O}(g)$ connects three sites; adjacency is due to the planar limit.

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[5]Without going into details, the oscillator representation introduced in [47] and applied in [48] is practically equivalent to the above representation.
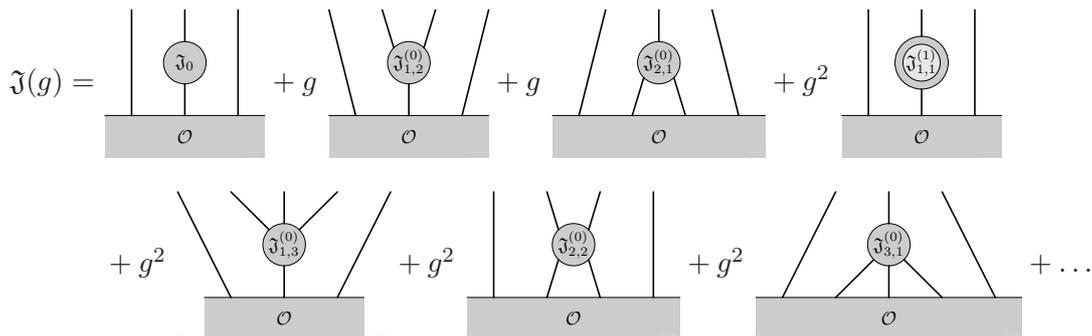
Figure 3. Expansion of quantum symmetry generators for local operators.

2.3 Higher-loop representation on scattering amplitudes

Now one could imagine that similar deformations apply to the representation of conformal symmetry on scattering amplitudes. Clearly the origin of the corrections is different: for local operators it is due to UV divergences whereas for scattering amplitudes it is due to IR divergences. This means that perhaps the representations are not exactly equivalent. Nevertheless one would expect the structural constraints to be the same because they merely originate from the structure of Feynman graphs (and the planar limit).

The action of deformations which involve several legs, but preserve their number should be self-evident. But what does it mean to change the number of legs? In particular, how can this be possible at all if each leg has a well-defined particle momentum? How can invariance of a scattering amplitude be interpreted? First of all, if the number of legs changes by the action of symmetry generators, then a single n-leg amplitude cannot be invariant by itself; it only makes sense to talk about invariance of all amplitudes at the same time.

Before we introduce a proper framework for the treatment of length-changes, let us discuss their effects qualitatively. Suppose a generator consists of the terms depicted in figure 3

$$\mathfrak{J}(g) = \mathfrak{J}_0 + g\mathfrak{J}_{1,2}^{(0)} + g\mathfrak{J}_{2,1}^{(0)} + g^2\mathfrak{J}_{1,1}^{(1)} + g^2\mathfrak{J}_{1,3}^{(0)} + g^2\mathfrak{J}_{2,2}^{(0)} + g^2\mathfrak{J}_{3,1}^{(0)} + \dots \qquad (2.9)$$

The first term is the free generator $\mathfrak{J}_0 = \mathfrak{J}_{1,1}^{(0)}$. The contributions $\mathfrak{J}_{1,2}^{(0)}, \mathfrak{J}_{1,3}^{(0)}$ increase the number of legs by one or two, respectively, while $\mathfrak{J}_{2,1}^{(0)}, \mathfrak{J}_{3,1}^{(0)}$ decrease it. The symbol $\mathfrak{J}_{1,1}^{(1)}$ represents the loop correction to the free generator and $\mathfrak{J}_{2,2}^{(0)}$ maps two legs to two legs. Suppose further that the set of amplitudes can be written as the linear combination

$$\mathcal{A}(g) = \sum_{n=4}^{\infty} g^{n-2} A_n(g) = \sum_{n=4}^{\infty} \sum_{\ell=0}^{\infty} g^{n-2+2\ell} A_n^{(\ell)}. \qquad (2.10)$$

Note that we have included a factor of g for each three-vertex in the underlying Feynman graph; this counting is compatible with the counting of g in the expansion of $\mathfrak{J}(g)$. Demanding invariance of all amplitudes, $\mathfrak{J}(g)\mathcal{A}(g) = 0$, and separating the terms according to their number of external legs as well as the power of g leads to the following invariance
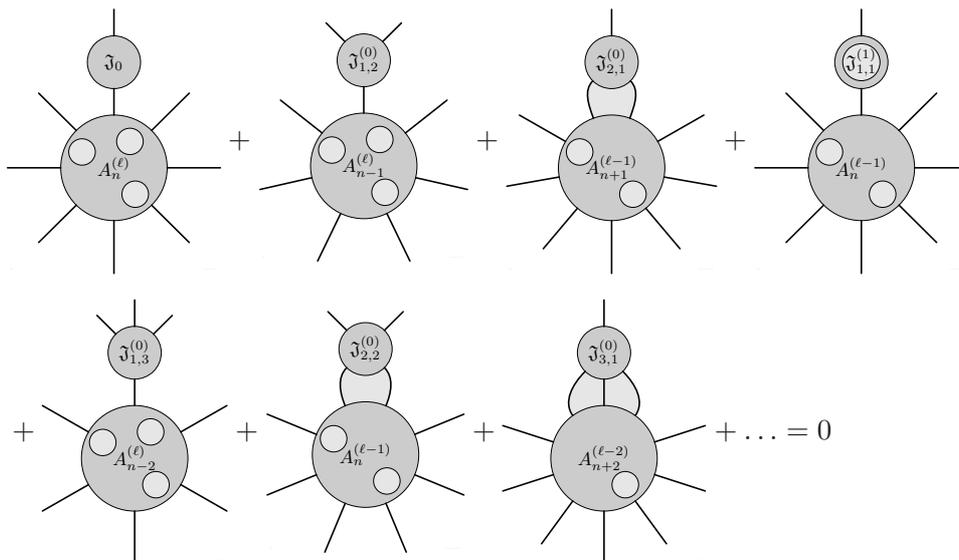
Figure 4. Expansion of quantum invariance of scattering amplitudes. Loops (light grey) can appear inside the amplitudes, inside the symmetry generator or in the connection of the two.

equation

$$\mathfrak{J}_{1,1}^{(0)}A_n^{(\ell)}+\mathfrak{J}_{1,2}^{(0)}A_{n-1}^{(\ell)}+\mathfrak{J}_{2,1}^{(0)}A_{n+1}^{(\ell-1)}+\mathfrak{J}_{1,1}^{(1)}A_n^{(\ell-1)}+\mathfrak{J}_{1,3}^{(0)}A_{n-2}^{(\ell)}+\mathfrak{J}_{2,2}^{(0)}A_n^{(\ell-1)}+\mathfrak{J}_{3,1}^{(0)}A_{n+2}^{(\ell-2)}+\ldots=0. \quad (2.11)$$

In particular the illustration of this equation in figure 4 shows that the loop counting includes loops within the *amplitude*, loops within the *symmetry generator* as well as loops formed by *connecting* the two.

Note that it makes sense to rescale n-leg amplitudes A_n by a factor of g^{2-n} such that all tree amplitudes are at $\mathcal{O}(g^0)$ and such that g^2 exclusively counts the number of loops in Feynman graphs. Thus we would use instead of (2.10) and (2.8)

$$\mathcal{A}(g) = \sum_{n=4}^\infty A_n(g) = \sum_{n=4}^\infty \sum_{\ell=0}^\infty g^{2\ell} A_n^{(\ell)}, \qquad \mathfrak{J}_\alpha(g) = \sum_{m,n=1}^\infty \sum_{\ell=0}^\infty g^{2(\ell+m-1)} (\mathfrak{J}_{m,n}^{(\ell)})_\alpha. \quad (2.12)$$

This is the normalisation that we shall use in the present work. Note that this leads to the same invariance equation (2.11).

The crucial observation one can make in (2.11) is that generators which act on a single leg and replace it by several legs, such as $\mathfrak{J}_{1,n}^{(0)}$, contribute to the same order as the free generator $\mathfrak{J}_0$. The conclusion would be that tree amplitudes in $\mathcal{A}(0)$ are not invariant under $\mathfrak{J}_0$, but rather under the combination $\mathfrak{J}_\alpha(0)$:

$$\mathfrak{J}_\alpha(0)\mathcal{A}(0) = 0 \qquad \text{with} \qquad \mathcal{A}(0) = \sum_{n=4}^\infty A_n^{(0)}, \qquad \mathfrak{J}_\alpha(0) = \sum_{n=1}^\infty (\mathfrak{J}_{1,n}^{(0)})_\alpha. \quad (2.13)$$

On the one hand this type of invariance is reasonable because terms like $\mathfrak{J}_{1,2}^{(0)} + \ldots$ are precisely the "non-linear" contributions to symmetries in the interacting *classical* theory. Naturally these would have to be the proper symmetries for tree level amplitudes and not

$$\mathcal{A}[J] = \frac{1}{4} \, {}_J\!-\!\!\left(A_4\right)\!\!-\!{}_J \; + \; \frac{1}{5} \; {}^J\!\left(A_5\right)^J + \; \frac{1}{6} \; \left(A_6\right) \; + \; \frac{1}{7} \, {}_J\!\left(A_7\right)\!{}_J \; + \dots .$$

Figure 5. The generating functional of colour-ordered scattering amplitudes. The prefactors $1/n$ are the appropriate symmetry factors for cyclicity of the trace.

their free truncations $\mathfrak{J}_0$. On the other hand, tree-level amplitudes at first sight do seem to be invariant under the free generators. Therefore the interacting correction terms $\mathfrak{J}_{1,2}^{(0)} + \dots$ would either have to be trivial or they would have to annihilate the amplitudes on their own and independently of $\mathfrak{J}_0$. Both alternatives are somewhat unsatisfactory and indeed there is a third: Tree-level amplitudes are *not invariant* under the free action $\mathfrak{J}_0$ of the symmetry. This violation of conformal symmetry is subtle and therefore is not immediately seen. For generic external momenta the amplitudes are indeed invariant under naive conformal symmetry. However, when the amplitudes are treated as distributions, the action of $\mathfrak{J}_0$ leaves certain contact terms when two adjacent momenta become collinear. Collinearity is essential because breaking up one massless particle into two by means of $\mathfrak{J}_{1,2}^{(0)} + \dots$ can only produce collinear momenta due to momentum conservation. In conclusion, it is conceivable that conformal symmetry has a representation under which the tree-level amplitudes are exactly invariant in a distributional sense. In particular, the length-changing effects would be crucial for this representation. It would also be the proper starting point for extending the symmetries to the loop level.

2.4 Amplitude generating functional

Before we consider concretely the length-changing contributions we shall first introduce a framework to deal with such terms.

On a technical level we can combine all scattering amplitudes into a single generating functional. Let $J(\lambda, \bar{\lambda}, \eta)$ be a source field corresponding to the superspace field $\Phi(\lambda, \bar{\lambda}, \eta)$. For clarity of notation we shall combine the bosonic and fermionic superspace coordinates into a single symbol $\Lambda = (\lambda^a, \bar{\lambda}^{\dot{a}}, \eta^A)$. The superspace measure is given through $d^{4|4}\Lambda := d^4\lambda \, d^4\eta$, see appendix A. The generating functional $\mathcal{A}$ of colour-ordered amplitudes A_n then reads simply, cf. figure 5 (see also [10])

$$\mathcal{A}[J] = \sum_{n=4}^{\infty} \int d^{4|4}\Lambda_1 \dots d^{4|4}\Lambda_n \, \frac{1}{n} \, \mathrm{Tr}\big(J(\Lambda_1) \dots J(\Lambda_n)\big) \, A_n(\Lambda_1, \dots, \Lambda_n). \tag{2.14}$$

Conversely, the n-particle amplitude can be extracted as the variation

$$A_n(\Lambda_1, \dots, \Lambda_n) = \frac{1}{N_{\mathrm{c}}^n} \, \mathrm{Tr}\left(\frac{\delta}{\delta J(\Lambda_n)} \dots \frac{\delta}{\delta J(\Lambda_1)}\right) \mathcal{A}[J]\bigg|_{J=0}. \tag{2.15}$$

Note that the traces incorporate the colour structure of colour-ordered amplitudes and $1/n$ is the proper symmetry factor.

For representations of $\mathfrak{psu}(2,2|4)$ the central charge of $\mathfrak{su}(2,2|4)$ must act trivially. This implies that the fields $\Phi(\lambda, \bar{\lambda}, \eta)$ are homogeneous functions under a simultaneous phase shift of the arguments

$$\Phi(e^{i\varphi}\Lambda) = e^{-2i\varphi}\Phi(\Lambda), \qquad e^{i\varphi}\Lambda := (e^{i\varphi}\lambda, e^{-i\varphi}\bar{\lambda}, e^{-i\varphi}\eta). \tag{2.16}$$

Consequently, the same must be true for each leg of the amplitude, $A_n(\ldots, e^{i\varphi}\Lambda_n, \ldots) = e^{-2i\varphi}A_n(\ldots, \Lambda_n, \ldots)$. The Jacobian of the measure also leads to a weight $d^{4|4}(e^{i\varphi}\Lambda) = e^{4i\varphi}d^{4|4}\Lambda$. We will not impose a homogeneity condition for the source fields $J(\Lambda)$ so that the variations $\delta/\delta J$ can be performed straight-forwardly. It is nevertheless clear that the generating functional (2.14) projects to the part of J with definite scaling

$$\hat{J}(\Lambda) := \frac{1}{2\pi} \int_0^{2\pi} d\varphi \, e^{2i\varphi} J(e^{i\varphi}\Lambda). \tag{2.17}$$

Therefore each factor of J in (2.14) can safely be replaced by $\hat{J}$; the integral over $d^{4|4}\Lambda$ contains a similar integral over $d\varphi$. Note that the projection $\hat{J}$ turns out to have the same homogeneity as Φ in (2.16).

In the framework of the generating functional the superconformal generators (2.3) take the form of variations, cf. [3, 50] for a similar representation. For convenience we shall abbreviate variation by an accent $\check{J}$ on the field J

$$\check{J}(\Lambda) := \frac{\delta}{\delta J(\Lambda)} \,. \tag{2.18}$$

Here we list only a few of the relevant generators

$$(\mathfrak{Q}_0)^{aB} = \int d^{4|4}\Lambda \, \mathrm{Tr}\, \lambda^a \eta^B J(\Lambda) \, \check{J}(\Lambda) \,, \quad (\mathfrak{S}_0)_{aB} = \int d^{4|4}\Lambda \, \mathrm{Tr}\, \partial_a \partial_B J(\Lambda) \, \check{J}(\Lambda) \,,$$

$$(\bar{\mathfrak{Q}}_0)^{\dot{a}}_B = -\int d^{4|4}\Lambda \, \mathrm{Tr}\, \bar{\lambda}^{\dot{a}} \partial_B J(\Lambda) \, \check{J}(\Lambda) \,, \quad (\bar{\mathfrak{S}}_0)^B_{\dot{a}} = -\int d^{4|4}\Lambda \, \mathrm{Tr}\, \eta^B \bar{\partial}_{\dot{a}} J(\Lambda) \, \check{J}(\Lambda) \,,$$

$$(\mathfrak{P}_0)^{a\dot{b}} = \int d^{4|4}\Lambda \, \mathrm{Tr}\, \lambda^a \bar{\lambda}^{\dot{b}} J(\Lambda) \, \check{J}(\Lambda) \,, \quad (\mathfrak{K}_0)_{a\dot{b}} = \int d^{4|4}\Lambda \, \mathrm{Tr}\, \partial_a \bar{\partial}_{\dot{b}} J(\Lambda) \, \check{J}(\Lambda) \,. \tag{2.19}$$

After performing the variations on $\mathcal{A}$ the derivatives should be integrated by parts to make them act on A_n as for (2.3). A classical contribution to add one leg takes the qualitative form $\mathfrak{J}^{(0)}_{(1,2)} \sim \int \mathrm{Tr}\, JJ\check{J}$. In practice, it acts by taking away one source term and replacing it by two. The precise form of such contributions will be worked out in the following section.

Let us note that the above expressions for the superconformal generators remain valid even for amplitudes without colour ordering and away from the planar limit or for generic gauge groups. Also the classical length-changing contributions are expected to remain valid at finite N_{c}. Conversely, a representation of the Yangian cannot be formulated using the generating functional because one needs a framework which can make explicit reference to specific legs, e.g. legs k and ℓ as in (2.7).

3 Superconformal invariance of MHV amplitudes

In this section we wish to use the known form of the tree level MHV amplitudes to determine the necessary deformations $\mathfrak{J}^{(0)}_{1,n}$ of the classical conformal symmetry generators (cf. (2.13)).

3.1 MHV amplitudes

Scattering amplitudes can be classified through their helicity. It is measured by the generator $\mathfrak{B}$ counting the number of η's

$$A_n = \sum_{k=2}^{n-2} A_{n,k}, \qquad \mathfrak{B}A_{n,k} = 4kA_{n,k}. \tag{3.1}$$

The number of η's ranges between 8 for MHV amplitudes and $4n-8$ for $\overline{\text{MHV}}$ amplitudes

$$A_n^{\text{MHV}} = A_{n,2}, \qquad A_n^{\overline{\text{MHV}}} = A_{n,n-2}. \tag{3.2}$$

The tree level MHV amplitudes of $\mathcal{N} = 4$ SYM have a simple form when written in terms of Lorentz invariant products of spinors [51] and particularly so in the manifestly supersymmetric formulation [45] using the appropriate on-shell superspace. They take the form[6]

$$A_n^{\text{MHV}} = \frac{\delta^4(P)\,\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}, \qquad P^{a\dot{b}} = \sum_{k=1}^{n} \lambda_k^a \bar{\lambda}_k^{\dot{b}}, \quad Q^{aB} = \sum_{k=1}^{n} \lambda_k^a \eta_k^B, \tag{3.3}$$

where the brackets are defined in appendix A.

In the physically relevant case the spacetime signature is $(3,1)$. It implies that the above expression for the amplitude cannot be entirely meaningful because it assumes that all particles have strictly positive energies while energy conservation requires the sum of all energies to vanish. For non-trivial amplitudes at least two particles should have negative energies. A particle k with negative energy is achieved by flipping the sign of $\bar{\lambda}_k$. For the time being we shall ignore the implications of overall momentum conservation and assume all energies to be positive. The minute modifications due to negative energy particles will be discussed in section 3.3.

We now act with the free superconformal generator $(\bar{\mathfrak{S}}_0)_{\dot{a}}^B = \sum_{k=1}^{n} \eta_k^B \bar{\partial}_{k,\dot{a}}$ as defined in (2.3) on the above amplitude. Except for the delta function, the amplitude is holomorphic in the λ_k. Thus, at first sight, the generator seems to act only on the delta function

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^B \delta^4(P) = \sum_{k=1}^{n} \eta_k^B \frac{\partial}{\partial\bar{\lambda}_k^{\dot{a}}} \delta^4(P) = \sum_{k=1}^{n} \eta_k^B \lambda_k^c \frac{\partial\delta^4(P)}{\partial P^{c\dot{a}}} = Q^{cB} \frac{\partial\delta^4(P)}{\partial P^{c\dot{a}}}. \tag{3.4}$$

The fermionic delta function $\delta^8(Q)$ ensures that the action vanishes $\bar{\mathfrak{S}}_0 A_n^{\text{MHV}} = 0$ [7].

In $(3,1)$ spacetime signature, however, λ and $\bar{\lambda}$ are related by complex conjugation, and thus there is the holomorphic anomaly [52]. It gives a non-trivial contribution when the derivative with respect to $\bar{\lambda}$ acts on poles in the variable λ (see appendix A). This gives rise to terms[7]

$$\frac{\partial}{\partial\bar{\lambda}^{\dot{a}}} \frac{1}{\langle\lambda,\mu\rangle} = \pi\delta^2(\langle\lambda,\mu\rangle)\,\varepsilon_{\dot{a}\dot{b}}\bar{\mu}^{\dot{b}}. \tag{3.5}$$

As can be immediately seen these anomaly terms coincide with the collinear singularities of the amplitude which as discussed previously is their physical origin.

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[6]We neglect an overall factor of $i(2\pi)^4$ in our definition of the amplitudes and similarly they are normalised so that there is no prefactor of the coupling.

[7]We thank Emery Sokatchev for reminding us of the precise form of the anomaly.

Here there is a crucial difference between the $(3, 1)$ physical Minkowski signature and $(2, 2)$ split signature used for considerations in twistor space: In $(3, 1)$ signature two light-like vectors are orthogonal if and only if they are collinear. Collinearity implies two constraints on the six degrees of freedom for two light-like vectors, it is thus a codimension-two condition. In the spinor formulation collinearity is equivalent to $\langle \lambda_k, \lambda_{k+1} \rangle = 0$ which is one complex or two real conditions, hence codimension two. Conversely for $(2, 2)$ signature $\langle \lambda_k, \lambda_{k+1} \rangle = 0$ is merely one real condition or codimension one. Equivalently orthogonality does not imply full collinearity, but only one constraint. The nature of the two types of singularities is rather different. The holomorphic anomaly only applies to codimension-two singularities. Codimension-one singularities can also have anomalies, but one has to define properly the distributional meaning of $(\langle \lambda_k, \lambda_{k+1} \rangle)^{-1}$. One could consider adding $\pm i\epsilon$ to the denominators, but it is not clear which sign to use (for each term). A principal value prescription appears to be the proper choice, but this leads to no anomaly. Altogether this consideration shows that the signature plays an important role for scattering amplitudes and we shall continue to work exclusively in Minkowski signature.

In the light of the holomorphic anomaly, there are extra terms in the action of $\bar{\mathfrak{S}}_0$,[8]

$$
\begin{aligned}
(\bar{\mathfrak{S}}_0)_{\dot{a}}^B A_n^{\mathrm{MHV}} &= \sum_{k=1}^n \eta_k^B \frac{\partial}{\partial \bar{\lambda}_k^{\dot{a}}} \frac{\delta^4(P)\,\delta^8(Q)}{\langle 1, 2 \rangle \dots \langle k-1, k \rangle \langle k, k+1 \rangle \dots \langle n, 1 \rangle} \\
&= -\pi \sum_{k=1}^n \varepsilon_{\dot{a}\dot{b}} (\bar{\lambda}_{k-1}^{\dot{b}} \eta_k^B - \bar{\lambda}_k^{\dot{b}} \eta_{k-1}^B) \frac{\delta^2(\langle \lambda_{k-1}, \lambda_k \rangle)\,\delta^4(P)\,\delta^8(Q)}{\langle 1, 2 \rangle \dots \langle k-1, k \rangle^0 \dots \langle n, 1 \rangle}.
\end{aligned}
\tag{3.6}
$$

The existence of extra terms is well-known and it has been employed successfully at the loop level [52]. At tree level, it has largely been ignored so far because the anomaly is restricted to singular momentum configurations.

It turns out to be convenient to cast this statement into the language of generating functionals. Let $\mathcal{A}_n^{\mathrm{MHV}}[J]$ be the generating functional of MHV amplitudes (3.3) with n legs in the sense of (2.14). Acting with the bare generator $\bar{\mathfrak{S}}_0$ as defined in (2.19) on $\mathcal{A}_n^{\mathrm{MHV}}[J]$ by performing the functional variations and integrating by parts we find

$$
\begin{aligned}
(\bar{\mathfrak{S}}_0)_{\dot{a}}^B \mathcal{A}_n^{\mathrm{MHV}}[J] = -\pi \int \prod_{k=1}^n d^{4|4}\Lambda_k \; & \mathrm{Tr}\big([J(\Lambda_1), J(\Lambda_2)] \dots J(\Lambda_n)\big) \\
& \times \varepsilon_{\dot{a}\dot{c}} \bar{\lambda}_1^{\dot{c}} \eta_2^B \frac{\delta^2(\langle 1, 2 \rangle)\,\delta^4(P)\,\delta^8(Q)}{\langle 2, 3 \rangle \dots \langle n, 1 \rangle}.
\end{aligned}
\tag{3.7}
$$

We have made use of the cyclicity of the amplitudes, the trace and the measure in order to collect n equivalent copies of the contribution in (3.6) which thus cancel the symmetry factor of $1/n$ in (2.14). The commutator term in the trace results from the difference term in the second line of (3.6) after interchanging Λ_1 and Λ_2.

We can partially perform the integrals over Λ_1 to remove the delta function imposing the collinearity of the 1 and 2 legs. A convenient change of the variables Λ_1, Λ_2 to this end

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[8]This fact and several of its implications discussed below have been found independently by James Drummond. Johannes Henn and Emery Sokatchev also pointed out the distributional non-invariance.

reads

$$\lambda_1 = e^{i\varphi}\lambda_{12}\sin\alpha, \qquad \eta_1 = e^{-i\varphi}(\eta_{12}\sin\alpha + \eta'\cos\alpha),$$
$$\lambda_2 = \lambda_{12}\cos\alpha + z\lambda', \qquad \eta_2 = \eta_{12}\cos\alpha - \eta'\sin\alpha. \tag{3.8}$$

The four complex variables λ_1^a, λ_2^a have been replaced by three complex variables λ_{12}^a, z and two real variables $\alpha \in [0, \frac{1}{2}\pi]$, $\varphi \in [0, 2\pi]$. The spinor λ' is a constant reference spinor. The integral over d^2z localises at $z = \bar{z} = 0$ and after evaluating the various Jacobians we get that

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^B \mathcal{A}_n^{\mathrm{MHV}}[J] = -\pi \int \prod_{k=3}^n d^{4|4}\Lambda_k \, d^{4|4}\Lambda_{12} \, d^4\eta' \, d\alpha \, d\varphi \; e^{3i\varphi}\varepsilon_{\dot{a}\dot{c}}\bar{\lambda}_1^{\dot{c}}\eta_2^B \times$$
$$\times \mathrm{Tr}\big([J(\Lambda_1), J(\Lambda_2)]\ldots J(\Lambda_n)\big)\frac{\delta^4(P')\delta^8(Q')}{\langle 12, 3\rangle \ldots \langle n, 12\rangle} \tag{3.9}$$

where $P' = \lambda_{12}\bar{\lambda}_{12} + \sum_{k=3}^n \lambda_k\bar{\lambda}_k$ and $Q' = \eta_{12}\lambda_{12} + \sum_{k=3}^n \eta_k\lambda_k$. Alternatively one can use the formula (A.6) to derive this result. Note that the integral over φ amounts to the projection $\hat{J}(\Lambda_1)$ in (2.17). Removing the phase in Λ_1 such that $\lambda_1 = \lambda_{12}\sin\alpha$ and $\eta_1 = \eta_{12}\sin\alpha + \eta'\cos\alpha$ we obtain the more compact expression

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^B \mathcal{A}_n^{\mathrm{MHV}}[J] = -2\pi^2 \int \prod_{k=3}^n d^{4|4}\Lambda_k \, d^{4|4}\Lambda_{12} \, d^4\eta' \, d\alpha \; \varepsilon_{\dot{a}\dot{c}}\bar{\lambda}_1^{\dot{c}}\eta_2^B \times$$
$$\times \mathrm{Tr}\big([\hat{J}(\Lambda_1), \hat{J}(\Lambda_2)]\ldots J(\Lambda_n)\big)\frac{\delta^4(P')\delta^8(Q')}{\langle 12, 3\rangle \ldots \langle n, 12\rangle}. \tag{3.10}$$

Note that the integrand is homogeneous in Λ_2 (2.16) and thus the second source term $J(\Lambda_2)$ was replaced by the projection $\hat{J}(\Lambda_2)$.

We observe that the anomalous variation (3.9) produces A_{n-1}^{MHV} with slight modifications merely on the first leg. Such a modification can be imposed through a variation of the sort $\int \mathrm{Tr}\, J J \check{J}$ acting on $\mathcal{A}^{\mathrm{MHV}}$. More precisely the form of the correction $\bar{\mathfrak{S}}_+ = \bar{\mathfrak{S}}_{1,2}^{(0)}$ reads

$$(\bar{\mathfrak{S}}_+)_{\dot{a}}^B = 2\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \, \varepsilon_{\dot{a}\dot{c}}\bar{\lambda}_1^{\dot{c}} \, \eta_2^B \, \mathrm{Tr}\big([\hat{J}(\Lambda_1), \hat{J}(\Lambda_2)]\check{J}(\Lambda)\big)$$
$$= -\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \, \varepsilon_{\dot{a}\dot{c}}\bar{\lambda}^{\dot{c}}\eta'^B \, \mathrm{Tr}\big([\hat{J}(\Lambda_1), \hat{J}(\Lambda_2)]\check{J}(\Lambda)\big) \tag{3.11}$$

with the following definitions for Λ_1, Λ_2

$$\lambda_1 = \lambda\sin\alpha, \qquad \eta_1 = \eta\sin\alpha + \eta'\cos\alpha,$$
$$\lambda_2 = \lambda\cos\alpha, \qquad \eta_2 = \eta\cos\alpha - \eta'\sin\alpha. \tag{3.12}$$

The second form in (3.11) is due to replacement of Λ_1, Λ_2 and making use of antisymmetry of the commutator. The plus in $\bar{\mathfrak{S}}_+$ signifies that the operator increases the helicity by $+2$ relative to $\bar{\mathfrak{S}}_0$. It was constructed such that

$$\bar{\mathfrak{S}}_0 \mathcal{A}_n^{\mathrm{MHV}} + \bar{\mathfrak{S}}_+ \mathcal{A}_{n-1}^{\mathrm{MHV}} = 0. \tag{3.13}$$

As can be seen we find a recursive pattern for the action of the generator on the amplitudes. We can ask what is the starting point for this action and the answer is straightforward:

$$\bar{\mathfrak{S}}_0 \mathcal{A}_4^{\text{MHV}} = 0. \tag{3.14}$$

This follows from the above calculation as

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^B A_4^{\text{MHV}} = -\pi \sum_{k=1}^{4} \varepsilon_{\dot{a}\dot{b}} \big(\bar{\lambda}_{k-1}^{\dot{b}} \eta_k^B - \bar{\lambda}_k^{\dot{b}} \eta_{k-1}^B \big) \frac{\delta^2(\langle \lambda_{k-1}, \lambda_k \rangle) \, \delta^4(P) \, \delta^8(Q)}{\langle 12 \rangle \dots \langle k-1, k \rangle^0 \dots \langle 41 \rangle}. \tag{3.15}$$

However now, after making use of the delta-function imposing collinearity between p_k and p_{k-1}, the momentum conservation implies that the three remaining momenta are collinear and the zero coming from the $\delta^8(Q)$ results in the right hand side being zero. This is essentially equivalent to the fact that in $(3,1)$ signature and for real momenta the three-point amplitude vanishes due to zero allowed phase space.

In conclusion, the corrected classical superconformal generator (relevant to MHV amplitudes) is

$$\bar{\mathfrak{S}}_{\dot{a}}^B = (\bar{\mathfrak{S}}_0)_{\dot{a}}^B + (\bar{\mathfrak{S}}_+)_{\dot{a}}^B \tag{3.16}$$

and it exactly annihilates the MHV functional $\mathcal{A}^{\text{MHV}}[J]$

$$\bar{\mathfrak{S}} \mathcal{A}^{\text{MHV}}[J] = 0. \tag{3.17}$$

Note that the cancellation is not restricted to the planar, but it holds for all N_c and even for generic gauge groups.

Although, and as we will show later, it can be determined from the algebra, it is perhaps worthwhile to directly calculate $\mathfrak{K}_+ = \mathfrak{K}_{1,2}^{(0)}$ from $\mathfrak{K}_0$ acting on MHV amplitudes; by a very similar calculation we find

$$(\mathfrak{K}_+)_{b\dot{a}} = -2\pi^2 \int d^{4|4} \Lambda \, d^4 \eta' \, d\alpha \, \varepsilon_{\dot{a}\dot{c}} \bar{\lambda}_1^{\dot{c}} \, \text{Tr}\big([\hat{J}(\Lambda_1), \partial_{2,b} \hat{J}(\Lambda_2)] \check{J}(\Lambda) \big) \tag{3.18}$$

with the same definitions as above.

3.2 Conjugate MHV amplitudes

Now we wish to find the deformation of the operator $\mathfrak{S}$ defined in (2.19) and the simplest method is to consider its action on $\overline{\text{MHV}}$ amplitudes. A convenient form of the tree-level $\overline{\text{MHV}}$ contribution to the n-point super-amplitude is given by [53]

$$A_n^{\overline{\text{MHV}}} = \delta^4(P) \, \delta^8(Q) F_n(\Lambda) \tag{3.19}$$

where

$$\delta^8(Q) F_n(\Lambda) = \int \prod_{i=1}^{n} \big(d^4 \bar{\eta}_i \, \exp(\eta^A \bar{\eta}_{i,A}) \big) \frac{\delta^8(\bar{Q})}{[12][23] \dots [n1]}, \qquad \bar{Q}_B^{\dot{a}} = \sum_{i=1}^{n} \bar{\lambda}_i^{\dot{a}} \bar{\eta}_{i,B}. \tag{3.20}$$

One can use the integral representation for the Graßmannian delta function

$$\delta^8(\bar{Q}) = \int d^8 \omega \, \exp \big(\omega_{\dot{a}}^B \bar{Q}_B^{\dot{a}} \big) \tag{3.21}$$

to write this as

$$\delta^8(Q)F_n(\Lambda) = \frac{1}{[12]\dots[n1]} \int d^8\omega \prod_{i=1}^n \delta^4(\eta_i - \bar{\lambda}_i^{\dot{a}}\omega_{\dot{a}}). \qquad (3.22)$$

Thus

$$(\mathfrak{S}_0)_{Ba}\mathcal{A}^{\overline{\mathrm{MHV}}}[J] = \pi \sum_{n=4}^{\infty} \int d^8\omega \prod_{k=1}^n \left(d^{4|4}\Lambda_k\, \delta^4(\eta_i - \bar{\lambda}_i^{\dot{a}}\omega_{\dot{a}})\right) \frac{\delta^2([1,2])\,\delta^4(P)}{[2,3]\dots[n,1]} \times$$
$$\times\, \varepsilon_{ab}\lambda_1^b\, \mathrm{Tr}\big([J(\Lambda_1), \partial_{2,B}J(\Lambda_2)]\dots J(\Lambda_n)\big). \quad (3.23)$$

As was previously done we can use the delta function to partially perform the Λ_n integral and rewrite the above expression in a form that makes the necessary generator correction apparent. After a little algebraic manipulation one can show that the correction $\mathfrak{S}_- = \mathfrak{S}_{1,2}^{(0)}$ of $\mathfrak{S}$ takes the form

$$(\mathfrak{S}_-)_{Ba} = -2\pi^2 \int d^{4|4}\Lambda\, d^4\eta'\, d\alpha\; \delta^4(\eta')\varepsilon_{ac}\lambda_1^c\, \mathrm{Tr}\big([\hat{J}(\Lambda_1), \partial_{2,B}\hat{J}(\Lambda_2)]\check{J}(\Lambda)\big)$$
$$= -2\pi^2 \int d^{4|4}\Lambda\, d\alpha\; \varepsilon_{ac}\lambda_1^c\, \mathrm{Tr}\big([\hat{J}(\Lambda_1), \partial_{2,B}\hat{J}(\Lambda_2)]\check{J}(\Lambda)\big)$$
$$= +\pi^2 \int d^{4|4}\Lambda\, d^4\eta'\, d\alpha\; \delta^4(\eta')\varepsilon_{ac}\lambda^c\partial_B'\, \mathrm{Tr}\big([\hat{J}(\Lambda_1), \hat{J}(\Lambda_2)]\check{J}(\Lambda)\big) \qquad (3.24)$$

where the minus in $\mathfrak{S}_-$ signifies a decrease of the helicity. As before Λ_1 and Λ_2 are defined in (3.12) but with $\eta' = 0$ for the second line. The complete classical expression for the superconformal generator

$$\mathfrak{S}_{Ba} = (\mathfrak{S}_0)_{Ba} + (\mathfrak{S}_-)_{Ba} \qquad (3.25)$$

annihilates the $\overline{\mathrm{MHV}}$ amplitudes $\mathfrak{S}\mathcal{A}^{\overline{\mathrm{MHV}}}[J] = 0$.

Similarly we find

$$(\mathfrak{K}_-)_{a\dot{a}} = -2\pi^2 \int d^{4|4}\Lambda\, d^4\eta'\, d\alpha\; \delta^4(\eta')\, \mathrm{Tr}\big[\varepsilon_{ab}\lambda_1^b\hat{J}(\Lambda_1), \bar{\partial}_{2,\dot{a}}\hat{J}(\Lambda_2)\big]\check{J}(\Lambda). \qquad (3.26)$$

3.3 Negative-energy particles

In the above discussion we have restricted ourselves to external particles with positive energy, that is $p_k^{a\dot{b}} = \lambda_k^a\tilde{\lambda}_k^{\dot{b}}$ with $\tilde{\lambda}_k = +\bar{\lambda}_k$ and thus $E_k = p_k^0 = \frac{1}{2}(\lambda_k^1\bar{\lambda}_k^1 + \lambda_k^2\bar{\lambda}_k^2) > 0$ for all particles k. As mentioned previously though, physical scattering amplitudes require that at least two external particles have negative energy, i.e. $\tilde{\lambda}_k = -\bar{\lambda}_k$ such that $p_k^{a\dot{b}} = -\lambda_k^a\bar{\lambda}_k^{\dot{b}}$ and hence $E_k < 0$. In the following we will extend our framework slightly in order to include negative-energy particles.

In the tree-level MHV amplitude (3.3), negative-energy particles only introduce a change of sign inside the momentum delta-function $\delta^4(P)$. The form of collinear singularities as in (3.6) is therefore not affected by the energy signs of the adjacent collinear particles. As we shall see below, only the splitting of the collective momentum into two collinear pieces as in (3.12) changes slightly when the adjacent particles have different energy signs.

For including particles with positive and negative energies into our framework, we introduce two types of source fields: $J^+(\Lambda)$ corresponds to positive-energy particles while $J^-(\Lambda)$ corresponds to negative-energy ones. The amplitude generating functional (2.14) comprising all possible particle configurations then becomes[9]

$$\mathcal{A}[J] = \sum_{n=4}^{\infty} \frac{1}{n} \int \prod_{k=1}^{n} d^{4|4}\Lambda_k \sum_{\substack{s_j=\pm \\ (1\leq j\leq n)}} \mathrm{Tr}\big(J^{s_1}(\Lambda_1)\ldots J^{s_n}(\Lambda_n)\big) A_n(\Lambda_1^{s_1},\ldots,\Lambda_n^{s_n}), \qquad (3.27)$$

where $A_n(\Lambda_1^{s_1},\ldots,\Lambda_n^{s_n})$ equals the amplitude $A_n(\Lambda_1,\ldots,\Lambda_n)$ with $\bar\lambda_k$ and η_k replaced by $s_k\bar\lambda_k$ and $s_k\eta_k$.[10]

When extending the formalism in this way, also the free symmetry generators (2.19) need to include variations with respect to particles of both energy signs. The generator $\bar{\mathfrak{S}}_0$ for example becomes

$$(\bar{\mathfrak{S}}_0)_{\dot a}^B = -\int d^{4|4}\Lambda\, \eta^B \,\mathrm{Tr} \sum_{s=\pm} \big(\bar\partial_{\dot a} J^s(\Lambda)\big)\check J^s(\Lambda). \qquad (3.28)$$

We will now calculate the classical non-linear correction to this generator. This generalises the treatment in section 3.1 and will result in correction terms $\bar{\mathfrak{S}}_{s_0\to\{s_1,s_2\}}^{(0)}$ that split a leg with sign s_0 into two collinear particles with signs s_1, s_2. Acting with $\bar{\mathfrak{S}}_0$ on $\mathcal{A}^{\mathrm{MHV}}[J]$ is completely analogous to (3.7) and yields

$$(\bar{\mathfrak{S}}_0)_{\dot a}^B \mathcal{A}^{\mathrm{MHV}}[J] = -\pi \sum_{n=4}^{\infty} \int \prod_{k=1}^{n} d^{4|4}\Lambda_k\, \varepsilon_{\dot a\dot c}\bar\lambda_1^{\dot c}\eta_2^B \frac{\delta^2(\langle 1,2\rangle)\delta^8(Q)}{\langle 2,3\rangle\cdots\langle n,1\rangle}$$
$$\times \sum_{s_j=\pm} \delta^4(P)\,\mathrm{Tr}\big([J_1^{s_1},J_2^{s_2}]J_3^{s_3}\cdots J_n^{s_n}\big), \quad (3.29)$$

where $P = \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} s_j\lambda_j\bar\lambda_j$ and $Q = \sum_{j=1}^{n} s_j\lambda_j\eta_j$. Now the terms in which $s_1 = s_2$, i.e. the part where the collinear particles have both positive or both negative energy is compensated by a non-linear correction $\bar{\mathfrak{S}}_+^= = \bar{\mathfrak{S}}_{s\to\{s,s\}}^{(0)}$ which looks exactly as (3.11) with J replaced by J^s and including a sum over $s=\pm$.

The terms of (3.29) in which the two collinear particles 1 and 2 have different energy signs we split into a part with $|E_1| < |E_2|$ and a part with $|E_1| > |E_2|$. In the former part, the momentum $\lambda_2\bar\lambda_2 - \lambda_1\bar\lambda_1$ carries the sign s_2 of particle 2, in the latter part it carries the opposite sign s_1. Using the fact that $\delta^2(\langle 12\rangle)/\langle 23\rangle\cdots\langle n,1\rangle$ is invariant under an exchange of the labels 1 and 2, we can exchange those labels in the latter part. It then combines

---

[9]An alternative to deal with negative-energy particles is to represent them through variations $\check J(\Lambda)$ where the sources $J(\Lambda)$ correspond to positive-energy particles only. The amplitude would thus be promoted to a variational operator. This picture is equivalent to the canonical quantum field theory framework where the S-matrix is an operator acting on the Fock space.

[10]Changing also the sign of the fermionic variable η is purely conventional.

with the former to

$$
-\pi \sum_{n=4}^{\infty} \int_{|E_1|<|E_2|} \prod_{k=1}^n d^{4|4} \Lambda_k \, \varepsilon_{\dot{a}\dot{c}} \frac{\delta^2(\langle 1, 2\rangle)\delta^8(Q)}{\langle 2,3\rangle \cdots \langle n,1\rangle}
$$
$$
\times \sum_{\substack{s_j=\pm \\ 2\le j \le n}} \delta^4(P)\big(\bar{\lambda}_1^{\dot{c}}\eta_2^B - \bar{\lambda}_2^{\dot{c}}\eta_1^B\big) \operatorname{Tr}\big([J_1^{-s_2}, J_2^{s_2}]J_3^{s_3}\cdots J_n^{s_n}\big), \quad (3.30)
$$

where $P = s_2(\lambda_2\bar{\lambda}_2 - \lambda_1\bar{\lambda}_1) + \sum_{j=3}^n s_j\lambda_j\bar{\lambda}_j$. As before we can now use the delta function $\delta^2(\langle 12\rangle)$ to partially perform the λ_1 integral by using (A.5), this time setting $\lambda_1 = e^{i\varphi}\lambda_2 \tanh\alpha$, rescaling $\lambda_2 \to \lambda_2' \cosh\alpha$ and integrating over φ and α instead of λ_1. Further including a rotation of η_1 and η_2, altogether we define the new set of variables Λ_2', η', α and φ through (cf. (3.8)):

$$
\begin{aligned}
\lambda_1 &= e^{i\varphi}\lambda_{12}\sinh\alpha, & \eta_1 &= e^{-i\varphi}(\eta_{12}\sinh\alpha + \eta'\cosh\alpha), \\
\lambda_2 &= \lambda_{12}\cosh\alpha, & \eta_2 &= \eta_{12}\cosh\alpha + \eta'\sinh\alpha,
\end{aligned} \quad (3.31)
$$

$$
\Rightarrow d^4\lambda_1\, d^4\lambda_2\, \delta^2(\langle 12\rangle) = d\lambda_{12}\, d\varphi\, d\alpha\, \sinh\alpha \cosh\alpha.
$$

With this change of variables, the part of $(\bar{\mathfrak{S}}_0)_{\dot{a}}^B \mathcal{A}^{\mathrm{MHV}}[J]$ where $s_1 = -s_2$ (3.30) becomes

$$
-\sum_{n=4}^{\infty} \int \prod_{k=3}^n d^{4|4}\Lambda_k\, d^{4|4}\Lambda_{12} \sum_{\substack{s_j=\pm \\ 1\le j \le n-1}} \frac{\delta^4(P')\delta^4(Q')}{\langle 12,3\rangle \cdots \langle n,12\rangle} \cdot \pi \int d^4\eta'\, d\varphi\, d\alpha\, e^{3i\varphi}
$$
$$
\times \varepsilon_{\dot{a}\dot{c}}(\bar{\lambda}_1^{\dot{c}}\eta_2^B - \bar{\lambda}_2^{\dot{c}}\eta_1^B) \operatorname{Tr}\big([J_1^{-s_2}, J_2^{s_2}]J_3^{s_3}\cdots J_n^{s_n}\big), \quad (3.32)
$$

where $P' = s_2\lambda_{12}\bar{\lambda}_{12} + \sum_{j=3}^n s_j\lambda_j\bar{\lambda}_j$ and $Q' = s_2\lambda_{12}\eta_{12} + \sum_{j=3}^n s_j\lambda_j\eta_j$. As in the purely positive-energy case (3.9), this produces something very reminiscent of A_{n-1}^{MHV} and can hence be compensated by adding a term $\bar{\mathfrak{S}}_{s\to\{+,-\}}^{(0)}$ to $\bar{\mathfrak{S}}$. In this case, the correction term splits a particle with sign s into two collinear particles with opposite energy signs.

The complete tree-level correction to the operator $\bar{\mathfrak{S}}$ thus reads

$$
\bar{\mathfrak{S}}_+ = \bar{\mathfrak{S}}_{s\to\{s,s\}}^{(0)} + \bar{\mathfrak{S}}_{s\to\{+,-\}}^{(0)} = \bar{\mathfrak{S}}_+^{=} + \bar{\mathfrak{S}}_+^{\ne}, \quad (3.33)
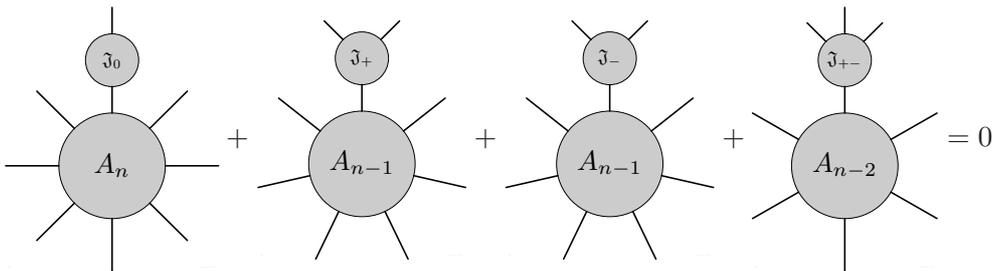$$

where $\bar{\mathfrak{S}}_+^{=}$ is given by (3.11) with J replaced by J^s and including a sum over $s = \pm$. After replacing J^s by the projection $\hat{J}^s$ (2.17) and removing the phase of Λ_n in (3.32), the further correction $\bar{\mathfrak{S}}_+^{\ne}$ is given by[11]

$$
(\bar{\mathfrak{S}}_+^{\ne})_{\dot{a}}^B = 2\pi^2 \int d^{4|4}\Lambda\, d^4\eta'\, d\alpha \sum_{s=\pm} \varepsilon_{\dot{a}\dot{c}}(\bar{\lambda}_1^{\dot{c}}\eta_2^B - \bar{\lambda}_2^{\dot{c}}\eta_1^B) \operatorname{Tr}\big([\hat{J}^{-s}(\Lambda_1), \hat{J}^s(\Lambda_2)]\check{J}^s(\Lambda)\big), \quad (3.34)
$$

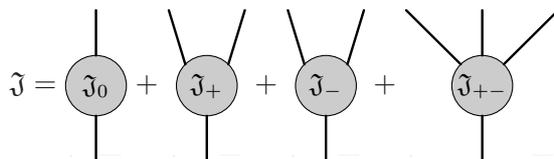where Λ_1 and Λ_2 are defined as

$$
\begin{aligned}
\lambda_1 &= \lambda\sinh\alpha, & \eta_1 &= \eta\sinh\alpha + \eta'\cosh\alpha, \\
\lambda_2 &= \lambda\cosh\alpha, & \eta_2 &= \eta\cosh\alpha + \eta'\sinh\alpha.
\end{aligned} \quad (3.35)
$$

---

[11]Note that the integral over α in (3.34) runs from 0 to ∞, while it runs from 0 to $\frac{1}{2}\pi$ in (3.11).

Figure 6. Statement of exact invariance of tree amplitudes under the deformed superconformal representation.

Figure 7. The free superconformal generators $\mathfrak{J}_0$ are deformed by contributions changing the number of particles and thereby relating scattering amplitudes with different numbers of legs to each other.

As can be seen from this example calculation, the contributions to the classical generators coming from the inclusion of negative-energy particles are obtained straightforwardly once the purely positive-energy corrections are known. Since the additional terms obscure notation though, we refrain from including them in the remainder of this work.

4 Closure of the algebra

In the previous section perturbative corrections to the superconformal generators $\mathfrak{S}_{aA}$ and $\bar{\mathfrak{S}}_{\dot{a}}^B$ of $\mathcal{N} = 4$ SYM theory were derived by requiring the generating functional of MHV scattering amplitudes (2.14) to be invariant under the action of these operators (cf. figure 6).

A priori, however, it is not clear that these deformations are complete because we have considered only a subset of amplitudes. An indication of completeness may come from algebra. We would like to show that the deformed generators still obey the $\mathfrak{psu}(2,2|4)$ superconformal algebra, which is also not clear a priori.

4.1 Classical representation

Looking ahead to section 4.3, the corrected generators are of the form (cf. figure 7)

$$\mathfrak{S} = \mathfrak{S}_0 + \mathfrak{S}_-, \qquad \bar{\mathfrak{S}} = \bar{\mathfrak{S}}_0 + \bar{\mathfrak{S}}_+, \qquad \mathfrak{K} = \mathfrak{K}_0 + \mathfrak{K}_+ + \mathfrak{K}_- + \mathfrak{K}_{+-}. \qquad (4.1)$$

All other generators remain undeformed. The correction terms to (2.19) were computed in (3.11), (3.24), (3.18), (3.26) and read

$$(\mathfrak{S}_-)_{Aa} = -2\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \; \delta^4(\eta') \, \mathrm{Tr}[\varepsilon_{ab}\lambda_1^b \hat{J}_1, \partial_{2,A}\hat{J}_2]\check{J},$$

$$(\bar{\mathfrak{S}}_+)_{\dot{a}}^A = +2\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \; \mathrm{Tr}\big[\varepsilon_{\dot{a}\dot{b}}\bar{\lambda}_1^{\dot{b}}\hat{J}_1, \eta_2^A \hat{J}_2\big]\check{J},$$

$$(\mathfrak{K}_-)_{a\dot{a}} = -2\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \; \delta^4(\eta') \, \mathrm{Tr}\big[\varepsilon_{ab}\lambda_1^b \hat{J}_1, \bar{\partial}_{2,\dot{a}}\hat{J}_2\big]\check{J},$$

$$(\mathfrak{K}_+)_{a\dot{a}} = -2\pi^2 \int d^{4|4}\Lambda \, d^4\eta' \, d\alpha \; \mathrm{Tr}\big[\varepsilon_{\dot{a}\dot{b}}\bar{\lambda}_1^{\dot{b}}\hat{J}_1, \partial_{2,a}\hat{J}_2\big]\check{J}, \tag{4.2}$$

where $J_k = J(\Lambda_k)$, $J = J(\Lambda)$. The term $\mathfrak{K}_{+-}$ can be found at the end of section 4.3. The spinor helicity coordinates Λ_1, Λ_2 are defined as follows (3.12)

$$\begin{aligned}
\lambda_1 &= \lambda \sin\alpha, & \eta_1 &= \eta \sin\alpha + \eta' \cos\alpha, \\
\lambda_2 &= \lambda \cos\alpha, & \eta_2 &= \eta \cos\alpha - \eta' \sin\alpha.
\end{aligned} \tag{4.3}$$

4.2 Algebra relations

It is straight-forward to read off the algebra relations from the representation (2.3) of the undeformed generators. The indices of a generator $\mathfrak{J}$ under Lorentz and internal symmetry transform as

$$\begin{aligned}
[\mathfrak{L}^a{}_b, \mathfrak{J}_c] &= -\delta_c^a \mathfrak{J}_b + \frac{1}{2}\delta_b^a \mathfrak{J}_c, & [\mathfrak{L}^a{}_b, \mathfrak{J}^c] &= \delta_b^c \mathfrak{J}^a - \frac{1}{2}\delta_b^a \mathfrak{J}^c, \\
[\mathfrak{R}^A{}_B, \mathfrak{J}_C] &= -\delta_C^A \mathfrak{J}_B + \frac{1}{4}\delta_B^A \mathfrak{J}_C, & [\mathfrak{R}^A{}_B, \mathfrak{J}^C] &= \delta_B^C \mathfrak{J}^A - \frac{1}{4}\delta_B^A \mathfrak{J}^C, \\
[\bar{\mathfrak{L}}^{\dot{a}}{}_{\dot{b}}, \mathfrak{J}_{\dot{c}}] &= -\delta_{\dot{c}}^{\dot{a}} \mathfrak{J}_{\dot{b}} + \frac{1}{2}\delta_{\dot{b}}^{\dot{a}} \mathfrak{J}_{\dot{c}}, & [\bar{\mathfrak{L}}^{\dot{a}}{}_{\dot{b}}, \mathfrak{J}^{\dot{c}}] &= \delta_{\dot{b}}^{\dot{c}} \mathfrak{J}^{\dot{a}} - \frac{1}{2}\delta_{\dot{b}}^{\dot{a}} \mathfrak{J}^{\dot{c}}.
\end{aligned} \tag{4.4}$$

All indices in the deformations (4.2) are contracted properly using only invariant symbols. Consequently all commutators with $\mathfrak{L}$, $\bar{\mathfrak{L}}$ and $\mathfrak{R}$ are unchanged using the free rotation generators $\mathfrak{L}_0$, $\bar{\mathfrak{L}}_0$ and $\mathfrak{R}_0$.

Commutators with the dilatation generator, $[\mathfrak{D}, \mathfrak{J}] = \dim(\mathfrak{J})\mathfrak{J}$, are specified through the conformal dimensions of the generators, the non-trivial ones being

$$\dim(\mathfrak{P}) = -\dim(\mathfrak{K}) = 1, \qquad \dim(\mathfrak{Q}) = \dim(\bar{\mathfrak{Q}}) = -\dim(\mathfrak{S}) = -\dim(\bar{\mathfrak{S}}) = \frac{1}{2}. \tag{4.5}$$

By power counting it is also straight-forward to show that $\mathfrak{D} = \mathfrak{D}_0$ yields the correct algebra.

It is the aim of this section to show that the additional non-trivial algebra relations given by

$$\begin{aligned}
\{\mathfrak{Q}^{aA}, \bar{\mathfrak{Q}}_B^{\dot{a}}\} &= \delta_B^A \mathfrak{P}^{a\dot{a}}, & \{\mathfrak{S}_{aA}, \bar{\mathfrak{S}}_{\dot{a}}^B\} &= \delta_A^B \mathfrak{K}_{a\dot{a}}, \\
[\mathfrak{P}^{a\dot{a}}, \mathfrak{S}_{bA}] &= \delta_b^a \bar{\mathfrak{Q}}_A^{\dot{a}}, & [\mathfrak{K}_{a\dot{a}}, \mathfrak{Q}^{bA}] &= \delta_a^b \bar{\mathfrak{S}}_{\dot{a}}^A, \\
[\mathfrak{P}^{a\dot{a}}, \bar{\mathfrak{S}}_{\dot{b}}^A] &= \delta_{\dot{b}}^{\dot{a}} \mathfrak{Q}^{aA}, & [\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{Q}}_A^{\dot{b}}] &= \delta_{\dot{a}}^{\dot{b}} \mathfrak{S}_{aA},
\end{aligned} \tag{4.6}$$

and

$$[\mathfrak{K}_{a\dot{a}}, \mathfrak{P}^{b\dot{b}}] = \delta_{\dot{a}}^{\dot{b}}\mathfrak{L}^b{}_a + \delta_a^b\bar{\mathfrak{L}}^{\dot{b}}{}_{\dot{a}} + \delta_a^b\delta_{\dot{a}}^{\dot{b}}\mathfrak{D},$$

$$\{\mathfrak{Q}^{aA}, \mathfrak{S}_{bB}\} = \delta_B^A\mathfrak{L}^a{}_b - \delta_b^a\mathfrak{R}^A{}_B + \delta_b^a\delta_B^A\left(\frac{1}{2}\mathfrak{D} + \frac{1}{4}\mathfrak{C}\right),$$

$$\{\bar{\mathfrak{Q}}_A^{\dot{a}}, \bar{\mathfrak{S}}_{\dot{b}}^B\} = \delta_A^B\bar{\mathfrak{L}}^{\dot{a}}{}_{\dot{b}} + \delta_{\dot{b}}^{\dot{a}}\mathfrak{R}^B{}_A + \delta_{\dot{b}}^{\dot{a}}\delta_A^B\left(\frac{1}{2}\mathfrak{D} - \frac{1}{4}\mathfrak{C}\right), \tag{4.7}$$

as well as all trivial commutators are not altered by the introduced corrections.

Since $\mathfrak{P}$ and $\mathfrak{K}$ are expressed in terms of $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$ and $\mathfrak{S}$, $\bar{\mathfrak{S}}$, respectively, the verification of the algebra reduces to a minimal set of commutation relations. These are the relations involving only the four latter operators. The remaining commutators then follow using the Jacobi identity as will be demonstrated at the end of the section.

4.3 The generator $\mathfrak{K}$

For a verification of the superconformal algebra it is not necessary to explicitly construct the generator $\mathfrak{K}$ of special conformal transformations. Nevertheless it is desirable to obtain an expression for its deformations with regard to the symmetries of scattering amplitudes. The corrections to the conformal generator take the form

$$\mathfrak{K} = \mathfrak{K}_0 + \mathfrak{K}_+ + \mathfrak{K}_- + \mathfrak{K}_{+-}$$
$$= \{\mathfrak{S}_0, \bar{\mathfrak{S}}_0\} + \{\mathfrak{S}_0, \bar{\mathfrak{S}}_+\} + \{\mathfrak{S}_-, \bar{\mathfrak{S}}_0\} + \{\mathfrak{S}_-, \bar{\mathfrak{S}}_+\}. \tag{4.8}$$

Employing the expressions for the corrections to $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ obtained in (3.11), (3.24), the above anti-commutators can be explicitly evaluated. We make use of the notation introduced in (3.12) and note the following set of useful identities for the evaluation of commutation relations

$$0 = \lambda^a\eta^A - \lambda_1^a\eta_1^A - \lambda_2^a\eta_2^A, \tag{4.9}$$

$$\eta_1^B\bar{\partial}_{1,\dot{a}}J_1 = (\eta^B + \cot\alpha\,\eta'^B)\bar{\partial}_{\dot{a}}J_1, \tag{4.10}$$

$$\eta_2^B\bar{\partial}_{2,\dot{a}}J_2 = (\eta^B - \tan\alpha\,\eta'^B)\bar{\partial}_{\dot{a}}J_2, \tag{4.11}$$

$$\bar{\lambda}_1^{\dot{a}}\partial_{1,A}J_1 = \bar{\lambda}^{\dot{a}}\partial_A J_1, \tag{4.12}$$

$$\bar{\lambda}_2^{\dot{a}}\partial_{2,A}J_2 = \bar{\lambda}^{\dot{a}}\partial_A J_2. \tag{4.13}$$

We first compute the anti-commutator of $\bar{\mathfrak{S}}_0$ and $\mathfrak{S}_-$ to find

$$\{(\mathfrak{S}_-)_{aA}, (\bar{\mathfrak{S}}_0)_{\dot{a}}^B\}\hat{J} = 2\pi^2\int d^4\eta'\,d\alpha\,\delta^4(\eta')\,\varepsilon_{ab}\lambda_1^b\Big\{-\eta^B\bar{\partial}_{\dot{a}}\big[\hat{J}_1, \partial_{2,A}\hat{J}_2\big]$$
$$+ \eta_1^B\big[\bar{\partial}_{1,\dot{a}}\hat{J}_1, \partial_{2,A}\hat{J}_2\big] - \big[\hat{J}_1, \partial_{2,A}\eta_2^B\bar{\partial}_{2,\dot{a}}\hat{J}_2\big]\Big\}. \tag{4.14}$$

Evaluating (4.10), (4.11) at $\eta' = 0$ yields $\{(\mathfrak{S}_-)_{aA}, (\bar{\mathfrak{S}}_0)_{\dot{a}}^B\} = \delta_A^B(\mathfrak{K}_-)_{a\dot{a}}$ with

$$(\mathfrak{K}_-)_{a\dot{a}}\hat{J} = -2\pi^2\int d^4\eta'\,d\alpha\,\delta^4(\eta')\,\big[\varepsilon_{ab}\lambda_1^b\hat{J}_1, \bar{\partial}_{2,\dot{a}}\hat{J}_2\big]. \tag{4.15}$$

In order to compute $\mathfrak{K}_+$ we consider

$$\{(\mathfrak{S}_0)_{aB}, (\bar{\mathfrak{S}}_+)^A_{\dot{a}}\}\hat{J} = 2\pi^2 \int d^4\eta'\, d\alpha\, \varepsilon_{\dot{a}b}\bar{\lambda}^{\dot{b}}_1 \Big\{ -\partial_a \partial_B \big[\hat{J}_1, \eta^A_2 \hat{J}_2\big]$$

$$- \eta^A_2 \big[\partial_{1,a}\partial_{1,B}\hat{J}_1, \hat{J}_2\big] - \eta^A_2 \big[\hat{J}_1, \partial_{2,a}\partial_{2,B}\hat{J}_2\big] \Big\}. \tag{4.16}$$

We add the following integration by parts term to the r.h.s.

$$2\pi^2 \int d^4\eta'\, d\alpha\, \partial'_B\, \varepsilon_{\dot{a}b}\bar{\lambda}^{\dot{b}}_1 \Big\{ -\cos\alpha\, \big[\partial_{1,a}\hat{J}_1, \eta^A_2 \hat{J}_2\big] + \sin\alpha\, \big[\hat{J}_1, \eta^A_2 \partial_{2,a}\hat{J}_2\big] \Big\} = 0 \tag{4.17}$$

in order to shift all fermionic derivatives to their bosonic counterparts. Using $\partial_A J_1 = \sin\alpha\, \partial_{1,A}J_1$ and $\partial'_A J_1 = \cos\alpha\, \partial_{1,A}J_1$, etc., we obtain $\{(\mathfrak{S}_0)_{aB}, (\bar{\mathfrak{S}}_+)^A_{\dot{a}}\} = \delta^A_B(\mathfrak{K}_+)_{a\dot{a}}$ with

$$(\mathfrak{K}_+)_{a\dot{a}}\hat{J} = -2\pi^2 \int d^4\eta'\, d\alpha\, \big[\varepsilon_{\dot{a}b}\bar{\lambda}^{\dot{b}}_1 \hat{J}_1, \partial_{2,a}\hat{J}_2\big], \tag{4.18}$$

which coincides with the result given in (3.18).

Finally, we want to show that $\{\mathfrak{S}_-, \bar{\mathfrak{S}}_+\}$ is a $\mathfrak{su}(4)$ singlet and hence defines $\mathfrak{K}_{+-}$ properly. To make the calculation more tractable, we introduce two sets of fermionic variables $\tilde{\theta}^A, \theta^A$ which we contract with the generators

$$\mathfrak{S}^a_- := \varepsilon^{ab}\tilde{\theta}^C(\mathfrak{S}_-)_{bC}, \qquad \bar{\mathfrak{S}}^{\dot{a}}_+ := \varepsilon^{\dot{a}\dot{b}}\varepsilon_{CDEF}\theta^D\theta^E\theta^F(\bar{\mathfrak{S}}_+)^C_{\dot{b}}. \tag{4.19}$$

The aim is to show that the commutator is totally antisymmetric in $\tilde{\theta}$ and the θ's[12]

$$\big[\mathfrak{S}^a_-, \bar{\mathfrak{S}}^{\dot{b}}_+\big] \sim \varepsilon_{CDEF}\tilde{\theta}^C\theta^D\theta^E\theta^F. \tag{4.20}$$

We can evaluate the action of the generators on a source by rewriting the fermionic integral in $\bar{\mathfrak{S}}^{\dot{a}}_+$ as $\int d^4\eta'\, \eta' \sim \partial'^3$

$$\mathfrak{S}^a_- \hat{J}(\Lambda) \sim +\lambda^a \int d\alpha\, \cos\alpha\, \big[\tilde{\partial}\hat{J}(\sin\alpha\Lambda), J(\cos\alpha\Lambda)\big]$$

$$- \lambda^a \int d\alpha\, \sin\alpha\, \big[\hat{J}(\sin\alpha\Lambda), \tilde{\partial}J(\cos\alpha\Lambda)\big],$$

$$\bar{\mathfrak{S}}^{\dot{a}}_+ \hat{J}(\Lambda) \sim +\bar{\lambda}^{\dot{a}} \int d\alpha\, \cos^3\alpha\, \big[\partial^3\hat{J}(\sin\alpha\Lambda), J(\cos\alpha\Lambda)\big]$$

$$- 3\bar{\lambda}^{\dot{a}} \int d\alpha\, \cos^2\alpha\sin\alpha\, \big[\partial^2\hat{J}(\sin\alpha\Lambda), \partial J(\cos\alpha\Lambda)\big]$$

$$+ 3\bar{\lambda}^{\dot{a}} \int d\alpha\, \cos\alpha\sin^2\alpha\, \big[\partial\hat{J}(\sin\alpha\Lambda), \partial^2 J(\cos\alpha\Lambda)\big]$$

$$- \bar{\lambda}^{\dot{a}} \int d\alpha\, \sin^3\alpha\, \big[\hat{J}(\sin\alpha\Lambda), \partial^3 J(\cos\alpha\Lambda)\big]. \tag{4.21}$$

---

[12]The fermionic variables θ turn the new generators $\mathfrak{S}^a_-$ and $\bar{\mathfrak{S}}^{\dot{a}}_+$ into bosonic operators. Consequently we should compute their commutator. Likewise all the objects in the following computation will turn out to be (conveniently) bosonic.

The above index-free partial derivatives are defined as $\partial := \theta^A \partial_A$ and $\tilde{\partial} := \tilde{\theta}^A \partial_A$, and they are bosonic. Applying the two generators to a source $J(\Lambda)$ results in three sources $J_{x,y,z} := J(x\Lambda, y\Lambda, z\Lambda)$ with spherical coordinates and measure

$$x = \sin\alpha\cos\beta, \quad y = \sin\alpha\sin\beta, \quad z = \cos\alpha, \qquad \int d^2\Omega = \int d\alpha\, d\beta\, \sin\alpha. \qquad (4.22)$$

The benefit of these coordinates is that they are fully interchangeable which allows for the Jacobi identity to be used easily.

Using this expression we can compute $[\mathfrak{S}_-^a, \bar{\mathfrak{S}}_+^{\dot{b}}]\hat{J}(\Lambda)$ and find 16 terms initially which can be grouped into 5 classes depending on how their derivatives are distributed. Some terms have to be converted by means of a Jacobi identity and permuting the coordinates x, y, z accordingly. It is now a matter of patience and care to show that all the derivatives ∂ and $\tilde{\partial}$ appear symmetrically and thus (4.20) holds.

There is however a slightly more convenient way to show the required property formally: We note that the terms in (4.20) follow a certain regular pattern. Let us therefore introduce some derivative operators ∂_1, ∂_2 acting on three sources $J_{x,y,z}$ according to

$$\partial_1 J_x = \frac{xz}{\sqrt{1-z^2}}\partial J_x, \qquad \partial_1 J_y = \frac{yz}{\sqrt{1-z^2}}\partial J_y, \qquad \partial_1 J_z = -\sqrt{1-z^2}\,\partial J_z,$$
$$\partial_2 J_x = \frac{y}{\sqrt{1-z^2}}\partial J_x, \qquad \partial_2 J_y = \frac{-x}{\sqrt{1-z^2}}\partial J_y, \qquad \partial_2 J_z = 0, \qquad (4.23)$$

It is easy to convince oneself that

$$\left[\mathfrak{S}_-^a, \bar{\mathfrak{S}}_+^{\dot{b}}\right]\hat{J}(\Lambda) \sim \lambda^a \bar{\lambda}^{\dot{b}} \int d^2\Omega \left(\tilde{\partial}_1(\partial_2)^3 - \tilde{\partial}_2(\partial_1)^3\right) \left[[\hat{J}_x, \hat{J}_y], \hat{J}_z\right]. \qquad (4.24)$$

The point is that $\mathfrak{S}_-^a \sim \partial_k$ and $\mathfrak{S}_+^{\dot{b}} \sim (\partial_k)^3$, cf. (4.21), and the index k tells whether the operator acts on the outer or the inner commutator. Note that the density factor $\sin\alpha$ of $d^2\Omega$ originates from rescaling λ^a or $\bar{\lambda}^{\dot{a}}$ in the second generator.

The above expression (4.24) is however not yet manifestly symmetric in tilded and untilded derivatives as required for (4.20). We have to use the Jacobi identity to achieve symmetry. It turns out that replacing

$$\left[[\hat{J}_x, \hat{J}_y], \hat{J}_z\right] \to \frac{2}{3}\left[[\hat{J}_x, \hat{J}_y], \hat{J}_z\right] - \frac{1}{3}\left[[\hat{J}_x, \hat{J}_z], \hat{J}_y\right] - \frac{1}{3}\left[[\hat{J}_z, \hat{J}_y], \hat{J}_x\right] \qquad (4.25)$$

achieves the goal. In order to make the three terms comparable, we have to permute the coordinates x, y, z. The permutations also transform the two derivative operators $\vec{\partial} = (\partial_1, \partial_2)$ using the permutation matrices

$$P_{xy}\vec{\partial} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}\vec{\partial},$$

$$P_{xz}\vec{\partial} = \frac{1}{\sqrt{1-x^2}\sqrt{1-z^2}} \begin{pmatrix} -xz & -y \\ -y & +xz \end{pmatrix}\vec{\partial},$$

$$P_{yz}\vec{\partial} = \frac{1}{\sqrt{1-y^2}\sqrt{1-z^2}} \begin{pmatrix} -yz & +x \\ +x & +yz \end{pmatrix}\vec{\partial}. \qquad (4.26)$$

In confirming the relation one can for convenience treat ∂_k as two bosonic variables and thus (4.24) is merely a quadratic polynomial in ∂_k.

Using the same notation we can formally write down $\mathfrak{K}_{+-}$

$$(\mathfrak{K}_{+-})_{b\dot{a}} \sim \int d^{4|4}\Lambda \, d^2\Omega \, d^4\theta \, \varepsilon_{bd}\varepsilon_{\dot{a}\dot{c}}\lambda^d\bar{\lambda}^{\dot{c}} \left(\partial_1(\partial_2)^3 - \partial_2(\partial_1)^3\right) \left[[\hat{J}_x, \hat{J}_y], \hat{J}_z\right]\check{J}. \tag{4.27}$$

Due to $\mathfrak{K} \sim \{\mathfrak{S}, \bar{\mathfrak{S}}\}$ the conformal generator inherits the property to annihilate the generating functional of scattering amplitudes (2.14) from $\mathfrak{S}$, $\bar{\mathfrak{S}}$. We thus consider it an unreasonable hardship to compute the precise prefactor of (4.27).

4.4 Commutators between $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$ and $\mathfrak{S}$, $\bar{\mathfrak{S}}$

In this section we demonstrate that the commutation relations of the generators $\mathfrak{S}$, $\bar{\mathfrak{S}}$ with $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$ are not altered by the perturbative corrections introduced above by acting on a source term $\hat{J}(\Lambda)$.

It is straight-forward to show that the anticommutator between $\mathfrak{Q}_0$ and $\bar{\mathfrak{S}}_+$ vanishes by means of (4.9)

$$\left\{(\mathfrak{Q}_0)^{aA}, (\bar{\mathfrak{S}}_+)_{\dot{a}}^B\right\} = 0. \tag{4.28}$$

Taking into account (4.12), (4.13), also the anti-commutator of $\bar{\mathfrak{Q}}_0$ and $\mathfrak{S}_-$ vanishes:

$$\begin{aligned}
\left\{(\bar{\mathfrak{Q}}_0)_A^{\dot{a}}, (\mathfrak{S}_-)_{aB}\right\}\hat{J} = 2\pi^2 \int d^4\eta' \, d\alpha \, \delta^4(\eta') \, \varepsilon_{ab}\lambda_1^b &\Big\{ \bar{\lambda}_1^{\dot{a}}\left[\partial_{1,A}\hat{J}_1, \partial_{2,B}\hat{J}_2\right] \\
&+ \bar{\lambda}_2^{\dot{a}}\left[\hat{J}_1, \partial_{2,A}\partial_{2,B}\hat{J}_2\right] - \bar{\lambda}^{\dot{a}}\partial_A\left[\hat{J}_1, \partial_{2,B}\hat{J}_2\right]\Big\} \\
&= 0.
\end{aligned} \tag{4.29}$$

Next we evaluate the anti-commutator of $\mathfrak{Q}_0$ and $\mathfrak{S}_-$ giving

$$\begin{aligned}
\left\{(\mathfrak{Q}_0)^{aA}, (\mathfrak{S}_-)_{bB}\right\}\hat{J} = -2\pi^2 \int d^4\eta' \, d\alpha \, \delta^4(\eta')\varepsilon_{bc}\lambda_1^c &\Big\{ \lambda_1^a\eta_1^A\left[\hat{J}_1, \partial_{2,B}\hat{J}_2\right] \\
&- \lambda_2^a\left[\hat{J}_1, \partial_{2,B}(\eta_2^A\hat{J}_2)\right] - \lambda^a\eta^A\left[\hat{J}_1, \partial_{2,B}\hat{J}_2\right]\Big\}.
\end{aligned} \tag{4.30}$$

Now (4.9) yields

$$\left\{(\mathfrak{Q}_0)^{aA}, (\mathfrak{S}_-)_{bB}\right\}\hat{J} = 2\pi^2\delta_B^A \int d^4\eta' \, d\alpha \, \delta^4(\eta') \, \varepsilon_{bc}\lambda_1^c\lambda_2^a\left[\hat{J}_1, \hat{J}_2\right], \tag{4.31}$$

and the integral is antisymmetric under the shift of the integration variable

$$\alpha \mapsto \frac{\pi}{2} - \alpha \quad \Rightarrow \quad \Lambda_1 \leftrightarrow \Lambda_2, \tag{4.32}$$

and does therefore vanish.

Last but not least we compute the anti-commutator of $\bar{\mathfrak{Q}}_0$ and $\bar{\mathfrak{S}}_+$ giving

$$\begin{aligned}
\left\{(\bar{\mathfrak{Q}}_0)_A^{\dot{a}}, (\bar{\mathfrak{S}}_+)_{\dot{b}}^B\right\}\hat{J} = -2\pi^2 \int d^4\eta' \, d\alpha \, \varepsilon_{\dot{b}\dot{c}}\bar{\lambda}_1^{\dot{c}} &\Big\{ -\bar{\lambda}^{\dot{a}}\partial_A\left[\hat{J}_1, \eta_2^B\hat{J}_2\right] \\
&- \eta_2^B\bar{\lambda}_1^{\dot{a}}\left[\partial_{1,A}\hat{J}_1, \hat{J}_2\right] - \eta_2^B\bar{\lambda}_2^{\dot{a}}\left[\hat{J}_1, \partial_{2,A}\hat{J}_2\right]\Big\}.
\end{aligned} \tag{4.33}$$

By means of (4.12), (4.13) we are left with an integral expression of the form

$$\{(\bar{\mathfrak{Q}}_0)^{\dot a}_A, (\bar{\mathfrak{S}}_+)^B_{\dot b}\}\hat J = 2\pi^2\delta^B_A\int d^4\eta'\,d\alpha\,\varepsilon_{\dot b\dot c}\bar\lambda^{\dot c}_1\bar\lambda^{\dot a}_2[\hat J_1,\hat J_2] = 0. \tag{4.34}$$

The integral, however, again vanishes being antisymmetric under a shift of integration variables:

$$\alpha\mapsto\frac{\pi}{2}-\alpha, \qquad \eta'\mapsto-\eta' \quad\Rightarrow\quad \Lambda_1\leftrightarrow\Lambda_2. \tag{4.35}$$

4.5 Commutators between $\mathfrak{S}$ and $\bar{\mathfrak{S}}$

The anticommutator of two $\mathfrak{S}$ vanishes in $\mathfrak{psu}(2,2|4)$ and the same is true for the tree superconformal representation $\mathfrak{S}_0$; similarly for $\bar{\mathfrak{S}}$.

Let us now compute the corrections due to $\mathfrak{S}_-$ by acting on the source $\hat J(\Lambda)$. Straightforward evaluation yields

$$\begin{aligned}
\{(\mathfrak{S}_0)_{aB}, (\mathfrak{S}_-)_{cD}\}\hat J = &-2\pi^2\int d\alpha\,\varepsilon_{ac}\sin^2\alpha\{\partial_{1,B}\hat J_1,\partial_{2,D}\hat J_2\} \\
&-2\pi^2\int d\alpha\,\varepsilon_{ce}\lambda^e\sin\alpha\cos^2\alpha\{\partial_{1,a}\partial_{1,B}\hat J_1,\partial_{2,D}\hat J_2\} \\
&+2\pi^2\int d\alpha\,\varepsilon_{ce}\lambda^e\sin^2\alpha\cos\alpha\{\partial_{1,B}\hat J_1,\partial_{2,a}\partial_{2,D}\hat J_2\} \\
&-2\pi^2\int d\alpha\,\varepsilon_{ac}\sin\alpha\cos\alpha[\hat J_1,\partial_{2,B}\partial_{2,D}\hat J_2] \\
&-2\pi^2\int d\alpha\,\varepsilon_{ce}\lambda^e\sin^3\alpha[\hat J_1,\partial_{2,a}\partial_{2,B}\partial_{2,D}\hat J_2] \\
&+2\pi^2\int d\alpha\,\varepsilon_{ce}\lambda^e\sin^2\alpha\cos\alpha[\partial_{1,a}\hat J_1,\partial_{2,B}\partial_{2,D}\hat J_2]. \tag{4.36}
\end{aligned}$$

The expansion of the anticommutator $\{\mathfrak{S}_{aB},\mathfrak{S}_{cD}\}$ contains the above anticommutator symmetrised over the pairs aB and cD. Note that each term in the above expression is manifestly antisymmetric in a,c or in B,D. Thus the final term must be antisymmetric in both a,c and B,D. We can make the antisymmetry in a,c manifest by pulling out ε_{ac}. After flipping some of the integral regions, $\alpha\mapsto\frac{1}{2}\pi-\alpha$, and rearranging some terms for later convenience, we obtain for $\{(\mathfrak{S}_0)_{aB},(\mathfrak{S}_-)_{cD}\} + \{(\mathfrak{S}_0)_{cD},(\mathfrak{S}_-)_{aB}\}$

$$\begin{aligned}
\ldots = &\,\pi^2\varepsilon_{ac}\int d\alpha\,(\cos^2\alpha-\sin^2\alpha)\{\partial_{1,B}\hat J_1,\partial_{2,D}\hat J_2\} \\
&+\pi^2\varepsilon_{ac}\int d\alpha\,\sin\alpha\cos\alpha\{\cot\alpha(2\lambda^e_1\partial_{1,e}+1)\partial_{1,B}\hat J_1,\partial_{2,D}\hat J_2\} \\
&+\pi^2\varepsilon_{ac}\int d\alpha\,\sin\alpha\cos\alpha\{\partial_{2,B}\hat J_1,-\tan\alpha(2\lambda^e_2\partial_{2,e}+1)\partial_{2,D}\hat J_2\} \\
&-\pi^2\varepsilon_{ac}\int d\alpha\,2\sin\alpha\cos\alpha[\hat J_1,\partial_{2,B}\partial_{2,D}\hat J_2] \\
&-\pi^2\varepsilon_{ac}\int d\alpha\,\sin^2\alpha[\cot\alpha(2\lambda^e_1\partial_{1,e}+2)\hat J_1,\partial_{2,B}\partial_{2,D}\hat J_2] \\
&-\pi^2\varepsilon_{ac}\int d\alpha\,\sin^2\alpha[\hat J_1,-2\tan\alpha\lambda^e_2\partial_{2,e}\partial_{2,B}\partial_{2,D}\hat J_2] \tag{4.37}
\end{aligned}$$

We would like to recast all these integrands in the form of a total derivative w.r.t. α. To this end we notice that terms like $\lambda^e \partial_e \hat{J}$ do appear in $\partial_\alpha \hat{J}$. Conversely, contributions of the sort $\bar{\lambda}^{\dot{e}} \bar{\partial}_{1,\dot{e}}$ and $\eta^E \partial_{1,E}$ which are also part of $\partial_\alpha \hat{J}_{1,2}$ to not appear. To resolve this problem we can make use of the identity

$$(\lambda^e \partial_e - \bar{\lambda}^{\dot{e}} \bar{\partial}_{\dot{e}} - \eta^E \partial_E + 2)\hat{J} = 0. \tag{4.38}$$

It holds by virtue of the definition (2.17) of $\hat{J}$ (total derivative) and it represents the central charge condition $\mathfrak{C}\hat{J} = 0$. This is also the reason why we started by acting on $\hat{J}$ representing the most general function with the property $\mathfrak{C}\hat{J} = 0$; our derivation only works for physical representations and the algebra closes only on when the central charge vanishes. The derivatives of $\hat{J}_{1,2}$ w.r.t. α thus yield

$$\frac{d\hat{J}_1}{d\alpha} = \cot\alpha(\lambda_1^e \partial_{1,e} + \bar{\lambda}_1^{\dot{e}} \bar{\partial}_{1,\dot{e}} + \eta_1^E \partial_{1,E})\hat{J}_1 = \cot\alpha(2\lambda_1^e \partial_{1,e} + 2)\hat{J}_1,$$
$$\frac{d\hat{J}_2}{d\alpha} = -\tan\alpha(\lambda_2^e \partial_{2,e} + \bar{\lambda}_2^{\dot{e}} \bar{\partial}_{2,\dot{e}} + \eta_2^E \partial_{2,E})\hat{J}_2 = -\tan\alpha(2\lambda_2^e \partial_{2,e} + 2)\hat{J}_2. \tag{4.39}$$

Notice that the term without derivatives is sensitive to the number of derivatives acting on J. For each fermionic derivative the number 2 is decreased by one unit. Altogether we can write

$$\ldots = \pi^2 \varepsilon_{ac} \int d\alpha \frac{d}{d\alpha} \left(\sin\alpha \cos\alpha \{\partial_{1,B}\hat{J}_1, \partial_{2,D}\hat{J}_2\} - \sin^2\alpha [\hat{J}_1, \partial_{2,B}\partial_{2,D}\hat{J}_2] \right)$$
$$= -\pi^2 \varepsilon_{ac} [\hat{J}(\Lambda), \partial_B \partial_D \hat{J}(0)] = \pi^2 \varepsilon_{ac} [\partial_B \partial_D \hat{J}(0), \hat{J}(\Lambda)]. \tag{4.40}$$

This has the form of a field-dependent gauge transformation of the gauge covariant object $\hat{J}(\Lambda)$ because it maps $\hat{J}(\Lambda) \mapsto [X, \hat{J}(\Lambda)]$ where X is the gauge variation parameter.

Finally we consider the anticommutator of two correction terms $\{(\mathfrak{S}_-)_{aB}, (\mathfrak{S}_-)_{cD}\}$. We apply the sequence of two $\mathfrak{S}_-$ to a source term $J(\Lambda)$

$$(\mathfrak{S}_-)_{cD}(\mathfrak{S}_-)_{aB}\hat{J}(\Lambda) = 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, y \{[\hat{J}_y, \partial_{x,D}\hat{J}_x], \partial_{z,B}\hat{J}_z\} \tag{4.41}$$

$$- 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, \frac{zy^2}{x^2+y^2} [\hat{J}_z, \{\partial_{y,B}\hat{J}_y, \partial_{x,D}\hat{J}_x\}]$$

$$- 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, \frac{zxy}{x^2+y^2} [\hat{J}_z, [\hat{J}_y, \partial_{x,B}\partial_{x,D}\hat{J}_x]].$$

Note that for the latter two lines we flipped the integration region $\alpha \mapsto \frac{1}{2}\pi - \alpha$ in order to achieve a common parametrisation. of $\Lambda_{x,y,z}$ where $\lambda_x = x\lambda$, $\eta_x = x\eta$, etc., with

$$x = \sin\alpha \cos\beta, \qquad y = \sin\alpha \sin\beta, \qquad z = \cos\alpha. \tag{4.42}$$

These are standard spherical coordinates and $d\alpha \, d\beta \, \sin\alpha$ is the corresponding measure. The integral is over the positive octant, $x, y, z > 1$, such that we can freely exchange the coordinates x, y, z. In the first line we exchange $y \leftrightarrow z$, multiply by $(x^2+y^2)/(x^2+y^2)$, and

exchange $x \leftrightarrow y$ for the part proportional to $x^2/(x^2 + y^2)$. Upon use of a Jacobi identity on the second line the result reads

$$(\mathfrak{S}_-)_{cD}(\mathfrak{S}_-)_{aB}\hat{J}(\Lambda) = 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, \frac{zy^2}{x^2 + y^2}\{[\hat{J}_z, \partial_{y,D}\hat{J}_y], \partial_{x,B}\hat{J}_x\} \quad (4.43)$$

$$- 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, \frac{zy^2}{x^2 + y^2}\{[\hat{J}_z, \partial_{y,B}\hat{J}_y], \partial_{x,D}\hat{J}_x\}$$

$$- 4\pi^4 \varepsilon_{ae}\varepsilon_{cf}\lambda^e\lambda^f \int d\alpha \, d\beta \, \sin\alpha \, \frac{zxy}{x^2 + y^2}[\hat{J}_z, [\hat{J}_y, \partial_{x,B}\partial_{x,D}\hat{J}_x]].$$

This expression is manifestly symmetric in a, c, but manifestly antisymmetric in B, D. The anticommutator $\{(\mathfrak{S}_-)_{cD}, (\mathfrak{S}_-)_{aB}\}$ thus vanishes.

In conclusion we find that $\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\}$ does not vanish for the interacting representation, but it closes onto a gauge transformation. Our proof depended crucially on the assumption of vanishing central charge for all objects we act upon. Let us introduce the generator of a gauge transformation with gauge parameter X

$$\mathfrak{G}[X] = \pi^2 \int d^{4|4}\Lambda \, \mathrm{Tr}\big([X, J(\Lambda)]\check{J}(\Lambda)\big). \quad (4.44)$$

Our final result reads

$$\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\} = \varepsilon_{ab}\mathfrak{G}[\partial_A\partial_B J(0)]. \quad (4.45)$$

We now turn to the commutator of two generators $\bar{\mathfrak{S}}$ acting on a source $\hat{J}(\Lambda)$. We consider the anti-commutator of $\bar{\mathfrak{S}}_0$ with $\bar{\mathfrak{S}}_+$ yielding

$$\{(\bar{\mathfrak{S}}_0)_{\dot{a}}^B, (\bar{\mathfrak{S}}_+)_{\dot{b}}^A\}\hat{J} = -2\pi^2 \int d^4\eta' \, d\alpha \, \varepsilon_{\dot{b}\dot{c}}\eta_2^A\Big\{ - \bar{\lambda}_1^{\dot{c}}\eta_1^B[\bar{\partial}_{1,\dot{a}}\hat{J}_1, \hat{J}_2]$$

$$- \bar{\lambda}_1^{\dot{c}}\eta_2^B[\hat{J}_1, \bar{\partial}_{2,\dot{a}}\hat{J}_2] + \eta^B(\bar{\partial}_{\dot{a}}\bar{\lambda}_1^{\dot{c}})[\hat{J}_1, \hat{J}_2] + \eta^B\bar{\lambda}_1^{\dot{c}}\bar{\partial}_{\dot{a}}[\hat{J}_1, \hat{J}_2]\Big\}. \quad (4.46)$$

Using (4.10), (4.11) this can be transformed to

$$\{(\bar{\mathfrak{S}}_0)_{\dot{a}}^B, (\bar{\mathfrak{S}}_+)_{\dot{b}}^A\}\hat{J} = -2\pi^2 \int d^4\eta' \, d\alpha \, \varepsilon_{\dot{b}\dot{c}}\eta_2^A\Big\{ - \bar{\lambda}_1^{\dot{c}}\eta'^B \cot\alpha \, [\bar{\partial}_{\dot{a}}\hat{J}_1, \hat{J}_2]$$

$$+ \bar{\lambda}_1^{\dot{c}}\eta'^B \tan\alpha \, [\hat{J}_1, \bar{\partial}_{\dot{a}}\hat{J}_2] + \delta_{\dot{a}}^{\dot{c}}\eta^B \sin\alpha \, [\hat{J}_1, \hat{J}_2]\Big\}. \quad (4.47)$$

The relevant term for the commutator $\{\bar{\mathfrak{S}}_{\dot{a}}^B, \bar{\mathfrak{S}}_{\dot{b}}^A\}$ is the sum

$$\{(\bar{\mathfrak{S}}_0)_{\dot{a}}^B, (\bar{\mathfrak{S}}_+)_{\dot{b}}^A\} + \{(\bar{\mathfrak{S}}_+)_{\dot{a}}^B, (\bar{\mathfrak{S}}_0)_{\dot{b}}^A\}. \quad (4.48)$$

We split the commutator into its symmetric and antisymmetric part

$$S_{\dot{a}\dot{b}}^{BA} = \{(\bar{\mathfrak{S}}_0)_{\dot{a}}^{(B}, (\bar{\mathfrak{S}}_+)_{\dot{b}}^{A)}\}, \qquad A^{BA} = \varepsilon^{\dot{a}\dot{b}}\{(\bar{\mathfrak{S}}_0)_{\dot{a}}^{[B}, (\bar{\mathfrak{S}}_+)_{\dot{b}}^{A]}\},$$

$$\{(\bar{\mathfrak{S}}_0)_{\dot{a}}^B, (\bar{\mathfrak{S}}_+)_{\dot{b}}^A\} + \{(\bar{\mathfrak{S}}_+)_{\dot{a}}^B, (\bar{\mathfrak{S}}_0)_{\dot{b}}^A\} = \varepsilon_{\dot{a}\dot{b}}A^{BA} + S_{\dot{a}\dot{b}}^{BA}, \quad (4.49)$$

where

$$X^{(AB)} = X^{AB} + X^{BA}, \qquad X^{[AB]} = X^{AB} - X^{BA}. \quad (4.50)$$

Expanding λ_1 and η_2 according to (3.12) and using antisymmetry under the shift

$$\alpha \mapsto \frac{\pi}{2} - \alpha \,, \qquad \eta' \mapsto -\eta' \,, \tag{4.51}$$

it is straightforward to show that the symmetric piece of (4.48) vanishes. The antisymmetric part reads

$$A^{BA}\hat{J} = \pi^2 \int d^4\eta' \, d\alpha \Big\{ \sin\alpha \, (\eta_2^A \eta'^B - \eta_2^B \eta'^A) \Big[\cot\alpha \, \bar{\lambda}^{\dot{c}}[\bar{\partial}_{\dot{c}}\hat{J}_1, \hat{J}_2] - \tan\alpha \, \bar{\lambda}^{\dot{c}}[\hat{J}_1, \bar{\partial}_{\dot{c}}\hat{J}_2] \Big] \\ - 2\sin\alpha \, (\eta_2^A \eta^B - \eta_2^B \eta^A)[\hat{J}_1, \hat{J}_2] \Big\} \,. \tag{4.52}$$

Expanding η_2 and using (anti-)symmetry of some parts of the integral under (4.51), this can be written as

$$A^{BA}\hat{J} = \pi^2 \int d^4\eta' \, d\alpha \Big\{ -\eta'^A \eta'^B \big(\cot\alpha \, \bar{\lambda}^{\dot{c}}[\bar{\partial}_{\dot{c}}\hat{J}_1, J_2] - \tan\alpha \, \bar{\lambda}^{\dot{c}}[\hat{J}_1, \bar{\partial}_{\dot{c}}\hat{J}_2] \big) \\ + (\eta'^A \eta^B - \eta'^B \eta^A)[\hat{J}_1, \hat{J}_2] \Big\} \,. \tag{4.53}$$

We can now use an analogue of (4.39) for $\eta' \neq 0$:

$$\frac{d\hat{J}_1}{d\alpha} = \cot\alpha(2\bar{\lambda}_1^{\dot{e}}\bar{\partial}_{1,\dot{e}} + 2\eta_1^E \partial_{1,E} - 2)\hat{J}_1 - \frac{1}{\sin\alpha\cos\alpha} \eta'^E \partial'_E \hat{J}_1,$$
$$\frac{d\hat{J}_2}{d\alpha} = -\tan\alpha(2\bar{\lambda}_2^{\dot{e}}\bar{\partial}_{2,\dot{e}} + 2\eta_2^E \partial_{2,E} - 2)\hat{J}_2 + \frac{1}{\sin\alpha\cos\alpha} \eta'^E \partial'_E \hat{J}_2. \tag{4.54}$$

Replacing $\bar{\lambda}^{\dot{c}}\bar{\partial}_{\dot{c}}$ in (4.53) by means of (4.54) and making use of the identities

$$\int d^4\eta' \, d\alpha \, \eta'^A \eta'^B \eta^E \big([\partial'_E \hat{J}_1, \hat{J}_2] + [\hat{J}_1, \partial'_E \hat{J}_2] \big) = -\int d^4\eta' \, d\alpha \, \eta'^{[A} \eta^{B]}[\hat{J}_1, \hat{J}_2], \tag{4.55}$$

$$\int d^4\eta' \, d\alpha \, \eta'^A \eta'^B \eta'^E \big([\partial_E \hat{J}_1, \hat{J}_2] - [\hat{J}_1, \partial_E \hat{J}_2] \big) = \int d^4\eta' \, d\alpha \, \eta'^A \eta'^B (\tan\alpha - \cot\alpha)[\hat{J}_1, \hat{J}_2], \tag{4.56}$$

we obtain

$$A^{BA}\hat{J} = -\pi^2 \int d^4\eta' \, d\alpha \, \frac{1}{2}\eta'^A \eta'^B \frac{d}{d\alpha}[\hat{J}_1, \hat{J}_2] = \pi^2 \int d^4\eta' \, \eta'^A \eta'^B [J(0, \eta'), J(\Lambda)], \tag{4.57}$$

which amounts to a gauge transformation, cf. (4.44).

We refrain from explicitly calculating $\{\bar{\mathfrak{S}}_+, \bar{\mathfrak{S}}_+\}$ since the result for $\{\bar{\mathfrak{S}}_{\dot{a}}^A, \bar{\mathfrak{S}}_{\dot{b}}^B\}$ can alternatively be obtained by conjugation of $\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\}$:

$$\{\bar{\mathfrak{S}}_{\dot{a}}^A, \bar{\mathfrak{S}}_{\dot{b}}^B\} = \varepsilon_{\dot{a}\dot{b}}\mathfrak{S}[\bar{\partial}^A \bar{\partial}^B \bar{J}(0)], \tag{4.58}$$

where $\bar{J}$ is a complex conjugate source field depending on conjugate odd variables $\bar{\eta}_A$. The latter are related to the original odd variables η^A through an odd Fourier transformation (cf. section 3.2)

$$\bar{J}(\bar{\Lambda}) = \int d^4\eta \, \exp(\eta^A \bar{\eta}_A) J(\Lambda). \tag{4.59}$$

Converting back to the original source J we obtain

$$\bar{\partial}^A \bar{\partial}^B \bar{J}(0) = \int d^4\eta\, \eta^A \eta^B J(0,\eta) = \frac{1}{2}\int d^4\eta\, \eta^A \eta^B \eta^C \eta^D \partial_C \partial_D J(0) = \frac{1}{2}\varepsilon^{ABCD}\partial_C \partial_D J(0) \tag{4.60}$$

and thus

$$\{\bar{\mathfrak{S}}_{\dot{a}}^A, \bar{\mathfrak{S}}_{\dot{b}}^B\} = \frac{1}{2}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_C \partial_D J(0)]. \tag{4.61}$$

Finally we should mention that the inclusion of negative-energy particles discussed in section 3.3 leads to additional gauge transformation terms.

4.6 Commutators involving $\mathfrak{P}$ and $\mathfrak{K}$

In order to evaluate the commutator of $\mathfrak{K}$ and $\mathfrak{Q}$ we can make use of $\mathfrak{K}_{a\dot{a}} = \frac{1}{4}\{\mathfrak{S}_{aB}, \bar{\mathfrak{S}}_{\dot{a}}^B\}$ and the Jacobi identity to find

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{Q}^{bA}\bigr] = -\frac{1}{4}\bigl[\{\bar{\mathfrak{S}}_{\dot{a}}^B, \mathfrak{Q}^{bA}\}, \mathfrak{S}_{aB}\bigr] - \frac{1}{4}\bigl[\{\mathfrak{Q}^{bA}, \mathfrak{S}_{aB}\}, \bar{\mathfrak{S}}_{\dot{a}}^B\bigr]. \tag{4.62}$$

The algebra of supercharges ensures that the first term vanishes and that the second term yields

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{Q}^{bA}\bigr] = -\frac{1}{4}\bigl[-\delta_a^b\mathfrak{R}^A{}_B + \frac{1}{2}\delta_a^b\delta_B^A\mathfrak{D}, \bar{\mathfrak{S}}_{\dot{a}}^B\bigr] = \delta_a^b\bar{\mathfrak{S}}_{\dot{a}}^A. \tag{4.63}$$

In other words this relation follows from consistency of the algebra and there is nothing to be shown concerning the corrections to $\mathfrak{K}$. The commutators of $\mathfrak{K}$ with $\bar{\mathfrak{Q}}$ can be derived analogously

$$\bigl[\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{Q}}_A^{\dot{b}}\bigr] = \delta_{\dot{a}}^{\dot{b}}\mathfrak{S}_{aA}. \tag{4.64}$$

Finally, the commutator of $\mathfrak{K}$ with $\mathfrak{P}$ follows expressing the latter in terms of $\mathfrak{Q}$ and $\bar{\mathfrak{Q}}$ and employing the Jacobi identity

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{P}^{b\dot{b}}\bigr] = \frac{1}{4}\bigl[\mathfrak{K}_{a\dot{a}}, \{\mathfrak{Q}^{bA}, \bar{\mathfrak{Q}}_A^{\dot{b}}\}\bigr] = \frac{1}{4}\bigl\{\mathfrak{Q}^{bA}, [\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{Q}}_A^{\dot{b}}]\bigr\} + \frac{1}{4}\bigl\{\bar{\mathfrak{Q}}_A^{\dot{b}}, [\mathfrak{K}_{a\dot{a}}, \mathfrak{Q}^{bA}]\bigr\}. \tag{4.65}$$

By means of the identities above this results in

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{P}^{b\dot{b}}\bigr] = \delta_{\dot{a}}^{\dot{b}}\mathfrak{L}^b{}_a + \delta_a^b\bar{\mathfrak{L}}^{\dot{b}}{}_{\dot{a}} + \delta_a^b\delta_{\dot{a}}^{\dot{b}}\mathfrak{D} \tag{4.66}$$

as expected.

For evaluating the commutator between $\mathfrak{K}$ and $\mathfrak{S}$, express $\mathfrak{K}$ in terms of $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ and use the Jacobi identity to find

$$\delta_A^C[\mathfrak{K}_{a\dot{a}}, \mathfrak{S}_{bB}] = \bigl[\{\mathfrak{S}_{aA}, \bar{\mathfrak{S}}_{\dot{a}}^C\}, \mathfrak{S}_{bB}\bigr] = -\bigl[\{\mathfrak{S}_{bB}, \mathfrak{S}_{aA}\}, \bar{\mathfrak{S}}_{\dot{a}}^C\bigr] - \bigl[\delta_B^C\mathfrak{K}_{b\dot{a}}, \mathfrak{S}_{aA}\bigr]. \tag{4.67}$$

By contracting once C with B and once C with A and taking a linear combination, we obtain

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{S}_{bA}\bigr] = \frac{1}{15}\bigl[\{\mathfrak{S}_{bB}, \mathfrak{S}_{aA}\} - 4\{\mathfrak{S}_{bA}, \mathfrak{S}_{aB}\}, \bar{\mathfrak{S}}_{\dot{a}}^B\bigr]. \tag{4.68}$$

Substituting the gauge transformation (4.45)

$$\bigl[\mathfrak{K}_{a\dot{a}}, \mathfrak{S}_{bA}\bigr] = \frac{1}{3}\varepsilon_{ab}\bigl[\mathfrak{G}[\partial_A \partial_B J(0)], \bar{\mathfrak{S}}_{\dot{a}}^B\bigr] = \varepsilon_{ab}\mathfrak{G}[\partial_A \bar{\partial}_{\dot{a}} J(0)]. \tag{4.69}$$

which amounts to a new gauge transformation $\mathfrak{G}[\partial_A \bar{\partial}_{\dot{a}} J(0)]$.

For the commutator between $\mathfrak{K}$ and $\bar{\mathfrak{S}}$ one finds in complete analogy with (4.67)

$$\delta_A^C[\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{S}}_{\dot{b}}^B] = \left[\{\mathfrak{S}_{aA}, \bar{\mathfrak{S}}_{\dot{a}}^C\}, \bar{\mathfrak{S}}_{\dot{b}}^B\right] = -\left[\{\bar{\mathfrak{S}}_{\dot{a}}^C, \bar{\mathfrak{S}}_{\dot{b}}^B\}, \mathfrak{S}_{aA}\right] - \left[\delta_A^B \mathfrak{K}_{a\dot{b}}, \bar{\mathfrak{S}}_{\dot{a}}^C\right]. \tag{4.70}$$

Again taking a linear combination of the two possible contractions of this equation and using (4.61), we obtain again a gauge transformation

$$\begin{aligned}
\left[\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{S}}_{\dot{b}}^A\right] &= \frac{1}{15}\left[\{\bar{\mathfrak{S}}_{\dot{a}}^A, \bar{\mathfrak{S}}_{\dot{b}}^B\} - 4\{\bar{\mathfrak{S}}_{\dot{a}}^B, \bar{\mathfrak{S}}_{\dot{b}}^A\}, \mathfrak{S}_{aB}\right] = \frac{1}{6}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}[\mathfrak{G}[\partial_C \partial_D J(0)], \mathfrak{S}_{aB}] \\
&= -\frac{1}{6}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_a \partial_B \partial_C \partial_D J(0)].
\end{aligned} \tag{4.71}$$

Finally, using the above results, we find that also $[\mathfrak{K}_{a\dot{a}}, \mathfrak{K}_{b\dot{b}}]$ amounts to a gauge transformation:

$$\begin{aligned}
\left[\mathfrak{K}_{a\dot{a}}, \mathfrak{K}_{b\dot{b}}\right] &= \frac{1}{4}\left[\mathfrak{K}_{a\dot{a}}, \{\mathfrak{S}_{bA}, \bar{\mathfrak{S}}_{\dot{b}}^A\}\right] = \frac{1}{4}\{\bar{\mathfrak{S}}_{\dot{b}}^A, [\mathfrak{K}_{a\dot{a}}, \mathfrak{S}_{bA}]\} + \frac{1}{4}\{\mathfrak{S}_{bA}, [\bar{\mathfrak{S}}_{\dot{b}}^A, \mathfrak{K}_{a\dot{a}}]\} \\
&= \frac{1}{4}\varepsilon_{ab}\{\bar{\mathfrak{S}}_{\dot{b}}^A, \mathfrak{G}[\partial_A \bar{\partial}_{\dot{a}} J(0)]\} + \frac{1}{24}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\{\mathfrak{S}_{bA}, \mathfrak{G}[\partial_a \partial_B \partial_C \partial_D J(0)]\} \\
&= \varepsilon_{ab}\mathfrak{G}[\bar{\partial}_{\dot{a}} \bar{\partial}_{\dot{b}} J(0)] + \frac{1}{24}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_a \partial_b \partial_A \partial_B \partial_C \partial_D J(0)]
\end{aligned} \tag{4.72}$$

To conclude we summarise the algebra relations closing onto gauge transformations (4.44)

$$\begin{aligned}
\{\mathfrak{S}_{aA}, \mathfrak{S}_{bB}\} &= \varepsilon_{ab}\mathfrak{G}[\partial_A \partial_B J(0)], \\
\{\bar{\mathfrak{S}}_{\dot{a}}^A, \bar{\mathfrak{S}}_{\dot{b}}^B\} &= \frac{1}{2}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_C \partial_D J(0)], \\
\left[\mathfrak{K}_{a\dot{a}}, \mathfrak{S}_{bA}\right] &= \varepsilon_{ab}\mathfrak{G}[\partial_A \bar{\partial}_{\dot{a}} J(0)], \\
\left[\mathfrak{K}_{a\dot{a}}, \bar{\mathfrak{S}}_{\dot{b}}^A\right] &= -\frac{1}{6}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_a \partial_B \partial_C \partial_D J(0)], \\
\left[\mathfrak{K}_{a\dot{a}}, \mathfrak{K}_{b\dot{b}}\right] &= \varepsilon_{ab}\mathfrak{G}[\bar{\partial}_{\dot{a}} \bar{\partial}_{\dot{b}} J(0)] + \frac{1}{24}\varepsilon_{\dot{a}\dot{b}}\varepsilon^{ABCD}\mathfrak{G}[\partial_a \partial_b \partial_A \partial_B \partial_C \partial_D J(0)].
\end{aligned} \tag{4.73}$$

The commutators of $\mathfrak{P}$ with the supercharges follow analogously to the above commutators with $\mathfrak{K}$. Note that momentum conservation is not quite sufficient to show the correct closure of these commutators.

5   Exact superconformal invariance

We would now like to extend the previous considerations, section 3.1 and section 3.2, to the case of general tree amplitudes. We expect to find the obvious generalisation

$$\mathfrak{J}_0 A_{n,k} + \mathfrak{J}_+ A_{n-1,k} + \mathfrak{J}_- A_{n-1,k-1} + \mathfrak{J}_{+-} A_{n-2,k-1} = 0. \tag{5.1}$$

This gives rise to the pattern of relations shown in figure 8 whereby a given amplitude is related to higher point amplitudes by the action of the deformed generators. As we have seen explicitly in the cases of MHV and $\overline{\text{MHV}}$ amplitudes, the anomalous terms arise from
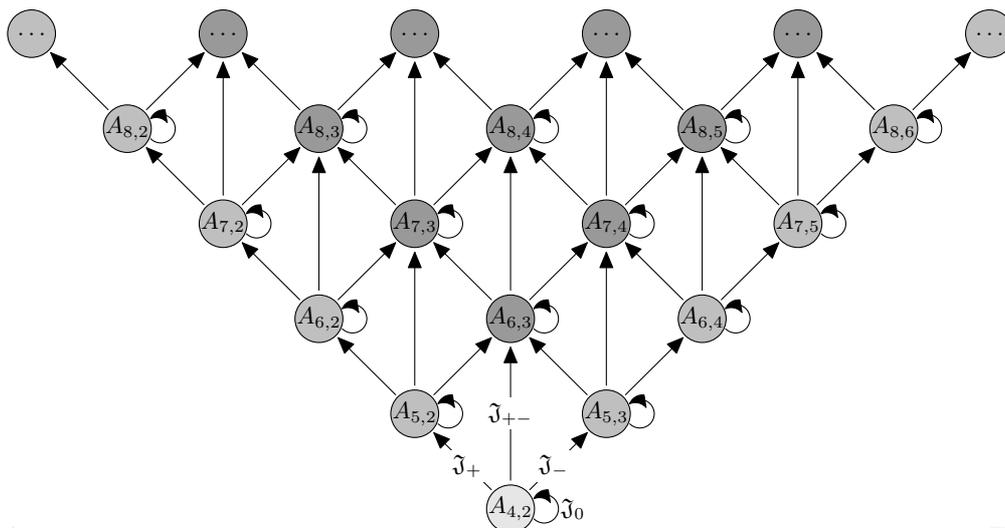
Figure 8. Illustration of the recursive action of the deformed generators.

collinear singularities seen by $\mathfrak{J}_0$ which are then removed by $\mathfrak{J}_+$ or $\mathfrak{J}_-$ as appropriate. In fact it is well known that the collinear behaviour is governed by the universal splitting functions and so we expect that the action of the deformed generators is easily extended to the most general case. There are in principle contributions from other kinematic singularities which would need to be considered however none of these turn out to be relevant for the action of the generators. We start our discussion with the concrete example of the six-point NMHV amplitude which as we will see has, in addition to the collinear singularities, multi-particle poles as well as apparent "spurious" (non-adjacent) singularities which are non-physical and merely due to the methods for deriving the expressions.

5.1 Six-point NMHV amplitudes

For the case, $A_{6,3} = A_6^{\mathrm{NMHV}}$, that is to say, of six-point NMHV amplitudes, we expect that the action of $\mathfrak{S}$ on the amplitude should be given by,

$$\mathfrak{S}_0 A_6^{\mathrm{NMHV}} + \mathfrak{S}_- A_5^{\mathrm{MHV}} = 0 \,, \tag{5.2}$$

where we note that the generator relates the six-point NMHV amplitude to the five-point MHV. We follow [39] (see also appendix A for relevant definitions) and write the six point NMHV amplitude as

$$A_6^{\mathrm{NMHV}} = A_6^{\mathrm{MHV}} \left(\frac{1}{2} R_{146} + \mathrm{cyclic} \right) \tag{5.3}$$

where there are several representations of R_{146}. One that is particularly useful is

$$R_{146} = c_{146} \delta^4 (\Xi_{146}) \tag{5.4}$$

where

$$
\begin{aligned}
c_{146} &= \frac{\langle 34 \rangle \langle 56 \rangle}{x_{14}^2 \langle 1 | x_{14} | 4] \langle 3 | x_{36} | 6] (\langle 45 \rangle \langle 61 \rangle)^3 [45][56]} \,, \\
\Xi_{146}^A &= \langle 61 \rangle \langle 45 \rangle (\eta_4^A [56] + \eta_5^A [64] + \eta_6^A [45]) \,,
\end{aligned}
\tag{5.5}
$$

which is a specific example (after a little manipulation) of the general formula

$$R_{pqr} = c_{pqr}\delta^4(\Xi_{pqr}),$$

$$c_{pqr} = \frac{\langle q-1,q\rangle\langle r-1,r\rangle}{x_{qr}^2\langle p|x_{pr}x_{rq-1}|q-1\rangle\langle p|x_{pr}x_{rq}|q\rangle\langle p|x_{pq}x_{qr-1}|r-1\rangle\langle p|x_{pq}x_{qr}|r\rangle},$$

$$\Xi_{pqr}^A = -\langle p|\left[x_{pq}x_{qr}\sum_{i=p}^{r-1}|i\rangle\eta_i^A + x_{pr}x_{rq}\sum_{i=p}^{q-1}|i\rangle\eta_i^A\right]. \tag{5.6}$$

Now we want to consider the action of $\mathfrak{S}$ on this amplitude and specifically the anomaly contribution coming from the action of ∂ on $1/\bar{\lambda}$ terms in the R_{pqr} terms. As always one can use cyclicity to consider a specific leg, for concreteness we consider the $\bar{\lambda}_6$ terms. There are several different possible contributions to the anomaly terms:

1. from multi-particle singularities which occur when linear combinations of momenta such as $(p_4 + p_5 + p_6)$ become null. These singularities are of the form $\sum\langle jk\rangle[jk]$ and so do not contribute to the anomaly.[13]

2. from singularities of the form $\langle 3|x_{46}|6]$ which occur when $p_4 + p_5$ is any linear combination of p_3 and p_6. In fact these singularities are spurious and cancel when we consider the full amplitude as can be explicitly seen in e.g. [20, 22, 54, 55]. For a recent discussion of these singularities in the twistor space approach see [11].

3. collinear singularities due to [56] type terms.

It is this last class that actually gives rise to the relevant physical singularities generating the anomaly terms and that we will consider. For completeness the full R terms are

$$\frac{1}{2}(R_{146} + R_{251} + R_{362}) = \frac{1}{2}\left[\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle}{x_{14}^2\langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]}\delta^4\left(\eta_4[56] + \eta_5[64] + \eta_6[45]\right)\right.$$

$$+ \frac{\langle 45\rangle\langle 61\rangle\langle 12\rangle\langle 56\rangle}{x_{25}^2\langle 2|x_{25}|5]\langle 4|x_{42}|1][56][61]}\delta^4\left(\eta_5[61] + \eta_6[15] + \eta_1[56]\right)$$

$$\left. + \frac{\langle 56\rangle\langle 12\rangle\langle 23\rangle\langle 61\rangle}{x_{36}^2\langle 3|x_{36}|6]\langle 5|x_{53}|2][61][12]}\delta^4\left(\eta_6[12] + \eta_1[26] + \eta_2[61]\right)\right] \tag{5.7}$$

and the anomaly term from the [61] denominator factors in the second and third lines, and from the [56] terms in the first and second lines give

$$(\mathfrak{S}_0)_{aB}A_6^{\text{NMHV}} = \frac{\pi}{2}\int\prod_{k=1}^6 d^{4|4}\Lambda_k\,\text{Tr}([J_6,\partial_{1B}J_1]J_2J_3J_4J_5)\frac{\delta^4(P_6)\delta^8(Q_6)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\delta^2(\langle 61\rangle)\varepsilon_{ab}\lambda_6^b$$

$$\left[\frac{\langle 45\rangle\langle 56\rangle\langle 12\rangle}{x_{25}^2\langle 2|x_{35}|5]\langle 4|x_{42}|1][56]}\delta^4(\eta_6[15] + \eta_1[56])\right.$$

$$\left. + \frac{\langle 56\rangle\langle 12\rangle\langle 23\rangle}{x_{36}^2\langle 3|x_{46}|6]\langle 5|x_{53}|2][12]}\delta^4(\eta_6[12] + \eta_1[26])\right]. \tag{5.8}$$

---

[13] At tree level it is safe to assume a principal part prescription for propagators and hence there are no further subtleties.

Using manipulations identical to previous sections this can be rewritten as

$$(\mathfrak{S}_0)_{aB} A_6^{\text{NMHV}} = 2\pi^2 \int \prod_{k=2}^{5} d^{4|4}\Lambda_k d^{4|4}\Lambda_1' d\alpha d^4\eta' \; \frac{\delta^4(P_5')\delta^8(Q_5')}{\langle 1'2\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51'\rangle} \delta^4(\eta')\varepsilon_{ab}\lambda_6^b$$
$$\times \; \text{Tr}([\hat{J}_6, \partial_{1,B}J_1]J_2 J_3 J_4 J_5) \,, \tag{5.9}$$

where we have evaluated the $\delta^2(\langle 16\rangle)$ and made use of the definitions $\lambda_1 = \lambda_1' \sin\alpha$, $\lambda_6 = \lambda_1' \cos\alpha$, $\eta_6 = \eta_1' \sin\alpha + \eta' \cos\alpha$, $\eta_1 = \eta_1' \cos\alpha - \eta' \sin\alpha$. This is consistent with $\mathfrak{S}_- \mathcal{A}_5^{\text{MHV}}$ using the expression (3.24) calculated from the action of $\mathfrak{S}$ on $\overline{\text{MHV}}$ amplitudes and thus we see that (5.2) does indeed hold. We now calculate the action of the undeformed generator $\bar{\mathfrak{S}}$ on the six-point NMHV amplitude. In this case we expect to find that

$$\bar{\mathfrak{S}}_0 A_6^{\text{NMHV}} + \bar{\mathfrak{S}}_+ A_5^{\text{NMHV}} = 0 \,. \tag{5.10}$$

It is convenient to choose a slightly different writing of the six-point amplitude using the formula (5.6)

$$A_n^{\text{NMHV}} = A_n^{\text{MHV}} \sum_{2 \leq s,t \leq n-1} R_{nst} \,, \tag{5.11}$$

where we sum over all s and t such that $s \neq t+1$ mod n. For the specific case of six-points we take

$$A_n^{\text{NMHV}} = A_n^{\text{MHV}}(R_{624} + R_{625} + R_{635}) \tag{5.12}$$

and look for anomalous terms arising from the action of $\bar{\partial}$ on inverse powers of λ. We use cyclic symmetry to consider only λ_6 and as in the previous case there are several possible sources for anomalous contributions, however, and again as in the previous discussion only those singularities arising from collinear singularities are relevant. Noting that $R_{624} \sim \langle 61\rangle$, $R_{625} \sim \langle 65\rangle\langle 61\rangle$ and $R_{635} \sim \langle 65\rangle$ we see that the only contribution from the singularity at $\lambda_6 \propto \lambda_5$ comes from the R_{624} term and similarly the only contribution from the $\lambda_6 \propto \lambda_1$ singularity comes from the R_{635} term. Thus we find,

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^A A_6^{\text{NMHV}} = -\pi \int \prod_{k=1}^{6} d^{4|4}\Lambda_k \, \text{Tr}(J_1 \dots J_6)\delta^4(P)\delta^8(Q)\eta_6^A$$
$$\times \left(\delta^2(\langle 56\rangle) \frac{\varepsilon_{\dot{a}\dot{b}}\bar{\lambda}_5^{\dot{b}}R_{624}}{\langle 61\rangle\langle 12\rangle \dots \langle 45\rangle} - \delta^2(\langle 16\rangle) \frac{\varepsilon_{\dot{a}\dot{b}}\bar{\lambda}_1^{\dot{b}}R_{635}}{\langle 12\rangle \dots \langle 56\rangle} \right). \tag{5.13}$$

Evaluating the delta functions, using $\delta^8(Q')R_{1'24} = \delta^8(Q')R_{1'35}$, relabelling the momenta and removing the phases we end up with

$$(\bar{\mathfrak{S}}_0)_{\dot{a}}^A \mathcal{A}_6^{\text{NMHV}} = -2\pi^2 \int \prod_{k=2}^{5} d^{4|4}\Lambda_k d^{4|4}\Lambda_1' d\alpha d^4\eta' \, \text{Tr}([\hat{J}_6, J_1] \dots J_5)$$
$$\times \delta^4(P')\delta^8(Q')\eta_1^A \left(\frac{\varepsilon_{\dot{a}\dot{b}}\bar{\lambda}_6^{\dot{b}}R_{1'24}}{\langle 1'2\rangle \dots \langle 51'\rangle} \right) \tag{5.14}$$

which is again consistent with the previous expressions for $\bar{\mathfrak{S}}_+$.

5.2 General tree amplitudes and splitting functions

It is useful to analyse the necessary behaviour of generic amplitudes so that our above results of section 5.1 generalise. As discussed, the important behaviour occurs when two particles become collinear. For concreteness we consider the case where particle n becomes collinear with particle 1 with the scaling

$$\lambda_n \to e^{i\varphi}\lambda_1' \sin\alpha, \qquad \lambda_1 \to \lambda_1' \cos\alpha \tag{5.15}$$

and the redefinitions

$$\eta_n = e^{-i\varphi}\eta_1' \sin\alpha + \eta'\cos\alpha, \qquad \eta_1 = \eta_1'\cos\alpha - e^{i\varphi}\eta'\sin\alpha. \tag{5.16}$$

We postulate that a generic amplitude scales as

$$
\begin{aligned}
A_{n,k}(\Lambda_1,\ldots,\Lambda_n)\,|_{1\|n} \simeq\ & \frac{e^{-i\varphi}\sec\alpha\csc\alpha}{\langle n1\rangle}\, A_{n-1,k}(\Lambda_1',\Lambda_2,\ldots,\Lambda_{n-1}) \\
& + \frac{e^{i\varphi}\sec\alpha\csc\alpha}{[n1]}\,\delta^4(\eta')A_{n-1,k-1}(\Lambda_1',\Lambda_2,\ldots,\Lambda_{n-1}) \\
& + \text{finite terms},
\end{aligned}
\tag{5.17}
$$

and with similar scaling in all other collinear limits. Particular collinear limits of superspace amplitudes were analysed in [18] using the BCFW recursion relations described below. Now assuming that the anomaly only receives contributions from the collinear singularities and that they scale as above it is straightforward to show that

$$
\begin{aligned}
(\bar{\mathfrak{S}}_0)_{\dot a}^A \mathcal{A}[J] = -2\pi^2 \int \sum_n \prod_{k=2}^{n-1} d^{4|4}\Lambda_k\, d^{4|4}\Lambda_1'\, d\alpha\, d^4\eta'\, \varepsilon_{\dot a\dot b}\bar\lambda_n^{\dot b}\eta_1^A A_{n-1,k}(\Lambda_1',\ldots,\Lambda_{n-1}) \\
\times \operatorname{Tr}([\hat J_n, J_1]J_2\ldots J_{n-1})
\end{aligned}
\tag{5.18}
$$

and similarly

$$
\begin{aligned}
(\mathfrak{S}_0)_{aA} \mathcal{A}[J] = 2\pi^2 \int \sum_n \prod_{k=2}^{n-1} d^{4|4}\Lambda_k\, d^{4|4}\Lambda_1'\, d\alpha\, d^4\eta'\, \varepsilon_{ab}\lambda_n^b\delta^4(\eta')A_{n-1,k-1}(\Lambda_1',\ldots,\Lambda_{n-1}) \\
\times \operatorname{Tr}([\hat J_n, \partial_{1,A} J_1]J_2\ldots J_{n-1})
\end{aligned}
\tag{5.19}
$$

which are both consistent with the expressions from the previous sections (as before we have removed the phases so that $\lambda_n = \lambda_1'\sin\alpha$, $\eta_n = \eta_1'\sin\alpha + \eta'\cos\alpha$ and by passing to the projection $\hat J_n$).

As previously mentioned, a convenient way to study arbitrary tree level amplitudes is to make use of the BCFW recursion relations [13] and for our purposes the superspace versions [14–17] are particularly useful. To verify the above collinear structure (5.17), one does not need to explicitly solve the recursion relations, as in [18], but can simply make use of an inductive argument with the initial step being provided by the MHV and $\overline{\text{MHV}}$ amplitudes considered previously. In the derivation of the BCFW relations one performs complex shifts of two of the external legs, say j and k, and studies the resulting singularities.
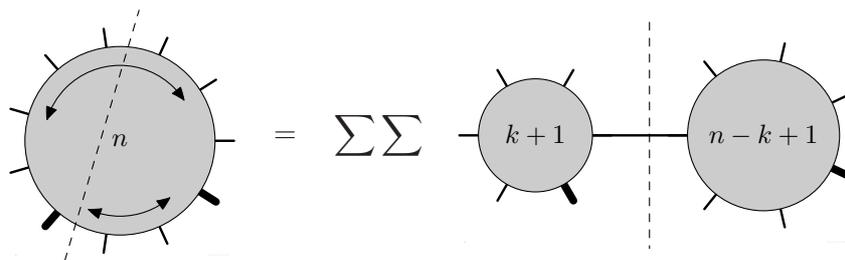
Figure 9. Schematic of the on-shell recursion relation for a general tree-level amplitude. The shifted momenta are denoted by thickened legs and the sum is over the product of subamplitudes which split the shifted legs.
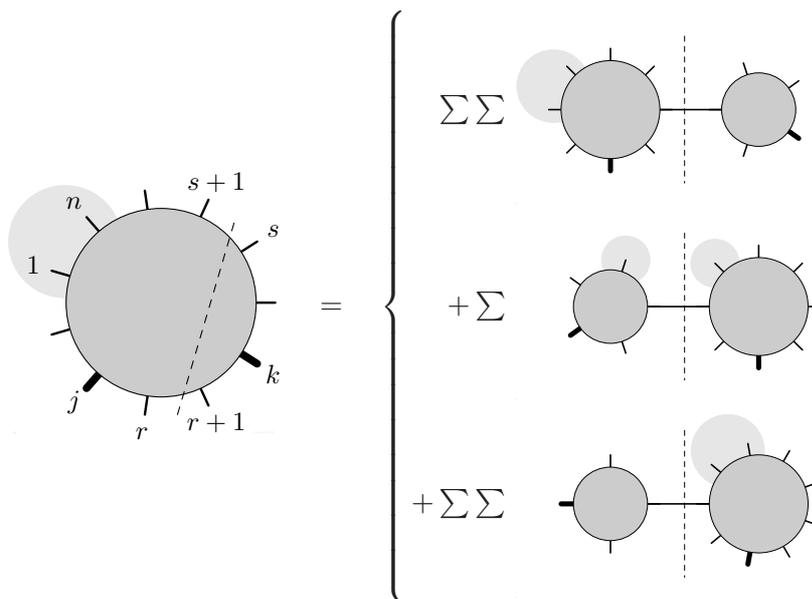
Figure 10. Illustration of the three positions of collinear legs in the recursion relations for the collinear legs chosen to be different than the shifted legs.

The resulting poles relate the amplitude to the sum over products of subamplitudes with the momenta suitably shifted, shown schematically in figure 9. Following the usual procedure we shift $\hat{\tilde{\lambda}}_j = \tilde{\lambda}_j + z_{rs}\tilde{\lambda}_k$, and $\hat{\lambda}_k = \lambda_k - z_{rs}\lambda_j$; In the superspace version we also shift the Graßmann variables so that $\hat{\eta}_j = \eta_j + z_{rs}\eta_k$. The resulting recursion relation can be written as

$$A_n(\Lambda_1, \ldots, \Lambda_n) = \int d^4 P_{rs} \int d^4 \eta_{rs} \sum_{r,s} A_L \frac{1}{P_{rs}^2} A_R. \tag{5.20}$$

with[14]

$$A_{\mathrm{L}} = A_{n-(s-r)+1}(\ldots, \hat{\Lambda}_j, \ldots, \Lambda_r, \bar{\Lambda}_{rs}, \Lambda_{s+1}),$$
$$A_{\mathrm{R}} = A_{s-r+1}(\Lambda_s, \Lambda_{rs}, \Lambda_{r+1}, \ldots, \hat{\Lambda}_k, \ldots). \tag{5.21}$$

---

[14]For this formula to be valid we must include in the sum three-point functions which are non-vanishing for complex momenta. We must thus extend our earlier definition of A_n to be $A_3 = A_{3,1} + A_{3,2}$ for this special case.

The shift parameter z_{rs} is determined by demanding the subamplitudes in each term to be on-shell. This is ensured to be the case if

$$(\hat{P}_{rs})^2 = \left(\sum_{\ell=r+1}^{s} \lambda_\ell \tilde{\lambda}_\ell - z_{rs} \lambda_j \tilde{\lambda}_k \right)^2 = 0. \tag{5.22}$$

We now want to consider the resulting behaviour as two legs become collinear and to show that if all n-point amplitudes have the required behaviour, all the $(n+1)$-point amplitudes will too. This is simply a rewriting of the known universality of the splitting functions governing the collinear limit, [55, 56], to the superspace notation via the BCFW recursion relations. In fact if the legs becoming collinear, again let us choose n and 1, are different than the shifted legs j and k, it is easy to see that the recursion relation (5.20) guarantees that this will be the case. There are three separate cases, as shown in figure 10; when both collinear legs are on the left hand subamplitude A_L which has the correct scaling by assumption, secondly when the collinear legs are on different subamplitudes so there are no singularities and this term is subleading, finally when the two collinear legs are on the right hand subamplitude A_R which again by assumption has the correct scaling.

6 Conclusions and outlook

In this paper we have considered superconformal invariance of scattering amplitudes in $\mathcal{N} = 4$ SYM at tree level. As the model is exactly superconformal, classically as well as quantum mechanically, observables ought to respect this symmetry. However, scattering amplitudes display collinear singularities which obscure the symmetries: At loop level they cause IR divergences which superficially break conformal symmetry. Further scrutiny reveals that collinear singularities even break naive conformal symmetry at tree level. This breakdown is easily overlooked because it only happens for singular configurations of the external momenta. In order to understand the symmetries of scattering amplitudes at loop level, it is crucial to first obtain complete understanding at tree level.

Here we have proposed to deform the *free* superconformal generators $\mathfrak{J}_0$ to *classical interacting* generators, cf. figure 7,

$$\mathfrak{J} = \mathfrak{J}_0 + \mathfrak{J}_+ + \mathfrak{J}_- + \mathfrak{J}_{+-}. \tag{6.1}$$

The correction terms cure the breaking of superconformal symmetry at collinear singularities. They are what is known as non-linear realisations of symmetry; as operators they act linearly, but they transform one field into several fields. For scattering amplitudes it means that anomalous terms in the action of the free generators are compensated by the interacting generators acting on amplitudes with *fewer legs*.

We should note that the structure of singularities in tree level scattering amplitudes is well understood. In general they correspond to internal propagators going on shell meaning that the overall momentum of a subset of the external particles becomes light-like. They can be classified into two-particle and multi-particle singularities: Multi-particle singularities are codimension-one and do not lead to a conformal anomaly. Conversely, two-particle

singularities in Minkowski signature require the particles to be collinear. Collinearity is a codimension-two momentum configuration which leads to the conformal anomaly. Collinear singularities can be expressed through splitting functions times an amplitude with one leg less. The conformal properties of splitting functions are understood. It is also known how certain soft momentum limits of the amplitudes are related to conformal symmetry. Arguably our proposal constitutes a reformulation of what has been known about conformal symmetry for a long time. In fact, the correction terms in (6.1) can be understood as the action of the free conformal generators on the splitting functions (5.17). Nevertheless we believe that it is a useful formalisation of classical conformal symmetry in view of extensions to the loop level.

Importantly we have shown that the deformations form a proper representation of $\mathfrak{psu}(2,2|4)$ superconformal symmetry. Actually, the algebra does not close exactly but only modulo field-dependent gauge transformations. This behaviour is not unexpected, it is rather very common in gauge field theories. Here only the commutators of special superconformal generators $\mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{K}$ yield gauge transformations. In a way this appears to be the dual of the very non-linear terms in classical interacting gauge covariant supersymmetry transformations, $\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{P}$. The latter act on the fields while our representation acts on the dual sources noting that an algebra automorphism maps between $\mathfrak{S}, \bar{\mathfrak{S}}, \mathfrak{K}$ and $\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{P}$.

An important insight is that conformal invariance not only constrains the functional form of the amplitudes, but it also constrains their singularities. In particular, invariance of the singularities requires cancellations between amplitudes with different numbers of legs, cf. figure 8. Hence, it does not make sense to consider an amplitude with a fixed number of legs on its own, but only all amplitudes at the same time, e.g. in the form of a generating functional (2.14). Therefore symmetry considerations can to some extent replace field theory computations which may become a very beneficial feature at higher loops.

Symmetries become even more powerful in the planar limit where the superconformal algebra apparently extends to an infinite-dimensional Yangian algebra. Yangian symmetry leads to further constraints which prohibit certain superconformal invariants. In fact, only a few invariants (up to anomalies) of the free Yangian are known [15, 39, 43]. The tree level amplitude can be written as a linear combination of these, but the coefficients are undetermined by symmetry. Although we have not shown this explicitly, we are confident that full *classical* Yangian symmetry (see [57] for interacting Yangians)

$$\widehat{\mathfrak{J}}_\alpha = \frac{1}{2} f_\alpha^{\beta\gamma} \sum_{1 \le k < \ell \le n} \mathfrak{J}_{k,\beta} \mathfrak{J}_{\ell,\gamma}, \tag{6.2}$$

where $\mathfrak{J}_{k,\beta}$ are the *classical* generators in (6.1), leads to a *unique* invariant which is precisely the tree scattering amplitude. The point is that the naive invariants of the free Yangian have spurious singularities which are due to some decomposition of the amplitude into partial fractions. Physicality requirements can be used to argue for the right linear combination. Our approach is different in that we merely rely on symmetry: Spurious singularities are seen by the free generators, but they are not cancelled by any interaction terms. Hence they should cancel among themselves leaving only the correct physical singularities. In fact, unique determination of the tree level amplitude is an essential pre-

requisite for complete algebraic determination of loop amplitudes: Tree-level invariants form the space of homogeneous solutions to the covariance equations at loop level, i.e. they can be added freely to loop amplitudes with arbitrary coefficients. If there is only a single invariant, it must be the physical tree-level amplitude. Adding it to the loop amplitude can be absorbed by changing the overall prefactor and redefining the coupling constant, both of which cannot be determined by algebraic means in any case. If there are multiple invariants, only one of them can be identified with the tree-level amplitude and thus the loop amplitude cannot be determined algebraically.

Note that we can easily argue for complete Yangian invariance of the tree scattering amplitude. According to (6.2) the level-one momentum generator $\widehat{\mathfrak{P}}$ (also known as the special dual conformal generator) relies only on the superconformal generators $\mathfrak{P}, \mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{L}, \bar{\mathfrak{L}}, \mathfrak{D}$. All of these are free from holomorphic anomalies and receive no classical corrections, thus $\widehat{\mathfrak{P}}$ equals its free representation for which invariance was shown in [15, 39, 43]. All the other Yangian generators are obtained from commutators with superconformal generators. Note that for completeness one should prove that (6.2) satisfies the Serre relations of the Yangian algebra. This would show that the closure of the algebra generated by (6.1), (6.2) is indeed a Yangian and not some other infinite-dimensional algebra.

Again, our interacting representation of the Yangian at tree level does not add much to what is known already. It would demonstrate its full power only when quantum corrections are included: If there is a unique invariant at tree level, we expect the same to hold true at loop level. This would imply a complete determination of scattering amplitudes in planar $\mathcal{N} = 4$ SYM at all loops. The price to be paid is the determination of corrections to the Yangian generators. This may or may not be simpler than determining the amplitude itself. Yet the formulation as a symmetry could ultimately enable certain non-perturbative statements, e.g. on the structure of singularities.

The possibility of a unique Yangian invariant scattering amplitude is also exciting for the spin chain point of view. When considered as a spin chain state, the scattering amplitude would be a representation of the unit operator of the quantum mechanical spin chain model. A Bethe ansatz based on this vacuum state could lead to a derivation of the exact spectrum of planar anomalous dimensions alternative to the proposal in [58] and follow-up works.

There are several issues deserving further investigation:

We did not consider conformal inversions in our work. These can be used to define conformal boosts as shifts conjugated by conformal inversions. Free shifts do not receive classical corrections, consequently conformal inversions should carry those corrections necessary for conformal boosts in this picture. It is however not a priori guaranteed that conformal inversions are exact symmetries. Are scattering amplitudes invariant under the superconformal group including inversions or merely under the component connected to the identity?

It would be desirable to prove that classical Yangian symmetry determined the tree scattering amplitude uniquely. Can one show that there is only a single invariant?

The proposed corrections to superconformal symmetry are based on the holomorphic anomaly which requires a spacetime with $(3, 1)$ Minkowski signature. Many works on tree

level scattering amplitudes make use of a twistor transform which can is most conveniently defined in $(2, 2)$ signature. It would be interesting to find out whether our results can also be formulated for this split signature. Clearly, the holomorphic anomaly would have to be replaced by something else. One could contemplate postulating the equivalent of (3.5). Alternatively one could try to find different anomalous terms in the action of the free generators. In the spinor helicity framework it is not immediately clear how to define such terms but in the twistor space representation the various signum factors [9, 10] do give rise to singular contributions when two spinors become collinear. Cancellations then might involve also three-leg and two-leg amplitudes in this signature. Moreover, we expect that $N^{-1}MHV$ amplitudes would play a role; like the three-leg amplitudes these have a restricted support in momentum space.

The interacting representation of superconformal symmetry does not rely on the planar limit or on integrability and therefore one may wonder if similar formulations can be obtained for field theories with less supersymmetry. In particular, all tree scattering amplitudes in pure $\mathcal{N} < 4$ supersymmetric gauge theories (including pure Yang-Mills at $\mathcal{N} = 0$) equal the restriction of the $\mathcal{N} = 4$ counterparts. Also the truncation of the classical $\mathfrak{psu}(2, 2|4)$ representation to $\mathfrak{su}(2, 2|\mathcal{N})$ is consistent. It is a proper representation that annihilates all truncated amplitudes. This appears to work independently of the conformal anomaly at one loop due to a non-trivial beta-function. It is however not immediately clear whether one can add massless matter to $\mathcal{N} < 4$ field theories and still obtain a proper representation of conformal symmetry which annihilates all tree amplitudes.

Finally, we would like to mention the possibility of establishing a similar framework for $\mathcal{N} = 8$ supergravity. In this model the $E_{7(7)}$ global symmetry has features reminiscent of the special conformal symmetries including relations between amplitudes with different numbers of legs, the behaviour in collinear and soft limits (see e.g. [16]) as well as the structure of generators and their algebra, (see e.g. [59]).

Acknowledgments

A   Conventions

- We will mostly consider the $(3, 1)$ signature $(-+++)$. The positive and negative chirality spinors are denoted by λ^a, $a = 1, 2$ and $\tilde{\lambda}^{\dot{a}}$, $\dot{a} = 1, 2$.

- We have for the antisymmetric two tensor: $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\varepsilon^{21} = -\varepsilon^{12} = 1$ so that $\varepsilon^{ab}\varepsilon_{bc} = \delta^a_c$. The antisymmetric four tensor ε^{ABCD} is defined such that $\varepsilon^{1234} = \varepsilon_{1234} = +1$.

- We define the positive chirality spinor brackets $\langle \lambda_1, \lambda_2 \rangle = \varepsilon_{ab}\lambda_1^a\lambda_2^b$ and the negative chirality brackets $[\tilde{\lambda}_1, \tilde{\lambda}_2] = \varepsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$. The same conventions apply for the abbreviations $\langle ij \rangle$ and $[ij]$.

- A four-vector p_μ can be thought of as a bi-spinor $p^{a\dot{a}} = (\sigma^\mu)^{a\dot{a}}p_\mu$ which for light-like vectors can be written as $p^{a\dot{a}} = \lambda^a\tilde{\lambda}^{\dot{a}}$ for some spinors λ, $\tilde{\lambda}$. In $(3,1)$ signature demanding that p_μ be real implies that $\tilde{\lambda} = \pm\bar{\lambda}$. When p_μ is a particle four-momentum, the sign corresponds to positive and negative energy.

- It is useful to introduce the dual variables $(x_i)^{a\dot{a}}$, $i = 1, \ldots, n$ defined by $x_i - x_{i+1} = p_i$ satisfying the condition $x_{n+1} = x_1$. We make use of the shorthand $x_{rs} = x_r - x_s = \sum_{i=r}^{s-1} p_i$. As well as

$$\langle p|x_{mn}|q] = \lambda_p^a(x_{mn})_{a\dot{a}}\tilde{\lambda}_q^{\dot{a}}$$
$$\langle p|x_{mn}x_{kl}|q\rangle = \lambda_p^a(x_{mn})_{a\dot{a}}(x_{kl})^{\dot{a}b}\varepsilon_{bc}\lambda_q^c \tag{A.1}$$

- For treating complex variables the convention for the measure is $d^2z = dxdy$ where $z = x + iy$. We define derivatives ∂ and $\bar{\partial}$ so that $\partial z = 1$ and $\bar{\partial}z = 0$ etc. We also define

$$\int d^2z \; \delta^2(z) = 1 \tag{A.2}$$

so that $\delta^2(z) = \delta(x)\delta(y)$. This implies for the holomorphic anomaly that[15]

$$\frac{\partial}{\partial\bar{z}}\frac{1}{z} = \pi\delta^2(z). \tag{A.3}$$

In other words, $1/z$ is the Green's function for the differential operator $\partial/\partial\bar{z}$. This can be easily seen, and the overall coefficient fixed, by making use of Green's theorem

$$\int_{\mathcal{R}} d^2z \; \frac{\partial}{\partial\bar{z}}\frac{1}{z} = -\frac{i}{2}\oint_{\partial\mathcal{R}} dz \; \frac{1}{z}\,. \tag{A.4}$$

- We assume that we are in $(3,1)$ signature and we treat the λ^a's as complex variables so that $d^4\lambda = d^2\lambda^1 \, d^2\lambda^2$. In particular it is defined so that

$$\delta^2(\langle\lambda,\mu\rangle) = \int d^2z \, \delta^4(\lambda - z\mu) \tag{A.5}$$

and

$$\int d^4\lambda \, \delta^2(\langle\lambda,\mu\rangle)f(\lambda,\bar{\lambda}) = \int d^4\lambda \, d^2z \, \delta^4(\lambda - z\mu)f(\lambda,\bar{\lambda}) = \int d^2z \, f(z\mu,\bar{z}\bar{\mu}). \tag{A.6}$$

- Graßmann integration is defined as $\int d\eta = 0$ and $\int d\eta \, \eta = 1$. The odd delta function is consequently defined as $\delta(\eta) = \eta$. Integral over all four η^A's is defined as $d^4\eta = d\eta^1 \, d\eta^2 \, d\eta^3 \, d\eta^4$ and the odd delta function such that $\int d^4\eta \, \delta^4(\eta) = 1$.

- The superspace integration measure, $d^{4|4}\Lambda$, is defined to be $d^{4|4}\Lambda = d^4\lambda \, d^4\eta$.

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[15]For distributions and this particular relation see e.g. [60].

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