Spontaneous breaking of conformal invariance and trace anomaly matching ☆

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Abstract

We argue that when conformal symmetry is spontaneously broken the trace anomalies in the broken and unbroken phases are matched. This puts strong constraints on the various couplings of the dilaton. Using the uniqueness of the effective action for the Goldstone supermultiplet for broken $\mathcal{N} = 1$ superconformal symmetry the dilaton effective action is calculated.

1. Introduction

The matching of chiral anomalies of the ultraviolet and infrared theories related by a massive flow plays an important role in understanding the dynamics of these theories. In particular using the anomaly matching the spontaneous breaking of chiral symmetry in QCD like theories was proven [1].

For supersymmetric gauge theories chiral anomaly matching provides constraints when different theories are related by "nonabelian" duality in the infrared [2]. The matching involves the equality of a finite number of parameters, "the anomaly coefficients" defined as the values of certain Green's function at a very special singular point in phase space. The Green's function themselves have very different structure at the two ends of the flow.

The massive flows relate by definition conformal theories in the ultraviolet and infrared but the trace anomalies of the two theories are not matched: rather the flow has the property that the a-trace anomaly coefficient decreases along it [3].

In this note we study a different set up. We consider a conformal field theory which admits vacua where the conformal symmetry is spontaneously broken. A typical example for such a theory in $d = 4$, which we will use as an illustration in the following, is the Coulomb branch of $\mathcal{N} = 4$ Super-Yang–Mills theory [4]. Then we can consider the trace anomalies in the broken and unbroken phases respectively. We will argue that the trace anomalies are matched in the two phases. We stress that the analytic structure of the amplitudes is different in the two phases: just the numerical values of the anomalies which depend on the value of the amplitudes at a very special point are matched.

An essential feature which allows the matching is the validity of the Ward identities related to the conservation and tracelessness of the energy–momentum tensor in both phases. This is analogous to the Ward identities following from global chiral invariance which are valid along the massive flow and allow the matching of the chiral anomalies. On the other hand the properties following from the $SO(d, 2)$ conformal symmetry are not valid in the broken phase since the vacuum is not invariant under some of the transformations belonging to the conformal group.

In the unbroken phase all the degrees of freedom are massless. In the broken phase some of the degrees of freedom become massive. The trace anomaly in the broken phase will be contributed by the massless degrees of freedom which include the ones which were not lifted by the spontaneous breaking and the "dilaton" the Goldstone boson related to the spontaneously broken conformal symmetry. The massive states play a role in determining through loop effects the couplings of the dilaton to energy momentum tensors such that the matching occurs.

We will give general arguments for the matching of trace anomalies. For type A anomalies (we use the terminology of [5]) the argument is rather similar to the one proving the matching of chiral anomalies [6]. For type B anomalies a more detailed argument based on an explicit calculation is needed.

When superconformal symmetry is spontaneously broken without breaking the global supersymmetry a short cut for the proof is available: since the trace anomalies (both type A and B) are related to chiral anomalies [7] the matching of chiral anomalies implies the matching of trace anomalies.

The couplings of the dilatons are summarized by the dilaton effective action in the presence of an external metric which acts as a source for the energy momentum tensor. Integrating out the dilaton field we get the generating functional for the correlators of energy momentum tensors incorporating the trace anomalies. It turns out that various generating functionals proposed [8] have analytic properties which identify them as describing the broken phase. They definitely cannot be used in the unbroken phase where the analytic structure of the correlators is different.

In order to get a "minimal" effective action for the dilaton we study spontaneously broken $\mathcal{N} = 1$ superconformal symmetry. In this case there is a supermultiplet of Goldstone bosons, the bosonic components being the dilaton and a Goldstone boson of the spontaneously broken $U(1)_{\mathcal{R}}$ symmetry. The "minimal" action for the $U(1)$ boson is well understood. Therefore, constructing a supersymmetric action for the Goldstone supermultiplet which reduces to the minimal one for the $U(1)$ boson, fixes the dilaton action. This action, after integrating out the dilaton, gives the generating functional for energy momentum tensors in the broken phase obeying all the constraints.

The analogue of anomaly matching for a chiral flow where in the infrared the chiral symmetry is preserved requires that the space of broken directions (moduli) has more than one point where conformal symmetry is unbroken.

We do not have a general proof that trace anomalies are matched when one adds to the space of moduli the space of truly marginal deformations of the conformal theory though on specific examples like $\mathcal{N} = 4$ Super-Yang–Mills this seems to be true.

The paper is organized as follows:

In Section 2 we give general arguments for trace anomaly matching and we perform a calculation for the simplest type B trace anomaly showing the mechanism.

In Section 3 we calculate the "minimal" effective action for the Goldstone supermultiplet for spontaneously broken superconformal supersymmetry.

In Section 4 we discuss the dilaton effective action following from the supersymmetric action calculated in Section 3. We integrate out the dilaton and we discuss the generating functional obtained, comparing it to various other actions. In Section 5 we summarize our results. We defer the explicit check of trace anomaly matching on the Coulomb branch of $\mathcal{N} = 4$ supersymmetric Yang–Mills both in the perturbative weak coupling regime and at strong coupling using the AdS/CFT correspondence to another publication [9].

2. Arguments for trace anomaly matching

A conformal field theory is characterized by an energy momentum tensor T_{mn} which is conserved

$$\partial^m T_{mn} = 0 \tag{2.1}$$

and traceless

$$T^m{}_m = 0 \tag{2.2}$$

at the quantum level. Since (2.1) and (2.2) are operatorial equations of motion, correlators of energy momentum tensors should obey Ward identities following from them independently of the vacuum on which the correlators are evaluated. An unavoidable violation of (2.2) in certain correlators gives the trace anomalies.

A very convenient way to study the correlators is to couple the energy momentum tensor to the perturbation h_{mn} around flat space of an external metric g_{mn}:

$$g_{mn} = \eta_{mn} + h_{mn} \tag{2.3}$$

Then if we define the generating functional $W(g)$ in this background, (2.1) is translated into the invariance of $W(g)$ under diffeomorphisms

$$\delta_\xi g_{mn} = \nabla_m \xi_n + \nabla_n \xi_m \tag{2.4}$$

while (2.2) into the invariance under Weyl transformations:

$$g_{mn} \to e^{2\sigma(x)} g_{mn} \tag{2.5}$$

If we are interested in correlators involving also other operators, we couple them to sources, the generating functional depending on the metric and the sources. We are using the "passive" convention i.e. the space–time coordinates do not transform under symmetry while the metric and the additional sources have variations at the same space–time point.

The transformations corresponding to the conformal group $SO(d, 2)$ are realized as diffeomorphisms on the flat metric compensated by Weyl transformations such that the flat metric is unchanged. This allows for a clear distinction between the Ward identities following from (2.1), (2.2) which should be valid (modulo anomalies) independent of the vacuum i.e. both in the unbroken and broken phases and the relations following from dilations and special conformal transformations, which are obeyed only in the unbroken phase.

The generating functional should be invariant (modulo anomalies) under Weyl transformations with parameter $\sigma(x)$ which goes to zero at infinity in both phases since this implies the validity of the Ward identities. On the other hand dilations and special conformal transformations correspond to diffeomorphisms which do not vanish at infinity and therefore also the Weyl transformations which compensate them have a parameter which does not vanish at infinity. Therefore the generating functional should be invariant under Weyl transformations which do not vanish at infinity only in the unbroken phase. We will call in the following Weyl transformations which vanish at infinity "local".

In $d = 4$ the generating functional $W(g)$ has the possible trace anomalies:

$$\delta_\sigma W = c \int d^4x \sqrt{g} \sigma C_{mnpq} C^{mnpq} - a \int d^4x \sqrt{g} \sigma E_4 \tag{2.6}$$

where E_4 is the $d = 4$ Euler density and C_{mnpr} is the Weyl tensor. The anomaly related to the Euler density is the only type A one while the second one is of type B. In more than four dimensions the number of type B anomalies increases and one can have even in $d = 4$ type B anomalies involving other sources. The dimensionless anomaly coefficients a and c are part of the characterization of the CFT.

If the conformal symmetry is spontaneously broken without breaking the Poincaré symmetry a Lorentz scalar primary $\mathcal{O}$ with dimension Δ has a vacuum expectation value:

$$\langle 0|\mathcal{O}|0\rangle = v^\Delta \tag{2.7}$$

where v has mass dimension 1. This scale is the only one in the broken phase characterizing all the dimensionful quantities appearing. As a consequence of the Goldstone theorem there is necessarily a massless Goldstone boson τ, the dilaton, which couples linearly to the energy momentum tensor:

$$\langle 0|T_{mn}|\tau; q\rangle = \frac{1}{3} f q_m q_n \tag{2.8}$$

where the dilaton τ is on mass shell with four momentum q. A normalization can be chosen for which f and v are equal and in the following we will use them interchangeably.

In the broken phase some of the states become massive with masses given by the scale v. This is happening though such that the energy momentum tensor still obeys (2.2). The trace anomaly in the broken phase is produced only by the massless states. These include states which remained massless who contribute through loops like in the unbroken phase. In addition there is now the dilaton: its contribution to the anomaly is given by tree diagrams where the coupling (2.8) plays an essential role and also a loop contribution where the dilaton acts like any other massless scalar.

The statement of anomaly matching is the equality of the a and c coefficients calculated in the broken phase as described above with their original values in the unbroken phase.

We now give arguments for the anomaly matching in decreasing order of generality and (hopefully) increasing rigor.

We start with a general argument: let us calculate the anomaly coefficients in the broken phase. For a given theory they can depend only on the scale parameter v characterizing the breaking.

However, a and c being dimensionless, cannot depend on v. As a consequence when v goes to zero and the unbroken phase is recovered, the values of a and c are unchanged. The possible loophole in this argument is a discontinuity in the limit.

Another general argument valid only for $\mathcal{N} = 1$ superconformal theories uses the fact [7] that the anomalies appear in supermultiplets and the coefficients a and c are related to chiral symmetry anomaly coefficients. Since chiral anomalies are matched so will be also the trace anomaly coefficients.

We give now more specific arguments. We start with the type A anomaly (the a coefficient). Type A anomalies are very similar in structure to the chiral anomalies: all the information in $d = 4$ is in the triangle diagram of three energy momentum tensors, there is no ultraviolet divergence involved and the anomaly has a topological form which obeys descent equations [10]. Also the argument for anomaly matching is very similar to the one for chiral anomalies [6,11].

Consider the correlator of three energy momentum tensors and decompose it in invariant amplitudes. Since the kinematic decomposition is rather formidable [12] we just give the relevant general structure. The special kinematical point which is responsible for the anomaly is when all the Lorentz invariants one can make of the momenta are zero:

$$q^2 = k_1^2 = k_2^2 = 0 \tag{2.9}$$

where q, k_1, k_2 are the momenta carried by the energy momentum tensors. The special point can be approached in different commuting orders e.g. taking all the invariants to be equal to q^2 and then sending $q^2 \to 0$. In this kinematical situation the Ward identities which follow from (2.2) and (2.1) reduce to [12]

$$A(q^2) = 0, \qquad A(q^2) - q^2 B(q^2) = 0 \tag{2.10}$$

Here A is an invariant amplitude of dimension zero which multiplies a kinematical structure corresponding to E_4 and B has dimension -2. The amplitude B, having negative dimension, is given unambiguously by

$$B(q^2) = \frac{a}{q^2} \tag{2.11}$$

Clearly both Ward identities (2.10) cannot be satisfied simultaneously and if the second is used to define $A = a$ there will be a "type A" anomaly with coefficient a.

So far the discussion has reflected the behavior around $q^2 = 0$ and it is equally valid in the broken and unbroken phases, but the coefficients are in principle different. From (2.11) we see, however, that also the high q^2 behavior of the amplitude B is determined. Since for $q^2 \gg v^2$ all effects of the spontaneous breaking should disappear, the high q^2 should match which implies the equality of the coefficients a in the broken and unbroken phases.

We stress that the correlators in the two phases are very different; in the unbroken phase all contributions come from loop diagrams (Fig. 1a) while in the broken phase there is the tree level contribution of the dilaton (Fig. 1b). In Section 4 we will study in more detail the dilaton couplings but the contribution of Fig. 1b has the form:

$$\frac{1}{3} f q_m q_n \frac{1}{q^2} f^\tau E_4 \tag{2.12}$$

where a coupling of strength f^τ of the dilaton to two energy momentum tensors with the kinematical structure of E_4 is needed. Then the contribution of (2.12) to the coefficient a is $\frac{1}{3} f f^\tau$ and knowing a and the other contributions fixes f^τ in terms of the scale f.
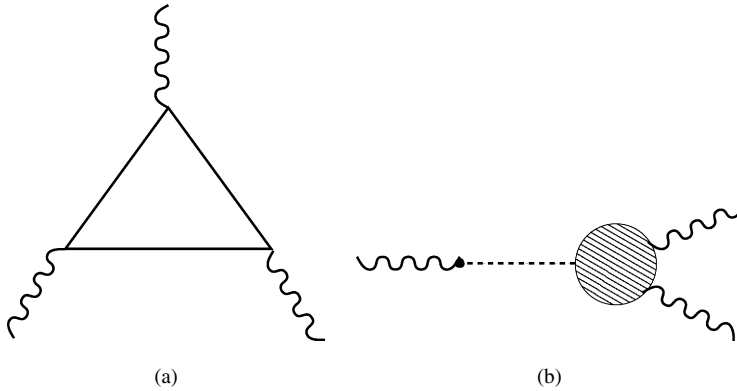
Fig. 1. Wavy line represents insertions of energy–momentum tensor and full line the propagator of a massless degree of freedom and the dotted the propagator of a dilaton.

The structure of (2.12), Fig. 1b, reflects the presence of a true pole at $q^2 = 0$, the dilaton, in the broken phase, while in the unbroken phase the B amplitude (2.11) does not have a pole: the analytic structure of Fig. 1a is a branch cut which degenerates at the special singular point into an apparent pole, but this pole is not present at generic q^2, k_1^2, k_2^2 and does not correspond to a state in the Hilbert space.

We now turn to the discussion of the matching for the coefficient c. The previous discussion cannot be generalized in a straightforward fashion due to the special features of type B anomalies. In the unbroken phase the anomaly appears in the UV finite part of the three point function but it is related to the logarithmic divergence of the two point function [13,14]. In order to calculate c one has to put in evidence this connection and therefore one approaches the singular point through values of the invariants imposed by the two point function i.e. $q^2 = 0$ and $k_1^2 = k_2^2 = k^2$.

On the other hand, in the broken phase the dilaton contribution does not have a direct connection to the two point function, the relevant diagram being again Fig. 1b. Therefore we should put $k^2 = 0$ at the beginning and keep $q^2 \neq 0$ in order to put in evidence the dilaton pole.

We will study in detail this mechanism trying again to avoid the very cumbersome kinematics of the energy momentum three point functions. We consider the kinematically simplest type B anomaly which singles out the relevant invariant three and two point amplitudes which are common to this class of anomalies.

Consider a $d = 4$ CFT with a dimension two primary, $\mathcal{O}^{(2)}$. This is the case for example in $\mathcal{N} = 4$ Super-Yang–Mills where the primary is the bilinear of the scalars in the protected traceless tensor irrep of $SO(6)$. The anomaly appears in the $\langle T_{mn} \mathcal{O}^{(2)} \mathcal{O}^{(2)} \rangle$ correlator. If we couple $\mathcal{O}^{(2)}$ to a source J which transforms under Weyl transformations as:

$$J(x) \rightarrow e^{-2\sigma(x)} J(x) \tag{2.13}$$

the anomaly has the form:

$$\delta_\sigma W(g, J) = \bar{c} \int d^4x \sqrt{g} \sigma(x) J^2(x) \tag{2.14}$$

Since there is an interplay between the three point function and the correlator of two $\mathcal{O}^{(2)}$ operators, we expand the generating functional to include these terms:

$$W(g, J) = \int d^4x \, d^4y \, \Gamma^{(2)}(x, y) J(x) J(y)$$
$$+ \int d^4x \, d^4y \, d^4z \, \Gamma^{(3)}_{mn}(x, y, z) h^{mn}(x) J(y) J(z) + \cdots \tag{2.15}$$

and the Ward identities which follow from diffeomorphisms and Weyl transformations are in momentum space:

$$q^m \Gamma^{(3)}_{mn}(q, k_1, k_2) = \frac{1}{2}(k_1)_n \Gamma^{(2)}(k_1^2) + \frac{1}{2}(k_2)_n \Gamma^{(2)}(k_2^2) \tag{2.16}$$

$$\eta^{mn} \Gamma^{(3)}_{mn}(q, k_1, k_2) = \Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2) \tag{2.17}$$

We have absorbed various factors of 2π into a convenient normalization.

The kinematical expansion of $\Gamma^{(3)}$ is

$$\Gamma^{(3)}_{mn} = \eta_{mn} \bar{A} + B q_m q_n + C(q_m r_n + q_n r_m) + D r_m r_n \tag{2.18}$$

with

$$r_m = (k_1)_m - (k_2)_m$$
$$q_m = (k_1)_m + (k_2)_m \tag{2.19}$$

The invariant amplitudes $\bar{A}, B, C, D$ depend on the invariants q^2, k_1^2, k_2^2. The two-point function $\Gamma^{(2)}$ depends on just one invariant.

Both $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are UV divergent, however in the combination

$$A = \bar{A} - \frac{1}{4}\left[\Gamma^{(2)}(k_1^2) + \Gamma^{(2)}(k_2^2)\right] \tag{2.20}$$

the UV divergence is removed. The amplitudes B, C, D, having dimension -2, are convergent from the beginning. We therefore rewrite the Ward identities (2.16) and (2.17) for this set of amplitudes:

$$A + q^2 B + q \cdot r C = 0 \tag{2.21}$$

$$q^2 C + q \cdot r D = \frac{1}{4}\left[\Gamma^{(2)}(k_1^2) - \Gamma^{(2)}(k_2^2)\right] \tag{2.22}$$

and

$$4A + q^2 B + 2q \cdot r C + r^2 D = 0 \tag{2.23}$$

These Ward identities have the generic structure of the relevant identities involving three and two energy momenta, respectively.

In the unbroken phase there are no singularities at $q^2 = 0$ for arbitrary positive k_1^2, k_2^2. Therefore we approach the singular point choosing also $k_1^2 = k_2^2 = k^2$ such that all the invariant amplitudes depend only on k^2. The Ward identities (2.21), (2.22) and (2.23) become:

$$A(k^2) = 0 \tag{2.24}$$

$$D(k^2) = \frac{1}{4} \frac{\partial \Gamma^{(2)}}{\partial k^2} \tag{2.25}$$

and

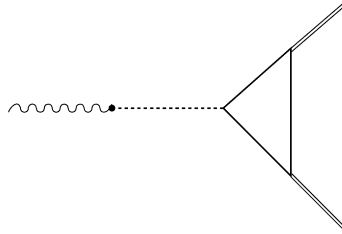$$4A(k^2) + 4k^2 D(k^2) = 0 \tag{2.26}$$

Fig. 2. The dilaton diagram for type B. The full line represents the massive constituent fields of $\mathcal{O}^{(2)}$ and the double line the insertion of $\mathcal{O}^{(2)}$.

The structure is characteristic for all type B anomalies: there is a "clash" between Eqs. (2.24), (2.26) showing that the anomaly appears in the three point function, however the normalization of the anomaly is related to the two point function by Eq. (2.25). Explicitly, the two point function in the unbroken phase is

$$\Gamma^{(2)}(k^2) = 2\bar{c}\log\left(\frac{k^2}{\Lambda^2}\right) \tag{2.27}$$

where Λ is the UV cut off. This leads to:

$$D(k^2) = \frac{\bar{c}}{4k^2} \tag{2.28}$$

the anomaly being $\bar{c}$, if we choose to put it in the trace, as usual.

We remark for later use that the singular point can also be reached if we give a mass m to all the fields ϕ in the loop (the scalar constituents of the operator $\mathcal{O}^{(2)}$ in our example) and putting all invariants to 0 from the beginning; there are no infrared singularities for this regulator. The identity (2.2) is now modified to:

$$T^m{}_m = m^2\phi^2 \tag{2.29}$$

and, with all kinematical invariants vanishing, (2.23) takes the form:

$$4A(0) = m^2 E(0) \tag{2.30}$$

where E, the correlator of two $\mathcal{O}^{(2)}$ operators and ϕ^2, is evaluated at the special point. Dimensionally E must have the form:

$$E(0) = \frac{\bar{c}}{m^2} \tag{2.31}$$

and the anomaly can be recovered in this way as the limit of the modified Ward identity when the regulator mass m is sent to zero. There is a nontrivial relation between the two point function $\Gamma^{(2)}$ and the three point function E which led to the same $\bar{c}$, since the limits are interchangeable.

In the above argument and in the following we do not keep track of the various numerical factors since Fig. 2 shows, as we will explain, how the results match at the Feynman diagram level.

We now evaluate the anomaly in the broken phase. The Ward identities (2.21), (2.22) and (2.23) continue to be valid. If we want to calculate the contribution of the dilaton we need to evaluate the diagram of Fig. 2. Since there is a pole at $q^2 = 0$ for any value of the invariants we have to go to the special point while keeping q^2 generic. We do not have a general argument

for normalizing the dilaton coupling to two J sources. We choose therefore a concrete realization in which the dilaton has a Lagrangian realization, i.e. the Coulomb branch of an $\mathcal{N} = 4$ Super-Yang–Mills theory with gauge group $SU(2)$.

For concreteness we pick:

$$\mathcal{O}^{(2)} \equiv \mathrm{tr}(\phi_5 \phi_6) \tag{2.32}$$

and ϕ_1 as the field which gets expectation value, the fields ϕ_i, $i = 1, \ldots, 6$ being the scalars.

We work in the unitary gauge:

$$\phi_1 = \begin{pmatrix} \tilde{\varphi} & 0 \\ 0 & -\tilde{\varphi} \end{pmatrix} \tag{2.33}$$

Due to our choice of $\mathcal{O}^{(2)}$ only the part of the Lagrangian involving the scalars will play a role:

$$\mathcal{L} = -\sum_{i=1}^{6} \mathrm{tr}(D_m \phi_i D^m \phi_i) + g^2 \sum_{i<j} \mathrm{tr}([\phi_i, \phi_j]^2) \tag{2.34}$$

Since the coupling g is a true parameter in the theory, we can systematically expand in it. We will calculate the correlator to leading order in g, which is $\mathcal{O}(g^0)$. The energy momentum tensor of the scalars to order g^0 is:

$$T_{mn} = -\sum_{i=1}^{6} \mathrm{tr}\left[\frac{1}{6} \partial^p \phi_i \partial_p \phi_i \eta_{mn} - \frac{2}{3} \partial_m \phi_i \partial_n \phi_i - \frac{1}{3} \phi_i \Box \phi_i \eta_{mn} + \frac{1}{3} \phi_i \partial_m \partial_n \phi_i \right] \tag{2.35}$$

In the broken phase $\tilde{\varphi}$ gets an expectation value v. The dilaton field is

$$\tau = \tilde{\varphi} - v \tag{2.36}$$

As a consequence of the breaking, the $\pm$ (in $SU(2)$) components of the scalars get massive with a mass

$$m = gv \tag{2.37}$$

The potential in (2.34) becomes

$$-g^2 \sum_{2 \leqslant i < j \leqslant 6} \mathrm{tr}([\phi_i, \phi_j]^2) + 2g^2(\tau + v)^2 \sum_{i=2}^{6} \phi_i^{(+)} \phi_i^{(-)} \tag{2.38}$$

The contribution of the massive fields to $\mathcal{O}^{(2)}$ is

$$\left(\phi_5^{(+)} \phi_6^{(-)} + \phi_5^{(-)} \phi_6^{(+)} \right) \tag{2.39}$$

and the energy momentum tensor has the linear term:

$$-\frac{1}{3} v (\partial_m \partial_n - \eta_{mn} \Box) \tau \tag{2.40}$$

which corresponds (on-shell) to (2.8). It is now clear how the matching works since in (2.38) we have the explicit coupling of the dilaton to massive fields in $\mathcal{O}^{(2)}$. The massive fields produce an amplitude like E and therefore the coupling of the dilaton to the two currents is

$$g^2 v \frac{\bar{c}}{m^2} = g^2 v \frac{\bar{c}}{g^2 v^2} = \frac{\bar{c}}{v} \tag{2.41}$$

Even though we used the coupling of the dilaton to the massive scalars to $\mathcal{O}(g^2)$, due to the m^2 dependence the coupling to $\mathcal{O}^{(2)}$ is $\mathcal{O}(g^0)$. Together with the coupling (2.40) the diagram Fig. 2 reproduces the coefficient $\bar{c}$ of the anomaly. We can see in this example the general mechanism by which the anomaly is matched in the broken phase: massless field which in the unbroken phase were circulating in the loop became massive after the breaking. They generate the dilaton coupling to the operators $\mathcal{O}^{(2)}$ and since their contribution can be viewed alternatively as the calculation of the anomaly in the unbroken phase with an infrared regulator, the normalizations match. The numerical factors will obviously match since from Fig. 2 it is evident that the same triangle gives the normalization of the anomaly in the unbroken phase and fixes the coupling of the dilaton in the broken phase.

3. The action for the Goldstone supermultiplet in spontaneously broken $\mathcal{N} = 1$ superconformal symmetry

Since there are ambiguities in the dilaton Lagrangians corresponding to a set of given trace anomalies we will try to find a "minimal" solution which satisfies all the constraints i.e. reproduces the anomalies and has the right analyticity.

We will do that by starting with a spontaneously broken chiral $U(1)$ theory. In this case the minimal action for the $U(1)$ Goldstone boson β is well understood: coupling the boson to a $U(1)$ gauge field A_m there is a kinetic term which is gauge invariant and a term reproducing the chiral anomaly of strength a

$$W(A, \beta) = -\int d^4x \left((\partial_m \beta - A_m)(\partial^m \beta - A^m) + a\beta F^{mn} \tilde{F}_{mn} \right) \tag{3.1}$$

Under gauge transformations, where $A_m \rightarrow A_m + \partial_m \alpha$, the Goldstone field β transforms as $\beta \rightarrow \beta + \alpha$ and the variation of (3.1) obviously reproduces the anomaly with coefficient a.

Integrating out β produces a generating functional depending on A_m only: the correlators of the $U(1)$ currents having the correct chiral anomaly follow and the analyticity of this nonlocal action reflects the broken phase, i.e. a zero mass pole corresponding to the Goldstone boson.

Now let us consider a spontaneously broken $\mathcal{N} = 1$ superconformal theory in which also the $U(1)_\mathcal{R}$ symmetry is spontaneously broken. In such a situation there is a chiral supermultiplet of Goldstone bosons, the axion β and the dilaton τ appearing together in its lowest component. If the action for this supermultiplet reduces for β to the minimal one, i.e. to (3.1), we will consider the action of τ also to be "minimal". To achieve this we follow a superspace version of the Wess–Zumino procedure in order to produce the anomalous term and then we will add to it a supersymmetry invariant kinetic term.

We start by reviewing the Wess–Zumino procedure i.e. the construction of the local action via integration of the anomaly [15]. Since it suffices for our purposes, we consider only abelian symmetries. The Wess–Zumino procedure is trivial in the case of the chiral $U(1)$, but nontrivial for the Weyl and super-Weyl anomalies.

Consider a quantum field theory which is invariant under an abelian symmetry. Gauge the symmetry, i.e. couple the QFT to an external gauge field, which is the source for the symmetry current. Integrating out the quantum fields leads to a nonlocal effective action $W(J)$, which is, in general, not invariant under the symmetry, thus signaling an anomaly $\mathcal{A}$. Here J is the external source. Under a symmetry transformation it transforms to $J^{(\sigma)}$, where $\sigma(x)$ is the parameter of the transformation. If T denotes the generator of the transformation,

$$(\sigma T)W(J) = \sigma \cdot \mathcal{A} \equiv \int dx \, \sigma(x) \mathcal{A}(J(x)) \tag{3.2}$$

where $\mathcal{A}$ is a local functional of the source. Let us now assume that $W(J)$ can be obtained via

$$e^{iW(J)} = \int D\phi \, e^{iW(J,\phi)} \tag{3.3}$$

where $W(J,\phi)$ is a local functional of the source and the Goldstone field ϕ which transforms as

$$\phi \to \phi + \sigma \tag{3.4}$$

such that $J^{(-\phi)}$ is invariant. Applying

$$e^{\sigma T} - 1 = \int_0^1 dt \, \sigma e^{t\sigma T} \tag{3.5}$$

to W gives

$$W(J^{(\sigma)}, \phi + \sigma) = W(J, \phi) + \int_0^1 dt \, \sigma \cdot \mathcal{A}(J^{(t\sigma)}) \tag{3.6}$$

The choice $\sigma = -\phi$ and the normalization $W(J, 0) = 0$ lead to

$$W(J, \phi) = \int_0^1 dt \, \phi \cdot \mathcal{A}(J^{(-t\phi)}) \tag{3.7}$$

The existence of W is guaranteed by the following observation:

$$\phi \mathcal{A}(J^{(-t\phi)}) = -\frac{d}{dt} W(J^{(-t\phi)}) \tag{3.8}$$

which immediately implies

$$W(J, \phi) = -\left(W(J^{(-\phi)}) - W(J)\right) \tag{3.9}$$

The r.h.s. should be viewed as a power series in ϕ. The first nonvanishing term is the anomaly. All higher order terms are therefore local.

Note that W is determined up to invariant terms. For instance, the invariant kinetic energy of the Goldstone field has to be provided by hand.

We will now turn to the construction of a local effective action, whose variation under super-Weyl transformation reproduces the superconformal anomaly. We will do this for $\mathcal{N} = 1$ supersymmetric conformal field theories and as such it also applies to $\mathcal{N} = 4$ SYM if we impose restrictions, such as $a = c$, which are implied by the extended SUSY.

The action W we want to construct depends on the Goldstone chiral supermultiplet Φ coupled to the supergravity gauge multiplet which acts as a source. The $U(1)$ gauge and Weyl anomalies belong now to supermultiplets of anomalies. These supermultiplets contain also the anomalies of the fermionic supercurrents but since these are of no interest to us we will put all the fermionic components of the fields to zero.

The calculation is most easily done in superspace, as we will now explain. Our discussion uses the notation, conventions and results of [16][1] and for reasons of notational convenience we use in this section a dimensionless Goldstone superfield. We will restore the physical dimensions in the next section.

The variation of the nonlocal effective action should give the anomaly

$$
\begin{aligned}
\delta_\Sigma W &= 2(a-c)\int d^8z\, \frac{E^{-1}}{R}\delta\Sigma\, W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + c.c. \\
&\quad - 2c\int d^8z\, E^{-1}(\delta\Sigma + \delta\bar\Sigma)\big(G^a G_a + 2R\bar R\big) \\
&= \int d^8z\, \frac{E^{-1}}{R}\delta\Sigma\, \frac{\delta W}{\delta\Sigma}
\end{aligned}
\tag{3.10}
$$

a and c are the anomaly coefficients. Some explanation of the notation is in order: $W_{\alpha\beta\gamma}$, R (both chiral) and G_a (real) are the gravitational superfields in terms of which the solutions of the Bianchi identities can be parametrized. E is the determinant of the supervielbein and the integrations are over $\mathcal{N}=1$ superspace. Σ is a chiral superfield which parametrizes the super-Weyl transformations under which the various superfields transform as $(G_{\alpha\dot\alpha} = \sigma^a_{\alpha\dot\alpha}G_a)$

$$
R \to e^{\bar\Sigma - 2\Sigma}\left(R - \frac{1}{4}\big(\bar{\mathcal{D}}^2\bar\Sigma + (\bar{\mathcal{D}}_{\dot\alpha}\bar\Sigma)(\bar{\mathcal{D}}^{\dot\alpha}\bar\Sigma)\big)\right)
$$

$$
G_{\alpha\dot\alpha} \to e^{-\frac{1}{2}(\Sigma+\bar\Sigma)}\left(G_{\alpha\dot\alpha} + \frac{1}{2}(\mathcal{D}_\alpha\Sigma)(\bar{\mathcal{D}}_{\dot\alpha}\bar\Sigma) - i\mathcal{D}_{\alpha\dot\alpha}(\Sigma-\bar\Sigma)\right)
$$

$$
W_{\alpha\beta\gamma} \to e^{-\frac{3}{2}\Sigma}W_{\alpha\beta\gamma}
$$

$$
E \to e^{-(\Sigma+\bar\Sigma)}E
$$

$$
\mathcal{D}_\alpha \to e^{\frac{1}{2}\Sigma - \bar\Sigma}\big(\mathcal{D}_\alpha - (\mathcal{D}^\beta\Sigma)M_{\beta\alpha}\big)
\tag{3.11}
$$

$M_{\alpha\beta}$ is the Lorentz transformation generator in the $(1/2, 0)$ representation. It acts as

$$
M_{\alpha\beta}(\psi_\gamma) = \frac{1}{2}(\epsilon_{\gamma\alpha}\psi_\beta + \epsilon_{\gamma\beta}\psi_\alpha)
\tag{3.12}
$$

The infinitesimal version of these are

$$
\delta_\Sigma R = (\bar\Sigma - 2\Sigma)R - \frac{1}{4}\bar{\mathcal{D}}^2\bar\Sigma, \qquad \delta_\Sigma\bar R = (\Sigma - 2\bar\Sigma)\bar R - \frac{1}{4}\mathcal{D}^2\Sigma
$$

$$
\delta_\Sigma G_{\alpha\dot\alpha} = -\frac{1}{2}(\Sigma + \bar\Sigma)G_{\alpha\dot\alpha} - i\mathcal{D}_{\alpha\dot\alpha}(\Sigma - \bar\Sigma)
\tag{3.13}
$$

and

$$
\delta_\Sigma\mathcal{D}_\alpha = \left(\frac{1}{2}\Sigma - \bar\Sigma\right)\mathcal{D}_\alpha, \qquad \delta_\Sigma\bar{\mathcal{D}}_{\dot\alpha} = \left(\frac{1}{2}\bar\Sigma - \Sigma\right)\bar{\mathcal{D}}_{\dot\alpha}
$$

$$
\delta_\Sigma\mathcal{D}_{\alpha\dot\alpha} = -\frac{1}{2}(\Sigma+\bar\Sigma)\mathcal{D}_{\alpha\dot\alpha} - \frac{i}{2}\big((\mathcal{D}_\alpha\Sigma)\bar{\mathcal{D}}_{\dot\alpha} + (\bar{\mathcal{D}}_{\dot\alpha}\bar\Sigma)\mathcal{D}_\alpha\big)
$$

$$
\delta_\Sigma\mathcal{D}^2 = (\Sigma - 2\bar\Sigma)\mathcal{D}^2 + 2(\mathcal{D}^\alpha\Sigma)\mathcal{D}_\alpha, \qquad \delta_\Sigma\bar{\mathcal{D}}^2 = (\bar\Sigma - 2\Sigma)\bar{\mathcal{D}}^2 + 2(\bar{\mathcal{D}}_{\dot\alpha}\bar\Sigma)\bar{\mathcal{D}}^{\dot\alpha}
\tag{3.14}
$$

---

[1] Except for the convention for the Riemann curvature where we use $[\nabla_m, \nabla_n]V^p = -R_{mn}{}^p{}_q V^q$ which differs by a sign from [16].

where (3.14) are valid if they act on a Lorentz scalar. $\mathcal{D}$ are supercovariant derivatives and $\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2i\mathcal{D}_{\alpha\dot\alpha}$.

One can check that (3.10) satisfies the Wess–Zumino consistency condition

$$(\delta_{\Sigma_1}\delta_{\Sigma_2} - \delta_{\Sigma_2}\delta_{\Sigma_1})W = 0 \tag{3.15}$$

To construct the local effective action, following [15], we need the variation of the anomaly under finite Weyl transformations. Using (3.11) we find

$$
\begin{aligned}
&\big[(E^{-1}/R)W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\big]^{(\Sigma)} = (E^{-1}/R)W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \\
&\big[E^{-1}(G^a G_a + 2R\bar{R})\big]^{(\Sigma,\bar{\Sigma})} \\
&= E^{-1}\Big\{ G_a G^a + 2R\bar{R} + iG_{\alpha\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}(\Sigma - \bar{\Sigma}) - \frac{1}{2}\big(R\mathcal{D}^2\Sigma + \bar{R}\bar{\mathcal{D}}^2\bar{\Sigma}\big) \\
&\quad - \frac{1}{2}G_{\alpha\dot\alpha}\big(\mathcal{D}^\alpha\Sigma\big)\big(\bar{\mathcal{D}}^{\dot\alpha}\bar{\Sigma}\big) - \frac{1}{2}\big(R(\mathcal{D}\Sigma)^2 + \bar{R}(\bar{\mathcal{D}}\bar{\Sigma})^2\big) + \frac{1}{2}\mathcal{D}_{\alpha\dot\alpha}(\Sigma - \bar{\Sigma})\mathcal{D}^{\alpha\dot\alpha}(\Sigma - \bar{\Sigma}) \\
&\quad + \frac{1}{8}\big(\mathcal{D}^2\Sigma\big)\big(\bar{\mathcal{D}}^2\bar{\Sigma}\big) + \frac{i}{2}(\mathcal{D}_\alpha\Sigma)(\bar{\mathcal{D}}_{\dot\alpha}\bar{\Sigma})\mathcal{D}^{\alpha\dot\alpha}(\Sigma - \bar{\Sigma}) \\
&\quad + \frac{1}{8}\big(\mathcal{D}^2\Sigma\big)(\bar{\mathcal{D}}\bar{\Sigma})^2 + \frac{1}{8}\big(\bar{\mathcal{D}}^2\bar{\Sigma}\big)(\mathcal{D}\Sigma)^2 \Big\}
\end{aligned}
\tag{3.16}
$$

Then the local effective action is

$$
\begin{aligned}
W_{\text{loc}} &= f^2 \int d^8z\, E^{-1}e^{-\Phi}e^{-\bar{\Phi}} \\
&\quad + 2(c-a)\int_0^1 dt \int d^8z\, \Phi\big[(E^{-1}/R)W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\big]^{(-t\Phi)} + \text{c.c.} \\
&\quad - 2a\int_0^1 dt \int d^8z\,(\Phi + \bar{\Phi})\big[G^a G_a + 2R\bar{R}\big]^{(-t\Phi,-t\bar{\Phi})} \\
&= f^2\int d^8z\, E^{-1}e^{-\Phi}e^{-\bar{\Phi}} \\
&\quad + 2(c-a)\int d^8z\,(E^{-1}/R)\Phi W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{c.c.} \\
&\quad - 2a\int d^8z\, E^{-1}\Big\{(\Phi+\bar{\Phi})(G^a G_a + 2R\bar{R}) - \frac{1}{2}G_{\alpha\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\bar{\Phi}\mathcal{D}^\alpha\Phi \\
&\quad - \frac{1}{4}\big(R(\mathcal{D}\Phi)^2 + \bar{R}(\bar{\mathcal{D}}\bar{\Phi})^2\big) - \frac{i}{4}\mathcal{D}^{\alpha\dot\alpha}(\Phi - \bar{\Phi})\bar{\mathcal{D}}_{\dot\alpha}\bar{\Phi}\mathcal{D}_\alpha\Phi \\
&\quad + \frac{1}{16}(\mathcal{D}\Phi)^2(\bar{\mathcal{D}}\bar{\Phi})^2\Big\}
\end{aligned}
\tag{3.17}
$$

The first line is the super-Weyl invariant kinetic energy term for the Goldstone field. It is added by hand. To obtain the last two lines we have repeatedly integrated by parts and used the superspace Bianchi identities. One may check that the super-Weyl variation of (3.17) is independent of Φ and reproduces the anomaly. Here one uses

$$\Phi \to \Phi^{(\Sigma)} = \Phi + \Sigma \tag{3.18}$$

We are interested in the effective action in terms of the component fields. They are defined as[2]

$$\Sigma| = S = \sigma + \frac{2i}{3}\alpha, \qquad -\frac{1}{4}\mathcal{D}^2\Sigma| = f$$

$$\Phi| = \phi = \tau + i\beta, \qquad -\frac{1}{4}\mathcal{D}^2\Phi| = F$$

$$E_a^m| = e_a^m, \qquad R| = \frac{1}{3}\bar{B}, \qquad G_a| = \frac{4}{3}A_a \tag{3.19}$$

Here $\Psi|$ means the $\theta = \bar{\theta} = 0$ component of the superfield $\Psi(z) = \Psi(x, \theta, \bar{\theta})$. As before, τ is the dilaton and β the axion. Fermions will be set to zero throughout (which leads to enormous simplifications).

Under (infinitesimal) super-Weyl transformations the component fields transform as

$$\delta e_m^a = \sigma e_m^a, \qquad \delta B = -(\Sigma + 3i\alpha)B - 3f, \qquad \delta A_m = \delta(e_m^a A_a) = \partial_m\alpha \tag{3.20}$$

Furthermore

$$\delta\phi = S, \qquad \delta F = f \tag{3.21}$$

We will use the auxiliary component f to transform away F. Furthermore, the auxiliary scalar B is, like A_m and e_m^a, an external source which we can fix by hand. The equation of motion for B gives relations for the auxiliary fields which never appear linearly in W_{loc}. Since we are not interested in these relations, we will set $B = F = 0$ in the component action.

For a chiral superspace Lagrangian $\mathcal{L}_c$ ($\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{L}_c = 0$), such as the first term in (3.10),

$$\int d^8z \frac{E^{-1}}{R}\mathcal{L}_c = \int d^4x \sqrt{g}\left(-\frac{1}{4}\mathcal{D}^2\mathcal{L}_c| + B\mathcal{L}_c|\right) \tag{3.22}$$

plus terms which contain the gravitino field. The second term in (3.22) obviously vanishes for $B = 0$. For a nonchiral Lagrangian $\mathcal{L}$, such as the second line in (3.10), one uses that $-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{L}$ is chiral and then (3.22).

The action in terms of the remaining bosonic component fields is

$$\begin{aligned}
W_{\text{loc}} = f^2 \int d^4x \sqrt{g}\, e^{-2\tau}\left(-\nabla^m\tau\nabla_m\tau + \frac{1}{6}R - \left(\nabla_m\beta - \frac{2}{3}A_m\right)\left(\nabla^m\beta - \frac{2}{3}A^m\right)\right) \\
- \int d^4x \sqrt{g}\left[a\tau E_4 + c\tau C^{mnpq}C_{mnpq} - \frac{8}{3}c\tau F^{mn}F_{mn}\right. \\
\left. + \beta\left(\frac{8}{9}(3c - 5a)F^{mn}\tilde{F}_{mn} + (a - c)R^{mnpq}\tilde{R}_{mnpq}\right)\right] \\
+ \frac{16}{9}a \int d^4x \sqrt{g}\left(3\left[-R^{mn}A_n + \frac{1}{6}RA^m + \frac{4}{9}A^2A^m\right]\nabla_m\beta + 2A^mA^n\nabla_m\nabla_n\tau\right) \\
- a \int d^4x \sqrt{g}\left\{\left[-\left(R - \frac{8}{9}A^2\right)g^{mn} + 2\left(R^{mn} - \frac{8}{9}A^mA^n\right)\right]\nabla_m\tau\nabla_n\tau\right. \\
+ \left[-\left(\frac{1}{3}R + \frac{8}{9}A^2\right)g^{mn} + 2\left(R^{mn} - \frac{8}{9}A^mA^n\right)\right]\nabla_m\beta\nabla_n\beta \\
\left. + \frac{16}{3}A^n\nabla^m\tau\nabla_n\nabla_m\beta\right\} + \cdots
\end{aligned} \tag{3.23}$$

Here R is the (torsionless) Ricci scalar (not to be confused with the superfield R), R_{mn} the Ricci tensor, etc., C_{mnpq} the Weyl tensor and $\sqrt{g}E_4$ is the Euler-density in $d = 4$, i.e.

$$E_4 = \tilde{R}^{mnpq}\tilde{R}_{mnpq} = R^{mnpq}R_{mnpq} - 4R^{mn}R_{mn} + R^2$$
$$= C^{mnpq}C_{mnpq} - 2R^{mn}R_{mn} + \frac{2}{3}R^2 \tag{3.24}$$

Finally, a tilde denotes dualization, i.e.

$$\tilde{R}_{mnpq} = \frac{1}{2}\epsilon_{mn}{}^{st}R_{stpq}, \qquad \tilde{F}_{mn} = \frac{1}{2}\epsilon_{mn}{}^{pq}F_{pq} \tag{3.25}$$

The first line are the Weyl and gauge invariant kinetic energies of the dilaton τ and of the axion β. The external metric and the gauge fields play the role of Stueckelberg fields. The dilaton term can be written in terms of the Weyl-invariant combination $\tilde{g}_{mn} = e^{-2\tau}g_{mn}$ as $\frac{1}{6}\int \sqrt{\tilde{g}}R(\tilde{g})$. The second and third lines contain the anomalies:

The first two terms in the second line are the type A and type B Weyl anomalies which reflect anomalous Ward identities in the three point function $\langle TTT \rangle$ of three energy momentum tensors. The last term in the second line is a type B Weyl anomaly which is visible in $\langle TJJ \rangle$ where J is the $U(1)$ vector current. The terms in the third line reflect anomalous Ward identities in $\langle J_5JJ \rangle$ and $\langle J_5TT \rangle$ where J_5 is the axial $U(1)_{\mathcal{R}}$ current. They are the chiral and Delbourgo–Salam anomalies, respectively. Due to supersymmetry all five anomalies are characterized by just two coefficients, a and c.

Note that the fourth line in (3.23) is the sum of Weyl and gauge variation with parameters τ and β of a local term proportional to

$$\int d^4x \sqrt{g}\left(R^{mn}A_mA_n - \frac{1}{6}RA^2 - \frac{2}{9}(A^2)^2\right) \tag{3.26}$$

The second and third lines of (3.23) have previously been derived in [7]. To derive the remaining terms we have also used results in [17], appropriately converted to the conventions of [16]. The terms of third and fourth order in the Goldstone field S, which we have not written down in (3.23), contain three and four derivatives, respectively.

The $U(1)_{\mathcal{R}}$ part of the action reproduces (in a different normalization of the gauge field) (3.1). Therefore the complete part depending only on the dilaton, which we now display, is the "minimal" dilaton action we were looking for[3]

$$W_{\text{loc}} = -f^2 \int d^4x \sqrt{g}e^{-2\tau}\left(\partial^m\tau\partial_m\tau - \frac{1}{6}R\right)$$
$$- \int d^4x \sqrt{g}\left\{a\tau E_4 + c\tau C^{mnpq}C_{mnpq} + 2a\left(R^{mn} - \frac{1}{2}Rg^{mn}\right)\partial_m\tau\partial_n\tau\right.$$
$$\left. + 4a(\partial^m\tau\partial_m\tau)\Box\tau - 2a(\partial^m\tau\partial_m\tau)^2\right\} \tag{3.27}$$

Besides the coupling to the anomalies the action has higher order terms in the dilaton field. These terms are the analogues of similar terms in chiral Lagrangians for nonabelian chiral symmetries, reflecting the nonabelian algebra obeyed by energy momentum tensors. When the external metric is flat there is still a self-interaction of the dilaton: this is the analogue of the Wess–Zumino–Witten term but the group being noncompact the term can be written in a local fashion in $d = 4$.

---

[3] This is, in fact, the same action which one obtains from $\int dt\, \tau \cdot (c[C^2]^{(-t\tau)} - a[E_4]^{(-t\tau)})$.

4. The dilaton effective action

We study now the "minimal" dilaton effective action constructed in the previous section. After integrating out the dilaton (at tree level) a generating functional depending only on the metric is produced. This functional reproduces by construction the anomalies. We will compare this functional with various other propositions in the literature. The functional corresponds to the broken phase: it obeys the Ward identities (modulo anomalies) but it violates the dilation and/or special conformal transformation part of the conformal group.

The action for the dilaton has the form:

$$S = - \int d^4x \, \sqrt{g} \left(e^{-2\tau/f} g^{mn} \partial_m \tau \partial_n \tau - \frac{1}{6} f^2 e^{-2\tau/f} R \right) + W \tag{4.1}$$

where the first, "kinetic" term respects the Ward identities while the second term

$$
\begin{aligned}
W &= W_a + W_c \\
&= -a \int d^4x \sqrt{g} \left\{ \frac{1}{f} \tau E_4 + \frac{2}{f^2} \left(R^{mn} - \frac{1}{2} R g^{mn} \right) \partial_m \tau \partial_n \tau \right. \\
&\quad \left. + \frac{4}{f^3} (\partial^m \tau \partial_m \tau) \Box \tau - \frac{2}{f^4} (\partial^m \tau \partial_m \tau)^2 \right\} \\
&\quad - c \int d^4x \sqrt{g} \frac{1}{f} \tau C^{mnpq} C_{mnpq}
\end{aligned}
\tag{4.2}
$$

reproduces the anomalies. The kinetic term contains the characteristic coupling $\frac{1}{3} f \tau R$ required by Goldstone theorem (2.8).

We start the integration out with the kinetic term. This will also serve to discuss the various subtleties which appear in this process.

We first make a change of variable from τ to a field φ:

$$\varphi \equiv f \left(1 - e^{-\tau/f} \right) \tag{4.3}$$

The kinetic part of the action becomes:

$$- \int d^4x \, \sqrt{g} \left(g^{mn} \partial_m \varphi \partial_n \varphi - \frac{1}{6} R \varphi^2 - \frac{1}{6} f^2 R + \frac{1}{3} f \varphi R \right) \tag{4.4}$$

We then perform the Gaussian integration over φ. Alternatively we can think about this step as calculating the tree diagrams contributed by the vertex linear in φ. The Jacobian of the transformation (4.3) does not influence the calculation and the result is:

$$\bar{S}^{(0)}(g) = \int d^4x \, \sqrt{g} R \Phi(g) \tag{4.5}$$

where

$$\Phi(g) \equiv \frac{1}{6} f^2 \left[1 - \frac{1}{6\Box + R} R \right] \tag{4.6}$$

The functional $\Phi(g)$ was introduced in [18]. It should be understood, together with similar expressions which we will encounter in the following, as a formal expansion in curvatures, representing tree diagrams with dilaton poles. $\Phi(g)$ obeys the equation:

$$(6\Box + R) \Phi = 0 \tag{4.7}$$

understood again as holding on the functional expanded in powers of R.

We now check the transformation of $\Phi(g)$ and $\bar{S}^{(0)}$ under Weyl transformations. We have to be careful with boundary conditions and integration by parts: as we discussed before, the Ward identities are related to "local" Weyl transformations while the dilation/special conformal transformations require Weyl transformations which do not vanish at infinity.

We start with the Weyl transformation of $\Phi(g)$. The action in the various expressions will be from left to right, inverting it requires an integration by part valid only for transformations which vanish at infinity. We use the transformations:

$$\delta_\sigma R = -2\sigma R + 6\Box\sigma$$
$$\delta\Box = -2\sigma\Box + 2\nabla^m\sigma\nabla_m \qquad (4.8)$$

where in the second line it is understood that $\Box$ acts on a scalar. The transformation of the propagator and of the propagator acting together with R follow:

$$\delta_\sigma\left(\frac{1}{\Box}\right) = -\frac{1}{\Box}\delta_\sigma\Box\frac{1}{\Box} = \frac{2}{\Box}\sigma - 2\frac{1}{\Box}(\nabla^m\sigma)\nabla_m\frac{1}{\Box} \qquad (4.9)$$

$$\delta_\sigma\left(\frac{1}{\Box}R\right) = -2\frac{1}{\Box}(\nabla^m\sigma)\nabla_m\frac{1}{\Box}R + 6\sigma - \frac{6}{\Box}\sigma\Box - \frac{12}{\Box}(\nabla^m\sigma)\nabla_m \qquad (4.10)$$

A general term in the expansion of $\Phi(g)$ transforms as

$$\left(\frac{1}{\Box}R\right)^N\delta\left(\frac{1}{\Box}R\right)\left(\frac{1}{\Box}R\right)^M$$
$$= -2\left\{\left(\frac{1}{\Box}R\right)^N\frac{1}{\Box}(\nabla^m\sigma)\nabla_m\left(\frac{1}{\Box}R\right)^{M+1} + 6\left(\frac{1}{\Box}R\right)^N\frac{1}{\Box}(\nabla^m\sigma)\nabla_m\left(\frac{1}{\Box}R\right)^M\right\}$$
$$+ 6\left\{\left(\frac{1}{\Box}R\right)^N\sigma\left(\frac{1}{\Box}R\right)^M - \left(\frac{1}{\Box}R\right)^{N+1}\sigma\left(\frac{1}{\Box}R\right)^{M-1}\right\} \qquad (4.11)$$

Writing $\Phi(g)$ as an expansion

$$\Phi(g) = \frac{f^2}{6}\left(1 - \frac{1}{6}\sum_{N=0}^\infty\left(-\frac{1}{6}\right)^N\left(\frac{1}{\Box}R\right)^{N+1}\right) \qquad (4.12)$$

and using (4.11), which induces term by term cancellations, we get finally

$$\delta_\sigma\Phi = -\sigma\Phi \qquad (4.13)$$

or, for finite transformations,

$$\Phi \to e^{-\sigma}\Phi \qquad (4.14)$$

The transformation (4.14) cannot be used for a constant σ or for any Weyl transformation which does not vanish at infinity. Using (4.14) we can calculate the variation of $\bar{S}^{(0)}$:

$$\delta_\sigma\bar{S}^{(0)} = \int d^4x\,\sqrt{g}\left[4\sigma R\Phi + (-2\sigma R + 6\Box\sigma)\Phi - R\sigma\Phi\right]$$
$$= \int d^4x\,\sqrt{g}\,\sigma\,(6\Box + R)\Phi = 0 \qquad (4.15)$$

where in the last step we used (4.7).

Therefore, the first term of the generating functional is invariant under "local" Weyl transformations: this implies that the Ward identities are obeyed. On the other hand, as it depends on the dimensionful parameter f, it is obviously not invariant under dilations.

We now include in the integration out the terms which are responsible for the anomalies, i.e. W_a and W_c. After the change of variables (4.3), τ in the Ws is replaced by $-f \log(1 - \varphi/f)$, understood as a formal expansion in powers of φ. We work at tree level, as it is usually done for the infrared effective actions, treating the contributions of W as additional vertices:

$$\sum_{n=0}^{\infty} c_n(g)\varphi^n \tag{4.16}$$

We classify the contributions by the number of vertices of type (4.16) which they contain, such that the term (4.5) corresponds indeed to $\bar{S}^{(0)}$.

Let us calculate then $\bar{S}^{(1)}$, i.e. the contribution of a single "anomaly vertex". Each φ line in (4.16) connects to a $-\frac{1}{3} f\varphi R$ vertex through a $\frac{1}{2[\Box + R/6]}$ propagator. As a result each φ in (4.16) gets replaced by:

$$\frac{1}{6} f \frac{1}{\Box + \frac{1}{6}R} R = f\left(1 - \frac{6\Phi(g)}{f^2}\right) \tag{4.17}$$

After resumming (4.16) we get W with the dilaton τ replaced by:

$$-f \log(1 - \varphi/f) \rightarrow -f \log\left(\frac{6\Phi}{f^2}\right) \tag{4.18}$$

Therefore the $\bar{S}^{(1)}$ contribution is simply:

$$\bar{S}^{(1)}(g) = W\left(\tau = -f \log\left(\frac{6\Phi}{f^2}\right), g\right) \tag{4.19}$$

Then $\bar{S}^{(1)}$ reproduces the anomalies by construction since following from (4.18) and (4.14) one has the correct transformation rule:

$$-f \log\left(\frac{6\Phi}{f^2}\right) \rightarrow -f \log\left(\frac{6\Phi e^{-\sigma}}{f^2}\right) = -f \log\left(\frac{6\Phi}{f^2}\right) + \sigma f \tag{4.20}$$

needed in the Wess–Zumino procedure.

We calculate now the contribution of diagrams with two vertices of type (4.16), $c_n(g)\varphi^n$ and $c_m(g)\varphi^m$. One line from each vertex is joined by a propagator while all the other lines join the $-\frac{1}{3} f\phi R$ vertex. There is an nm symmetry factor. Resumming all the terms the symmetry factors can be reabsorbed into a functional derivative with respect to φ while φ gets replaced as before by (4.17). We get, therefore, the contribution

$$\bar{S}^{(2)} = \int d^4x \sqrt{g} \frac{1}{\sqrt{g}} \frac{\delta W}{\delta \varphi}\bigg| \frac{-1}{2(\Box + \frac{1}{6}R)} \frac{1}{\sqrt{g}} \frac{\delta W}{\delta \varphi}\bigg| \tag{4.21}$$

where

$$\frac{\delta W}{\delta \varphi}\bigg| \equiv \frac{\delta W}{\delta \varphi}\left(\varphi = f\left(1 - \frac{6\Phi}{f^2}\right), g\right) \tag{4.22}$$

The functional derivative (4.22) can be evaluated by the chain rule:

$$\frac{\delta W}{\delta \varphi} = \frac{\delta W}{\delta \tau}\frac{\delta \tau}{\delta \varphi} = \frac{\delta W}{\delta \tau}\frac{1}{1 - \varphi/f} = \frac{f^2}{6}\frac{1}{\varPhi(g)}\frac{\delta W}{\delta \tau} \tag{4.23}$$

We now prove that

$$\frac{\delta W}{\delta \tau} \tag{4.24}$$

is Weyl invariant: the generator of Weyl transformations

$$\frac{\delta}{\delta \sigma} = \frac{\delta}{\delta \tau} + 2 g_{mn}\frac{\delta}{\delta g_{mn}} \tag{4.25}$$

commutes with

$$\frac{\delta}{\delta \tau} \tag{4.26}$$

It follows that

$$\frac{\delta}{\delta \sigma}\frac{\delta W}{\delta \tau} = \frac{\delta}{\delta \tau}\frac{\delta W}{\delta \sigma} = \frac{\delta}{\delta \tau}\mathcal{A}(g) = 0 \tag{4.27}$$

where we used the fact that the Wess–Zumino construction reproduces the anomaly, which depends only on the metric. Using (4.14) and the transformation

$$\frac{1}{6\Box + R} \to e^{-\sigma}\frac{1}{6\Box + R}e^{3\sigma} \tag{4.28}$$

it is easy to prove that $\bar{S}^{(2)}$ is Weyl invariant. A similar treatment shows that all the higher $\bar{S}^{(n)}$ are Weyl invariant. We have therefore complete information about the generating functional associated with the minimal dilaton action $\bar{S}$. If we decompose it according to the number n of contributing "anomaly vertices"

$$\bar{S}(g) = \sum_{n=0}^{\infty}\bar{S}^{(n)}(g) \tag{4.29}$$

the $\bar{S}^{(1)}$ term reproduces the anomalies while all the other terms are invariant under local Weyl transformations. The term $\bar{S}^{(1)}$ was proposed as a generating functional in [18].

   All terms have dimensionful parameters and therefore break dilation invariance. An exception is the three curvature term in $\bar{S}^{(1)}$ which, however, does not respect the constraints following from special conformal transformations. The generating functional has the correct analyticity for the broken phase, i.e. tree diagrams with dilaton poles.

   The appearance of an infinite number of terms (tree diagrams) is not surprising: if one integrated out the Goldstone boson field in a chiral Lagrangian, reproducing the nonabelian chiral anomalies, one would get rather similar expressions.

   We discuss now the relation of this generating functional to the one introduced in [8], which also reproduces the trace anomalies. We first rederive this Lagrangian starting from an alternative dilaton effective action.

   Consider again the Wess–Zumino procedure, but this time applied to a type A anomaly which is modified by a cohomologically trivial piece, i.e. replace E_4 by $E_4 - \frac{2}{3}\Box R$. Then one obtains an action (without any additional kinetic part):

$$\tilde{S} = a \int d^4 x \sqrt{g} \left\{ \frac{1}{2} \tilde{\tau} \Box^2 \tilde{\tau} - 2 \nabla^m \tilde{\tau} \left(R_{mn} - \frac{1}{3} g_{mn} R \right) \nabla^n \tilde{\tau} + \tilde{\tau} \left(E_4 - \frac{2}{3} \Box R \right) \right\}$$
$$+ c \int d^4 x \sqrt{g} \tilde{\tau} C_{mnpq} C^{mnpq} \tag{4.30}$$

It is easy to check that this action reproduces the anomalies (including the cohomologically trivial piece of course) with the field $\tilde{\tau}$ having the transformation:

$$\tilde{\tau} \to \tilde{\tau} + \sigma \tag{4.31}$$

Interestingly the action has a quadratic form: in this sense one does not need to add any kinetic term. On the other hand the kinetic term starts with a $\Box^2$ which raises doubts about the analytic structure of the generating functional. This problem was studied in [19]. We reanalyze the analytic structure of the RFT generating functional [8], starting from the dilaton effective action (4.30).

We calculate the terms cubic in curvatures which follow from (4.30). We treat the first term as the kinetic part, which gives a $1/(a\Box^2)$ propagator, and the other terms as interactions. For the cubic terms one $\Box$ always cancels and the terms have a "normal" analytic structure. For example, the contribution originating from the terms in (4.30) linear in the curvature becomes:

$$(-2) \frac{4}{9} a \Box R \frac{1}{\Box^2} \overleftarrow{\nabla}^m \left(R_{mn} - \frac{1}{3} g_{mn} R \right) \nabla^n \frac{1}{\Box^2} \Box R$$
$$= -\frac{8}{9} a R \frac{1}{\Box} \overleftarrow{\nabla}^m \left(R_{mn} - \frac{1}{3} g_{mn} R \right) \overrightarrow{\nabla}^n \frac{1}{\Box} R \tag{4.32}$$

The same cubic terms can be reproduced from the "minimal" dilaton action (4.2). The $\mathcal{O}(\tau^3)$ contribute and lead to the relative coefficient $1/3$ between the two terms in (4.32). The cubic terms which carry the information about the anomalies coincide, however they are "completed" in different ways by the two actions.

If we continue integrating out of the RFT action, at the four curvature order we find contributions in which the $\Box^{-2}$ propagator remains uncanceled. We have to conclude therefore that the analytic structure of the RFT action is not compatible with the general requirements, even though the cubic terms responsible for the anomalies are correctly reproduced.

5. Conclusions

We presented various arguments that the trace anomaly coefficients are matched in the two phases of a spontaneously broken conformal field theory.

The matching happens through the contributions of the dilaton in the broken phase replacing the contributions of the massless states of the unbroken phase which became massive in the broken one. The matching determines the dilaton couplings to various combinations of energy momentum tensors in terms of the trace anomaly coefficients characterizing the conformal theory.

The couplings are summarized by the dilaton effective action. A minimal form for this action was calculated relating it through supersymmetry to the minimal $U(1)$ Goldstone boson action. Integrating out the dilaton we obtained the generating functional for energy momentum correlators in the broken phase. This generating functional has the correct and expected analyticity structure, i.e. tree diagrams with simple Goldstone boson poles.

A natural question to ask is what is the form of the generating functional for the unbroken phase. This action should have an analytic structure completely different from the ones we discussed, in particular it should not have any poles. The difference should appear already at the three curvature level since the terms produced by (4.32) differ from the three energy momentum tensor correlators calculated in [20] assuming an invariant vacuum i.e. in the unbroken phase. The generating functional produced by massless free matter, expanded in powers of the curvature, was computed in [21].

An explicit check in a nontrivial set up of the trace anomaly matching proposed in this paper can be done for the Coulomb branch of $\mathcal{N} = 4$ SUSY Yang–Mills theories. The matching can be checked both at weak coupling, perturbatively and at strong coupling using AdS/CFT duality [9].

Acknowledgements

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