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Abstract

The Hilbert space of loop quantum gravity is usually described in terms of cylindrical functionals of the gauge connection, the electric fluxes acting as non-commuting derivation operators. It has long been believed that this non-commutativity prevents a dual flux (or triad) representation of loop quantum gravity to exist. We show here, instead, that such a representation can be explicitly defined, by means of a non-commutative Fourier transform defined on the loop gravity state space. In this dual representation, flux operators act by $\star$-multiplication and holonomy operators act by translation. We describe the gauge invariant dual states and discuss their geometrical meaning. Finally, we apply the construction to the simpler case of a U(1) gauge group and compare the resulting flux representation with the triad representation used in loop quantum cosmology.

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1. Introduction

Loop quantum gravity (LQG) [1, 2] is now a solid and promising candidate framework for a quantum theory of gravity in four spacetime dimensions. It is based on the canonical quantization of the phase space of general relativity in the Ashtekar formulation, using rigorous functional techniques as well as ideas and tools from lattice gauge theory. Diffeomorphism invariance of the classical theory is a crucial ingredient of the construction, both conceptually and mathematically, and background independence is the guiding principle inspiring it. The main achievement to date in this framework is the complete definition of the kinematical space of (gauge and diffeomorphism invariant) states of quantum geometry, based on the conjugate

pair of variables given by holonomies $h_e[A]$ of the Ashtekar SU(2) connection A, and fluxes of the Ashtekar electric field E (densitized triads) across 2-surfaces. These states are described in terms of the so-called cylindrical functionals $\Psi[A]$ of the connection, which depend on A via holonomies along graphs. Under suitable assumptions involving a requirement of diffeomorphism invariance, the representation of the algebra generated by holonomies and fluxes, hence the definition of the state space, is unique [3].

A crucial, and somewhat surprising fact is that the flux variables, even at the classical level, do not (Poisson) commute [4, 5]. This non-commutativity is generic and necessary, once holonomies of the Ashtekar connection are chosen as their conjugate variables. In the simplest case, for a given fixed graph, fluxes across surfaces dual to a single edge act as invariant vector fields on the group, and have the symplectic structure of the $\mathfrak{su}(2)$ Lie algebra. Thus, the phase space associated with a graph is a product over the edges of cotangent bundles $T^*\mathrm{SU}(2) \simeq \mathrm{SU}(2) \times \mathfrak{su}(2)$ on the gauge group. For this case the Poisson structure for one edge e (variables associated with different edges will commute) is simply given by

$$
\begin{aligned}
\{h[A], h[A]\} &= 0 \\
\{E^i, h[A]\} &= \tau^i h[A] \\
\{E^i, E^j\} &= -\epsilon^{ijk} E^k.
\end{aligned}
\tag{1}
$$

Here, E^i is the flux through an elementary (i.e. dual to a single edge e) surface S_e with unit smearing function in a neighborhood of the intersection point $e \cap S_e$.[4] Recent works have shown that the structure of this phase space can also be understood from a simplicial geometric point of view [6–8].

The fact that non-commutative structures are at the very root of the LQG formalism is well known for a long time [4]. However, to our knowledge, it has not been built upon to any extent in the LQG literature, and the full implications of it, as well as the consequent links between the LQG approach and non-commutative geometry ideas and tools, have remained unexplored. In fact, it is often believed that non-commutativity of the fluxes implies that the framework has no flux (or triad) representation (for earlier attempts, see e.g [9]). The goal of this paper is to show, instead, that this non-commutativity is naturally encoded in a definition of a non-commutative Fourier transform and $\star$-product, and that these can be used to build up a well-defined non-commutative flux representation for generic LQG states.

The idea if defining a non-commutative flux representation for LQG originates from developments in the spin foam context [10–12], and especially in the context of group field theory [13]. Much of the recent progress in spin foam models stemmed from the use of a coherent state basis [14–20] to express both quantum states and amplitudes. This basis has the advantage, as compared to the standard spin network basis in LQG, of a clearer and more direct geometric interpretation of the labels that characterize it, in terms of metric variables. This allowed a more consistent encoding of geometric constraints in the definition of the spin foam amplitudes, a nice characterization of the corresponding boundary states and of the semi-classical limit of the same amplitudes, relating them with simplicial gravity actions. The same aim also motivated recent work attempting to introduce metric variables in the group field theory framework [21, 22]. This line of research has resulted in a new representation of group field theory in terms of non-commutative metric variables [23], which could in fact be directly interpreted as discrete (smeared) triads (in the SU(2) case). In this representation, where non-commutativity of metric variables is brought to the forefront and used in the very definition of the group field theory model, the Feynman amplitudes have the form of simplicial gravity path integrals in the same metric variables. These results suggest to explore a similar

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[4] Note that we are working with rescaled flux variables. Thus, the Immirzi parameter γ is implicitly hidden in the relation between Ashtekar's electric field and the triad through $E_j^a = \frac{1}{\gamma} \sqrt{\det q}\, e_j^a$.

metric representation for LQG states, since the group field can be interpreted as the (second quantized) wavefunction for a LQG spin network vertex. We exhibit such a representation here, and show that the whole construction of the LQG Hilbert space can be performed in this new representation as well.

We expect this new non-commutative flux representation to be useful in many respects. First of all it would help clarifying the quantum geometry of LQG states, including the relation with simplicial geometry [6, 7]. Thanks to this, it may facilitate the definition of the dynamics of the theory, both in the canonical (Hamiltonian or Master constraint) [1] and covariant (spin foam or GFT) setting [23], and the coupling of matter fields [24–28]. Further down the line, it offers a new handle for tackling the issue of the semi-classical limit of the theory. All these advantages of a metric representation are in fact shown already in the simpler context of loop quantum cosmology (LQC), where such a representation has already been developed and used successfully [29, 30]. Obviously, the new representation brings LQG closer to the language and framework of non-commutative geometry [31], thus possibly fostering further progress.

The paper is organized as follows. In section 2, in order to make this paper self-contained, we review the standard construction of the kinematical Hilbert space of LQG in the connection representation. The careful mathematical treatment of this review section will reveal useful for the rigorous construction of the new representation.

In section 3, we define the Fourier transform underlying the flux representation. The key technical ingredient is a generalization of the group Fourier transform [32, 33] to the whole LQG space of connections. In section 4, we further describe the new representation: we give the action of the fundamental operators and we discuss properties of the gauge invariant dual states, clarifying their geometric meaning and the relation with the spin network basis. Finally, in section 5, we discuss the analogous construction in the simpler case of U(1) and comment on its relation with the triad representation used in LQC. We conclude with a brief outlook on possible further developments.

2. The Hilbert space of loop gravity

Kinematical (gauge covariant) states in LQG are functions on a space $\bar{\mathcal{A}}$ of suitably generalized connections [34]. A cornerstone of the framework is the fact that the state space $\mathcal{H}_0$ can be defined by induction from a family of the Hilbert spaces $\mathcal{H}_\gamma = L^2(\mathcal{A}_\gamma, \mathrm{d}\mu_\gamma)$, labeled by graphs embedded in the spatial manifold σ. For a given graph γ with n edges, $\mathcal{A}_\gamma$ is a space of (distributional) connections on γ, naturally identified with the product G^n of n copies of the gauge group; $\mathrm{d}\mu_\gamma$ is the product Haar measure on G^n. The construction stems from a characterization of $\bar{\mathcal{A}}$ as a projective limit of the spaces $\mathcal{A}_\gamma$.

In this section, we briefly recall this standard construction, as we will use it to define the Fourier transform in section 3. We will assume that G is any compact group, though having in mind the cases $G = \mathrm{SU}(2)$ or SO(3) relevant to gravity. Further details can be found in the original articles [35–37] or in the textbook [1].

2.1. Generalized connections

Given any smooth connection A on Σ, one can assign a group element A_e to each path e in Σ, by considering the holonomy of A along e. This assignment respects composition and inversion of paths:

$$A_{e_1 \circ e_2} = A_{e_1} A_{e_2}, \qquad A_{e^{-1}} = A_e^{-1}.$$

In other words, the connection gives a morphism from the groupoid of paths to the gauge group G. The space $\bar{\mathcal{A}}$ of 'generalized connections' is defined as the set $\mathrm{Hom}(\mathcal{P}, G)$ of all
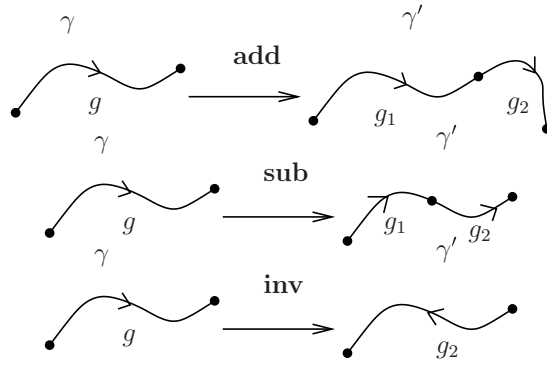
Figure 1. Elementary moves relating ordered graphs.

such morphisms. It contains the smooth connections, but also distributional ones. $\bar{\mathcal{A}}$ shows up as the quantum configuration space in LQG.

An independent and very useful characterization of $\bar{\mathcal{A}}$ makes use of projective techniques [34], based on the set of embedded *graphs*. A graph $\gamma = (e_1, \ldots, e_n)$ is a finite set of analytic paths with 1- or 2-endpoint boundary, such that every two distinct paths intersect only at one or two of their endpoints. The path components e_i are called the edges of γ; the endpoints of an edge are called vertices. The set of all graphs has the structure of a partially ordered and directed set: we say γ' is larger than γ, and we write $\gamma' \geqslant \gamma$, when every edge of γ can be obtained from a sequence of edges in γ' by composition and/or orientation reversal; then, for any two graphs γ_1, γ_2, there exists a graph γ_3 such that $\gamma_3 \geqslant \gamma_1, \gamma_2$.

For a given graph γ, let $\mathcal{A}_\gamma := \mathrm{Hom}(\bar{\gamma}, G)$ be the set of all morphisms from the subgroupoid $\bar{\gamma} \subset \mathcal{P}$ generated by all the n edges of γ, to the group G. $\mathcal{A}_\gamma$ is naturally identified with G^n, both set-theoretically and topologically. For any two graphs such that $\gamma' \geqslant \gamma$, $\bar{\gamma}$ is a subgroupoid of $\bar{\gamma}'$; we thus have a natural projection $p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$, restricting to γ any morphism in $\mathcal{A}_{\gamma'}$. These projections are surjective and satisfy the rule

$$p_{\gamma\gamma'} \circ p_{\gamma'\gamma''} = p_{\gamma\gamma''}, \qquad \forall \gamma'' \geqslant \gamma' \geqslant \gamma. \tag{2}$$

This defines a projective structure for the spaces $\mathcal{A}_\gamma$. It can be shown [37] that the space $\bar{\mathcal{A}}$ coincides with the projective limit of the family $(\mathcal{A}_\gamma, p_{\gamma\gamma'})$; namely, a generalized connection can be viewed as one of those elements $\{A_\gamma\}_\gamma$ of the direct product $\times_\gamma \mathcal{A}_\gamma$ such that

$$p_{\gamma\gamma'} A_{\gamma'} = A_\gamma, \qquad \forall \gamma' \geqslant \gamma.$$

Such a characterization allows us to endow $\bar{\mathcal{A}}$ with the topology of a compact Hausdorff space.

Let us close this section with a property of the projections $p_{\gamma\gamma'}$ that will be useful for us. Given any two ordered graphs $\gamma' \geqslant \gamma$, the larger one γ' may be obtained from the smaller one γ by a sequence of three elementary moves: (i) *adding* an edge, (ii) *subdividing* an edge by adding a new vertex and (iii) *inverting* an edge (see figure 1).

Together with the consistency rule (2), this means that the projections $p_{\gamma\gamma'}$ can be decomposed into the following elementary projections onto the space $\mathcal{A}_e$ of connections on a single edge e:

$$
\begin{aligned}
p_{\mathrm{add}} &: \mathcal{A}_{e,e'} \to \mathcal{A}_e; & (g, g') &\mapsto g \\
p_{\mathrm{sub}} &: \mathcal{A}_{e_1,e_2} \to \mathcal{A}_e; & (g_1, g_2) &\mapsto g_1 g_2 \\
p_{\mathrm{inv}} &: \mathcal{A}_e \to \mathcal{A}_e; & g &\mapsto g^{-1},
\end{aligned}
\tag{3}
$$

where we have used the identification $A_\gamma := (g_1, \ldots, g_n)$ of $\mathcal{A}_\gamma$ with G^n.

2.2. Inductive structure of $\mathcal{H}_0$

Having understood the projective structure of the space of generalized connections:

$$\bar{\mathcal{A}} \simeq \{\{A_\gamma\}_\gamma \in \times_\gamma \mathcal{A}_\gamma : \quad p_{\gamma\gamma'} A_{\gamma'} = A_\gamma \quad \forall \gamma' \geqslant \gamma\},$$

we now illustrate how to define the LQG state space $\mathcal{H}_0$ by an appropriate 'glueing' of the much more tractable spaces $\mathcal{H}_\gamma = L^2(\mathcal{A}_\gamma, \mathrm{d}\mu_\gamma)$. The idea is to define functions on $\mathcal{A}$ as equivalence classes of elements in $\cup_\gamma \mathcal{H}_\gamma$ for a certain equivalence relation which reflects the projective structure of $\mathcal{A}$.

Let us introduce the family of injective maps $p^*_{\gamma'\gamma} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma'}, \gamma' \geqslant \gamma$, obtained by pull back of the projections $p_{\gamma\gamma'} : \mathcal{A}_{\gamma'} \to \mathcal{A}_\gamma$ defined in section 2.1. Thus, $p^*_{\gamma'\gamma}$ acts on $f_\gamma \in \mathcal{H}_\gamma$ as

$$p^*_{\gamma'\gamma} : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma'}, \quad (p^*_{\gamma'\gamma} f_\gamma)[A_{\gamma'}] = f_\gamma[p_{\gamma\gamma'} A_{\gamma'}]. \tag{4}$$

These injective maps satisfy a rule analogous to (2):

$$p^*_{\gamma''\gamma'} \circ p^*_{\gamma'\gamma} = p^*_{\gamma''\gamma}, \qquad \forall \gamma'' \geqslant \gamma' \geqslant \gamma. \tag{5}$$

Just as for the projections $p_{\gamma\gamma'}$, the maps $p^*_{\gamma\gamma'}$ can be decomposed into three elementary injections $\mathbf{add} := p^*_{\mathrm{add}}, \mathbf{sub} := p^*_{\mathrm{sub}}$ and $\mathbf{inv} := p^*_{\mathrm{inv}}$, which encode the transformation of the functions when adding, subdividing and inverting an edge of a graph. These elementary injections act on the space $\mathcal{H}_e$ associated with a single edge as

$$\mathbf{add} : \mathcal{H}_e \to \mathcal{H}_{e,e'};$$
$$f(g) \mapsto (\mathbf{add}\, f)(g, g') := f(g)$$
$$\mathbf{sub} : \mathcal{H}_e \to \mathcal{H}_{e_1, e_2};$$
$$f(g) \mapsto (\mathbf{sub}\, f)(g_1, g_2) := f(g_1 g_2) \tag{6}$$
$$\mathbf{inv} : \mathcal{H}_e \to \mathcal{H}_e;$$
$$f(g) \mapsto (\mathbf{inv}\, f)(g) := f(g^{-1}),$$

where we have used once again the identification $A_\gamma := (g_1, \ldots, g_n)$ of $\mathcal{A}_\gamma$ with G^n. Using these elementary maps, as well as the translation and inversion invariance and the normalization of the Haar measure, it can be checked that the $p^*_{\gamma\gamma'}$ are isometric embeddings $\mathcal{H}_\gamma \hookrightarrow \mathcal{H}_{\gamma'}$, namely injective maps preserving the inner product. This expresses the fact that $(\mathcal{H}_\gamma, p^*_{\gamma'\gamma})_{\gamma' \geqslant \gamma}$ defines an inductive family of Hilbert spaces.

We now define an equivalence relation on $\cup_\gamma \mathcal{H}_\gamma$ by setting

$$f_{\gamma_1} \sim f_{\gamma_2} \quad \Longleftrightarrow \quad \exists \gamma_3 \geqslant \gamma_1, \gamma_2, \quad p^*_{\gamma_3\gamma_1} f_{\gamma_1} = p^*_{\gamma_3\gamma_2} f_{\gamma_2}.$$

The quotient space can be endowed with an inner product which naturally extends the inner products $\langle , \rangle_\gamma$ of each $\mathcal{H}_\gamma$. Let indeed $f_{\gamma_1}, f_{\gamma_2}$ be two functions in $\cup_\gamma \mathcal{H}_\gamma$. The set of graphs is directed, so we may pick a graph γ_3 such that $\gamma_3 \geqslant \gamma_1, \gamma_2$. It can then easily be shown using the rule (5) and the fact that the maps $p^*_{\gamma'\gamma}$ preserve the inner products that the quantity

$$\langle f_{\gamma_1}, f_{\gamma_2} \rangle := \langle p^*_{\gamma_3\gamma_1} f_{\gamma_1}, p^*_{\gamma_3\gamma_2} f_{\gamma_2} \rangle_{\gamma_3}$$

does not depend on the chosen larger graph γ_3, and is well defined on the equivalence classes $f_1 := [f_{\gamma_1}]$ and $f_2 := [f_{\gamma_2}]$. Hence, it defines an inner product on the quotient space $\cup_\gamma \mathcal{H}_\gamma / \sim$. The completion of this quotient space with respect to the inner product is called the inductive limit of the inductive family $(\mathcal{H}_\gamma, p^*_{\gamma'\gamma})_{\gamma' \geqslant \gamma}$. It can be shown (see for example [1]) that the limit

$$\mathcal{H}_0 = \overline{\cup_\gamma \mathcal{H}_\gamma / \sim} \tag{7}$$

coincides with the space $L^2(\bar{\mathcal{A}}, \mathrm{d}\mu_0)$ of square integrable functions on $\bar{\mathcal{A}}$, with respect to a gauge and diffeomorphism invariant measure—the so-called Ashtekar–Lewandowski measure [37]. This is the kinematical (gauge covariant) state space of LQG.

2.3. Quantum theory on $\mathcal{H}_0$

Let us fix a graph $\gamma = (e_1, \ldots, e_n)$, and identify $\mathcal{H}_\gamma$ with $L^2(G^n)$, where the L^2-measure is the product Haar measure. The fundamental operators arising from the quantization, on $\mathcal{H}_\gamma$, of a classical phase space given by a cotangent bundle T^*G^n act respectively by multiplication by a smooth function φ_γ of G^n, and as generators of (right) actions of G in (a dense subset of) $\mathcal{H}_\gamma$:

$$(\widehat{\varphi}_\gamma \, f_\gamma)(g_1, \ldots, g_n) := \varphi_\gamma(g_1, \ldots, g_n) f_\gamma(g_1, \ldots, g_n) \tag{8}$$

$$(\widehat{L}_e^i \, f_\gamma)(g_1, \ldots, g_n) := \left. \frac{\mathrm{d}}{\mathrm{d}t} f_\gamma(g_1, \ldots, g_e \, \mathrm{e}^{t\tau_i}, \ldots, g_n) \right|_{t=0}, \tag{9}$$

where τ^i is a basis of $\mathfrak{su}(2)$, say i times the Pauli matrices, $\tau_i = \mathrm{i}\sigma_i$. $\widehat{L}_e^i$ is the left-invariant vector field on the copy of G associated with the edge e. This provides the quantum theory on the graph γ, with well-defined momenta operators, whose algebra has the structure of $\mathfrak{su}(2)^n$.

The action (8) can easily be extended to the quotient $\cup_\gamma \mathcal{H}_\gamma / \sim$. For φ_{γ_1} and f_{γ_2} associated with different graphs, pick a graph γ larger than both γ_1 and γ_2, and define $\widehat{\varphi}_{\gamma_1} f_{\gamma_2}$ as the equivalence class $[\widehat{\varphi}_\gamma f_\gamma]$ of (8). This action does not depend on the representatives chosen in the equivalence classes $\varphi := [\varphi_{\gamma_1}]$ and $f := [f_{\gamma_2}]$; it defines the action of the holonomy operator $\widehat{\varphi}$ on generic states of $\mathcal{H}_0$. The operator (9) should be interpreted as the flux $E_{S_e}^i := E(S_e, \tau_i)$ of the electric field across an 'elementary' surface[5] S_e cut by the edge e. More generally, the LQG flux operator across a surface S acts on $f = [f_\gamma]$ as a sum of left-invariant derivatives on $f_{\gamma'}$, where $\gamma' \geqslant \gamma$ cuts S at its vertices, with only outgoing edges, the sum being over all the intersection points of $\gamma' \cap S$ and their adjacent edges:

$$\widehat{E}_S^i \, f_\gamma = \sum_{v \in \gamma' \cap S} \sum_{e \supset v} \epsilon(S, e) \widehat{L}_e^i \, f_\gamma,$$

where $\epsilon(S, e) = \pm$ depends on the relative orientation of the edge and the surface.

One can also define, on each $\mathcal{H}_\gamma$, operators $\widehat{g_v}$ generating gauge transformations at each vertex of $v \in \gamma$. These act on a state f_γ as

$$(\widehat{g_v} \, f_\gamma)(g_1, \ldots, g_n) = f_\gamma\big(g_{s_1}^{-1} g_1 g_{t_n}, \ldots, g_{s_n}^{-1} g_n g_{t_n}\big), \tag{10}$$

where s_e and t_e denote source and target vertices of the oriented graph e, respectively. Gauge invariance is thus imposed by acting with the gauge averaging operator

$$\mathcal{P}_\gamma := \bigotimes_v \int \mathrm{d}g_v \widehat{g_v}.$$

It can be checked that the action of such operators is well defined on equivalence classes.

Finally, the so-called spin network basis of $\mathcal{H}_0$ is a very convenient one for actual computations. Such a basis is obtained by harmonic analysis on the gauge group: using the Peter–Weyl theorem, a state $f_\gamma \in \mathcal{H}_\gamma$ can be decomposed into a product of Wigner functions $D_{m_e n_e}^{j_e}(g_e)$ for each edge, labeled by irreducible representations of G ($j \in \frac{1}{2}\mathbb{N}$ for $G = \mathrm{SU}(2)$ or $j \in \mathbb{N}$ for $\mathrm{SO}(3)$), and magnetic numbers $-j_e \leqslant m_e \leqslant j_e$ and $-j_e \leqslant n_e \leqslant j_e$.

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[5] Actually, there exist different proposals to which classical quantities the quantum flux operators should correspond. In [5] it was shown that they can also be interpreted as quantum versions of a different set of classical functions involving the holonomies *and* the triads. The construction performed there is based on a family of graphs γ *and* dual graphs γ^* and the classical continuum phase space is understood as a certain generalized projective limit of graph–phase spaces of the form $T^*\mathrm{SU}(2)^n$. In section 4.2, we will see that this interpretation is also favored from the dual (Fourier transformed) point of view.

These quantum numbers are usually interpreted as encoding geometric variables; in particular, the spin j labels the eigenvalues of area operators. In the next section, we define a Fourier transform on $\mathcal{H}_0$ that will provide an alternative decomposition of the LQG states into functions of *continuous* Lie algebra variables naturally interpreted as flux (triad) variables.

3. Fourier transform on the LQG state space

Here, we define the non-commutative Fourier transform that will give the dual flux representation. This transform generalizes the 'group Fourier transform' introduced in [32, 33, 38] to theories of connections. We first recall the main features of the group Fourier transform and use it to construct a family of Fourier transforms $\mathcal{F}_\gamma$ defined on $\mathcal{H}_\gamma$. We then show how this family extends to a transform $\mathcal{F}$ defined on the whole space $\mathcal{H}_0$. We emphasize that, to avoid unnecessary complications, we will work from now on with the gauge group $G = \mathrm{SO}(3)$. With more work, the construction can be extended to the SU(2) case, using the SU(2) group Fourier transform spelled out in [38].

3.1. Group Fourier transform

The SO(3) Fourier transform $\mathcal{F}$ maps isometrically $L^2(\mathrm{SO}(3), \mathrm{d}\mu_H)$, equipped with the Haar measure $\mathrm{d}\mu_H$, onto a space $L^2_\star(\mathbb{R}^3, \mathrm{d}\mu)$ of functions on $\mathfrak{su}(2) \sim \mathbb{R}^3$ equipped with a non-commutative $\star$-product, and the standard Lebesgue measure $\mathrm{d}\mu$. Just as for the standard Fourier transform on $\mathbb{R}^n$, the construction of $\mathcal{F}$ stems from the definition of plane waves:

$$\mathrm{e}_g : \mathfrak{su}(2) \sim \mathbb{R}^3 \to \mathrm{U}(1), \qquad \mathrm{e}_g(x) = \mathrm{e}^{\mathrm{i}\vec{p}_g \cdot \vec{x}}$$

depending on a choice of coordinates $\vec{p}_g$ on the group manifold. For a given choice of such coordinates, $\mathcal{F}$ is defined on $L^2(\mathrm{SO}(3))$ as

$$\mathcal{F}(f)(x) = \int \mathrm{d}g\, f(g)\, \mathrm{e}_g(x), \tag{11}$$

where $\mathrm{d}g$ is the normalized Haar measure on the group[6].

Let us fix our conventions and notations. In what follows we will identify functions of $\mathrm{SO}(3) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$ with functions of SU(2) which are invariant under the transformation[7] $g \to -g$. We denote by τ_i, $i = 1, 2, 3$, the generators of $\mathfrak{su}(2)$ algebra, chosen to be i times the (Hermitian) Pauli matrices. They are normalized as $(\tau_i)^2 = -\mathbb{1}$ and satisfy $[\tau_i, \tau_j] = -2\epsilon_{ijk}\tau_k$. We choose coordinates on SU(2) given by

$$\vec{p}_g = -\tfrac{1}{2}\,\mathrm{Tr}(|g|\vec{\tau}), \qquad |g| := \mathrm{sign}(\mathrm{Tr}\,g)\,g,$$

where 'Tr' is the trace in the fundamental representation. The presence of the factor $\mathrm{sign}(\mathrm{Tr}\,g)$ ensures that $\vec{p}_g = \vec{p}_{-g}$. Using these conventions, writing $x = \vec{x} \cdot \vec{\tau}$ and $g = \mathrm{e}^{\theta\vec{n}\cdot\vec{\tau}}$ with $\theta \in [0, \pi]$ and $\vec{n} \in \mathcal{S}^2$, the plane waves take the form

$$\mathrm{e}_g(x) = \mathrm{e}^{-\frac{i}{2}\mathrm{Tr}(|g|x)} = \mathrm{e}^{\mathrm{i}\epsilon_\theta \sin\theta\, \vec{n}\cdot\vec{x}} \tag{12}$$

with $\epsilon_\theta = \mathrm{sign}(\cos\theta)$. Note that we may identify SO(3) to the upper hemisphere of $\mathrm{SU}(2) \sim \mathcal{S}^3$, parametrized by $\theta \in [0, \pi/2]$ and $\vec{n} \in \mathcal{S}^2$; on this hemisphere, we have $\epsilon_\theta = 1$.

The image of the Fourier transform (11) has a natural algebra structure inherited from the addition and the convolution product in $L^2(\mathrm{SO}(3))$. The product is defined on plane waves as

$$\mathrm{e}_{g_1} \star \mathrm{e}_{g_2} = \mathrm{e}_{g_1 g_2} \qquad \forall g_1, g_2 \in \mathrm{SU}(2) \tag{13}$$

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[6] Since SO(3) is compact, $L^2(\mathrm{SO}(3)) \subset L^1(\mathrm{SO}(3))$. Therefore, the Fourier transform is well defined.

[7] Here, g is understood as an element of the fundamental representation of G. The transformation $g \to -g := h_\pi g$ can be understood as the action of an appropriate h_π, also an element of the fundamental representation of G.

and extended by linearity to the image of $\mathcal{F}$. Using the identity

$$\int \mathrm{d}^3x\, e_g(x) = 4\pi [\delta_{SU(2)}(g) + \delta_{SU(2)}(-g)] := 8\pi \delta_{SO(3)}(g) \tag{14}$$

for the delta function on the group, with d^3x being the standard Lebesgue measure on $\mathbb{R}^3$, one may prove the inverse formula

$$f(g) = \frac{1}{8\pi} \int \mathrm{d}^3x\, (\mathcal{F}(f) \star e_{g^{-1}})(x),$$

which shows that $\mathcal{F}$ is invertible. Next, let us denote by $L^2_\star(\mathbb{R}^3)$ the image of $\mathcal{F}$ endowed with the following Hermitian inner product:

$$\langle u, v \rangle_\star := \frac{1}{8\pi} \int \mathrm{d}^3x\, (\overline{u} \star v)(x). \tag{15}$$

Writing $u = \mathcal{F}(f)$, $v = \mathcal{F}(h)$, the quantity $\langle u, v \rangle_\star$ can be written as

$$\begin{aligned}
\langle \mathcal{F}(f), \mathcal{F}(h) \rangle_\star &= \frac{1}{8\pi} \int \mathrm{d}g_1\, \mathrm{d}g_2\, \overline{f(g_1)} h(g_2) \int \mathrm{d}^3x\, (\overline{e_{g_1}} \star e_{g_2})(x) \\
&= \frac{1}{8\pi} \int \mathrm{d}g_1\, \mathrm{d}g_2\, \overline{f(g_1)} h(g_2) \int \mathrm{d}^3x\, e_{g_1^{-1}g_2}(x) \\
&= \int \mathrm{d}g\, \overline{f(g)} h(g),
\end{aligned}$$

where in the second line we used that $\overline{e_g(x)} = e_{g^{-1}}(x)$ as well as the identity (14). This establishes in one stroke that the inner product (15) is well defined, since f and h are square integrable, and that the Fourier transform defines a unitary equivalence $L^2(SO(3)) \simeq L^2_\star(\mathbb{R}^3)$.

There are alternative ways to characterize the image $L^2_\star(\mathbb{R}^3)$ of the Fourier transform. To do so, we may recast the transform (11) into a standard $\mathbb{R}^3$ Fourier transform, in terms of the coordinates $\vec{p}_g = \sin\theta \vec{n}$, with $\theta \in [0, \pi/2]$. Writing the Haar measure as $\mathrm{d}g = \frac{1}{\pi} \sin^2\theta\, \mathrm{d}\theta\, \mathrm{d}^2\vec{n}$, where $\mathrm{d}^2\vec{n} = \frac{1}{2}(\partial_i\vec{n} \times \partial_j\vec{n}, \vec{n}) \mathrm{d}x^i \wedge \mathrm{d}x^j$ $(i, j \in \{1, 2\})$ is the normalized measure on the unit sphere $\mathcal{S}^2$, leads to the integral formula

$$\mathcal{F}(f)(x) = \frac{1}{\pi} \int_{|p| \leqslant 1} \frac{\mathrm{d}^3\vec{p}}{\sqrt{1 - p^2}} f(g(\vec{p}))\, \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}}.$$

We thus see that the map $\mathcal{F}$ hits functions of $\mathbb{R}^3$ that have *bounded* Fourier modes $|\vec{p}_g| \leqslant 1$ for the standard $\mathbb{R}^3$ Fourier transform.

We may also think of elements of $L^2_\star(\mathbb{R}^3)$ as equivalence classes of functions of $\mathbb{R}^3$, for the relation identifying two functions with the same $\mathbb{R}^3$-Fourier coefficients for (almost-every) low modes $|\vec{p}| \leqslant 1$. Loosely speaking, this means that the elements of $L^2_\star(\mathbb{R}^3)$ 'probe' the space $\mathbb{R}^3$ with a finite resolution.

It is worth noting that the image of the Fourier transform has a discrete basis, as shown by taking the Fourier transform of the Peter–Weyl formula:

$$\widehat{f}(x) = \sum_{j,m,n} f^j_{mn} \widehat{D}^j_{mn}(x), \tag{16}$$

expressed in terms of the matrix elements of the dual Wigner matrices $\widehat{D}^j(x) = \int \mathrm{d}g\, e_g(x) D^j(g)$ in the SO(3) representation j.

3.2. Fourier transform on $\mathcal{H}_\gamma$

The extension of the above construction to functions of several copies of the group is straightforward and gives the Fourier transform $\mathcal{F}_\gamma$ on the space $\mathcal{H}_\gamma \simeq L^2(\mathrm{SO}(3)^n)$ associated with any graph with n edges. Given $\mathbf{g} := (g_1, \ldots, g_n) \in \mathrm{SO}(3)^n$, we define the plane waves $E_{\mathbf{g}}^{(n)} : \mathfrak{su}(2)^n \to \mathrm{U}(1)$ as a product of SO(3) plane waves:

$$E_{\mathbf{g}}^{(n)}(\mathbf{x}) := \prod_{i=1}^{n} e_{g_i}(x_i).$$

The Fourier transform $\mathcal{F}_\gamma$ is defined on $\mathcal{H}_\gamma$ by

$$\mathcal{F}_\gamma(f)(\mathbf{x}) = \int \prod_{i=1}^{n} \mathrm{d}g_i \, f(\mathbf{g}) E_{\mathbf{g}}^{(n)}(\mathbf{x}).$$

The $\star$-product acts on plane waves as

$$\left(E_{\mathbf{g}}^{(n)} \star E_{\mathbf{g}'}^{(n)} \right)(\mathbf{x}) := E_{\mathbf{g}\mathbf{g}'}^{(n)}(\mathbf{x}) = \prod_{i=1}^{n} e_{g_i g_i'}(x_i)$$

and is extended by linearity to the image of $\mathcal{F}_\gamma$. This image, endowed with the inner product

$$\langle u, v \rangle_{\star,\gamma} = \frac{1}{(8\pi)^n} \int \prod_{i=1}^{n} \mathrm{d}^3 x_i (\overline{u} \star v)(\mathbf{x}),$$

is a Hilbert space $L_\star^2(\mathbb{R}^3)^{\otimes n} := \mathcal{H}_{\star,\gamma}$. The Fourier transform provides an unitary equivalence between the Hilbert spaces $\mathcal{H}_\gamma$ and $\mathcal{H}_{\star,\gamma}$.

3.3. Cylindrical consistency and fourier transform on $\mathcal{H}_0$

We have defined a family of unitary equivalences $\mathcal{F}_\gamma : \mathcal{H}_\gamma \to \mathcal{H}_{\star,\gamma}$ labeled by graphs γ. In this section, we show the key technical result of this paper: this family extends to a map defined on the whole LQG state space

$$\mathcal{H}_0 = \overline{\cup_\gamma \mathcal{H}_\gamma / \sim},$$

defined in section 2.2.

First, the family $\mathcal{F}_\gamma$ gives a map $\cup_\gamma \mathcal{H}_\gamma \to \cup_\gamma \mathcal{H}_{\star,\gamma}$. In order to project it onto a well-defined map on the equivalence classes, we introduce the equivalence relation on $\cup_\gamma \mathcal{H}_{\star,\gamma}$ which is 'pushed forward' by $\mathcal{F}_\gamma$:

$$\forall u_{\gamma_i} \in \mathcal{H}_{\star,\gamma_i}, \quad u_{\gamma_1} \sim u_{\gamma_2} \iff \mathcal{F}_{\gamma_1}^{-1}(u_{\gamma_1}) \sim \mathcal{F}_{\gamma_2}^{-1}(u_{\gamma_2}).$$

For simplicity, we use the same symbol $\sim$ for the equivalence relation in the source and target space. We thus have a map $\widetilde{\mathcal{F}}$ making the following diagram commute:

$$
\begin{array}{ccc}
\cup_\gamma \mathcal{H}_\gamma & \xrightarrow{\;\;\mathcal{F}_\gamma\;\;} & \cup_\gamma \mathcal{H}_{\star,\gamma} \\
{\scriptstyle \pi} \downarrow & & \downarrow {\scriptstyle \pi_\star} \\
\cup_\gamma \mathcal{H}_\gamma / \sim & \xrightarrow{\;\;\widetilde{\mathcal{F}}\;\;} & \cup_\gamma \mathcal{H}_{\star,\gamma} / \sim
\end{array}
\qquad (17)
$$

where π and $\pi_\star$ are the canonical projections. Next, the quotient space $\cup_\gamma \mathcal{H}_{\star,\gamma} / \sim$ is endowed with a Hermitian inner product inherited from the inner products $\langle \, , \, \rangle_{\star,\gamma}$ on each $\mathcal{H}_{\star,\gamma}$. This is also the inner product which is 'pushed forward' by $\widetilde{\mathcal{F}}$. The inner product of two elements

u and v of the quotient space with representatives $u_{\gamma_1} \in \mathcal{H}_{\star,\gamma_1}$ and $v_{\gamma_2} \in \mathcal{H}_{\star,\gamma_2}$ is specified by choosing a graph $\gamma_3 \geqslant \gamma_1, \gamma_2$ and two elements $u_{\gamma_3} \sim u_{\gamma_1}$ and $v_{\gamma_3} \sim v_{\gamma_1}$ in $\mathcal{H}_{\star,\gamma_3}$, and by setting

$$\langle u, v \rangle_\star := \langle u_{\gamma_3}, v_{\gamma_3} \rangle_{\star,\gamma_3}. \tag{18}$$

In fact, we know by unitarity of $\mathcal{F}_{\gamma_3}$ that the right-hand side coincides with $\langle \mathcal{F}_{\gamma_3}^{-1}(u_{\gamma_1}), \mathcal{F}_{\gamma_3}^{-1}(v_{\gamma_2}) \rangle$, hence does not depend on the representatives $u_{\gamma_1}, v_{\gamma_2}$ nor on the graph γ_3.

It is worth giving a more concrete characterization of the space $\cup_\gamma \mathcal{H}_{\star,\gamma}/\sim$, by making the equivalence relation and the inner product more explicit without using $\mathcal{F}_\gamma$. As explained in section 2, there are three generators of equivalence classes in $\cup_\gamma \mathcal{H}_\gamma$, induced by the action on the set of graphs, consisting of adding, subdividing or changing the orientation of an edge. These generators are encoded into the operators **add**, **sub** and **inv** defined on $L^2(SO(3))$. To characterize the equivalence classes in the target space, we thus need to compute the dual action of these operators on $L^2_\star(\mathbb{R}^3)$. We will need to introduce the following family of functions:

$$\delta_x(y) := \frac{1}{8\pi} \int \mathrm{d}g \, e_{g^{-1}}(x) \, e_g(y). \tag{19}$$

These play the role of Dirac distributions in the non-commutative setting, in the sense that

$$\int \mathrm{d}^3 y (\delta_x \star f)(y) = \int \mathrm{d}^3 y (f \star \delta_x)(y) = f(x).$$

However, let us emphasize that $\delta_x(y)$, seen as a function of $y \in \mathbb{R}^3$, is *not* distributional; this is a regular function[8] peaked on $y = x$, with a non-zero width, normalized as $\int \mathrm{d}^3 y \, \delta_x(y) = 1$. We will denote by δ_0 the function of this family obtained for the value $y = 0$.

Simple calculations show that the dual action of **add**, **sub** and **inv** is given by

$$\textbf{add} : L^2_\star(\mathbb{R}^3) \to L^2_\star(\mathbb{R}^3)^{\otimes 2}$$
$$(\textbf{add}\, u)(x_1, x_2) := 8\pi u(x_1)\delta_0(x_2)$$
$$\textbf{sub} : L^2_\star(\mathbb{R}^3) \to L^2_\star(\mathbb{R}^3)^{\otimes 2}$$
$$(\textbf{sub}\, u)(x_1, x_2) := 8\pi (\delta_{x_1} \star u)(x_2)$$
$$\textbf{inv} : L^2_\star(\mathbb{R}^3) \to L^2_\star(\mathbb{R}^3)$$
$$(\textbf{inv}\, u)(x) := u(-x).$$

Thus, when adding an edge, the function depends on the additional Lie algebra variables x_2 via $\delta_0(x_2)$; taking the inner product of this function with any other function $v(x_1, x_2)$ of $L^2_\star(\mathbb{R}^3)^{\otimes 2}$ will project it onto its value $v(x_1, 0)$. When subdividing an edge into two parts, the two variables x_1, and x_2 on the two sub-edges get identified (under inner product) via the function $\delta_{x_1}(x_2)$. Finally, when changing the orientation of the edge, the sign of the variable x is flipped.

These rules describe recursively all the elements equivalent to u. By an obvious extension of these rules to functions on a graph with an arbitrary number of edges, they generate all the equivalence classes in $\cup_\gamma \mathcal{H}_{\star,\gamma}$. It is instructive to check directly that the inner product given in (18) is well defined on equivalence classes. This amounts to showing that the linear maps **add**, **sub** and **inv** acting on $L^2_\star(\mathbb{R}^3)$ are unitary. For example, writing the inner product in $L^2_\star(\mathbb{R}^3)^{\otimes 2}$ as $\langle \, , \, \rangle_{\star,2}$, we easily check that, given $u, v \in L^2_\star(\mathbb{R}^3)$, we have

$$\langle \textbf{add}\, u, \textbf{add}\, v \rangle_{\star,2} = \int \mathrm{d}^3 x_1 \mathrm{d}^3 x_2 (\overline{u} \star v)(x_1)(\overline{\delta_0} \star \delta_0)(x_2)$$
$$= \langle u, v \rangle_\star,$$

---

[8] An explicit calculation using expression (12) of the plane waves gives in fact $\delta_x(y) = \frac{1}{8\pi} \frac{J_1(|x-y|)}{|x-y|}$ where J_1 is the Bessel function of the first kind J_n for $n = 1$.

where the second equality follows from the fact that $\delta_0 = \overline{\delta_0}$ is a $\star$-projector: $\delta_0 \star \delta_0 = \frac{1}{8\pi}\delta_0$, normalized to 1. Analogous calculations show the unitarity of **sub** and **inv**.

Coming back to the construction (17), we now have a map $\widetilde{\mathcal{F}}$ between two pre-Hilbert spaces, which, by construction, is invertible and unitary. Since $\cup_\gamma \mathcal{H}_\gamma$ is dense in its completion $\overline{\cup_\gamma \mathcal{H}_\gamma}$, there is a unique linear extension of $\widetilde{\mathcal{F}}$ to a map

$$\mathcal{F} : \overline{\cup_\gamma \mathcal{H}_\gamma / \sim} \longrightarrow \overline{\cup_\gamma \mathcal{H}_{\star,\gamma} / \sim}$$

between the completion of the two pre-Hilbert spaces. This defines our Fourier transform. $\mathcal{F}$ is invertible and unitary, so that it gives a unitary equivalence between the LQG Hilbert space $\mathcal{H}_0 = \overline{\cup_\gamma \mathcal{H}_\gamma / \sim}$ and the Hilbert space $\mathcal{H}_\star = \overline{\cup_\gamma \mathcal{H}_{\star,\gamma} / \sim}$.

4. Flux representation

We now describe the representation obtained by applying the non-commutative Fourier transform onto the LQG state space. We derive the dual action of holonomy and flux operators, analyze the geometrical interpretation of this dual space and investigate its relation to the standard spin network basis.

4.1. Dual action of holonomy and flux operators

For a given fixed graph γ, consider an elementary surface S_e intersecting γ at a single point of an edge e. The action of the flux operators $E_{S_e}^i$ on $\mathcal{H}_\gamma$ coincides with the action (9) of left- or right-invariant vector fields $\widehat{L}^i$, $\widehat{R}^i$ on SO(3), depending on the respective orientation of e and S_e (see for example [1]). They act dually on $L_\star^2(\mathbb{R}^3)$ as $\widehat{L}^i u := \mathcal{F}(\widehat{L}^i f)$ and $R^i u := \mathcal{F}(\widehat{R}^i f)$, where $u = \mathcal{F}(f)$. Now, since

$$\mathcal{F}(\widehat{R}^i f)(x) = \int dg\,(\widehat{R}^i f)(g)\,e_g(x)$$

$$= \int dg \left[\frac{d}{dt} f(e^{t\tau^i} g) \right]_{t=0} e_g(x)$$

$$= \int dg\, f(g) \left[\frac{d}{dt} e_{e^{-t\tau^i} g}(x) \right]_{t=0},$$

we only need to determine the action of the operators on the plane waves $e_g(x)$, for almost every g. By definition of the $\star$-product, $e_{e^{-t\tau^i} g} = e_{e^{-t\tau^i}} \star e_g(x)$. Thanks to the relation

$$\left[\frac{d}{dt} e_{e^{-t\tau^i}}(x) \right]_{t=0} = -\frac{1}{2} \text{Tr}(x\tau^i) = -i x^i,$$

we conclude that $\widehat{R}^i e_g = -i\hat{x}^i \star e_g$, where $\hat{x}^i(x) = -\frac{1}{2}\text{Tr}(x\tau^i)$ is the coordinate function on $\mathfrak{su}(2)$. This shows that

$$\mathcal{F}(\widehat{R}^i f)(x) = -i\hat{x}^i \star \mathcal{F}(f). \tag{20}$$

There is an analogous formula for the left-invariant vector field, which acts by $\star$-multiplication on the right. Thus, the invariant vector fields on SO(3), and hence the elementary flux operator $E_{S_e}^i$, act dually by $\star$-multiplication.

Next, we investigate the dual action of holonomy operators. We have seen that any function $\varphi(g)$ defines a multiplication operator $\widehat{\varphi}$ on $L^2(\text{SO}(3))$. Let us consider the elementary operators $\widehat{e}(a)$, labeled by Lie algebra variables $a \in \mathfrak{su}(2)$, generated by the plane waves $g \mapsto e_g(a)$. Let $u \in L_\star^2(\mathbb{R}^3)$, and assume $u = \mathcal{F}(f)$. The dual action of $\widehat{e}(a)$ on u is given by

$$(\widehat{e}(a)\,u)(x) := \mathcal{F}(\widehat{e}(a)\,f)(x) = \int dg\, e_g(a) f(g)\, e_g(x).$$

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Using the fact that $e_g(a)e_g(x) = e_g(x + a)$, we obtain

$$(\widehat{e}(a) u)(x) = \mathcal{F}(f)(x + a) = u(x + a).$$

Hence, elementary holonomy operators act by *translation* on the states in the dual representations. More generally, any function φ on the image $L^2_\star(\mathbb{R}^3)$ of the Fourier transform defines an operator $\widehat{\varphi}$ acting on f as

$$(\widehat{\varphi} f)(x) = \int d^3 a (\varphi \star_a f^x)(a),$$

where $f^x(a) := f(x + a)$.

As $\mathcal{F} : \mathcal{H}_0 \to \mathcal{H}_\star$ is a unitary transformation, it preserves the spectra of all operators. In particular, geometrical quantities such as area or volume are quantized the same way as in the standard representation of LQG. For instance, the area operator associated with an elementary surface S_e is given by

$$\hat{A}[S_e] := \gamma \sqrt{\delta_{ij} \hat{x}^i \star \hat{x}^j \star}, \tag{21}$$

where the coordinate operators under the square root act by $\star$-multiplication and where the square root is defined via the spectral theorem. Note that, just as in the standard representation, we have the quantization ambiguity associated with Immirzi's parameter γ.

4.2. Gauge invariant dual states

For a given graph γ, a gauge transformation at a vertex v generated by a group element g_v corresponds to the action of the operator $\widehat{g_v}$ on $\mathcal{H}_\gamma$ given by equation (10). Consider a dual state $u_\gamma = \mathcal{F}_\gamma(f_\gamma)$, Fourier transform of a function f_γ. The dual action of $\widehat{g_v}$ on u_γ is defined as $\widehat{g_v} u_\gamma := \mathcal{F}_\gamma(\widehat{g_v} f_\gamma)$ and reads

$$(\widehat{g_v} u_\gamma)(x_1, \dots, x_n) = u_\gamma\left(g_{t_1}^{-1} x_1 g_{s_1}, \dots, g_{t_n}^{-1} x_n g_{s_n}\right).$$

Gauge invariance is imposed by acting with the gauge averaging operator $\mathcal{P}_\gamma := \bigotimes_v \int dg_v \, \widehat{g_v}$. The averaging over gauge transformation at a vertex v, assuming it has only outgoing edges, takes the form

$$\left(\int dg_v \widehat{g_v} u_\gamma \right)(\mathbf{x}) = (\widehat{C}_v \star u)(x_i, \dots, x_n),$$

where $\widehat{C}_v$ is a 'closure' constraint at the vertex v:

$$\widehat{C}_v(x_i) := \int dg \prod_{e_i \supset v} e_g(x_i) = 8\pi \delta_0 \left(\sum_{e_i \supset v} x_i \right).$$

As emphasized in the previous section, the functions δ_0 act as Dirac distribution for the $\star$-product; in particular, $\delta_0 \star f = f \star \delta_0 = f(0)\delta_0$. Hence, gauge invariance corresponds to a *strong* closure constraint for the $\mathfrak{su}(2)$ variables x_i of the edges incident at v.

More generally, the gauge invariant state $\mathcal{P}_\gamma u_\gamma$ is obtained by $\star$-multiplication of the function u_γ with a product of closure constraints at each vertex $\widehat{C}_v = 8\pi \delta_0\left(\sum_{e_i \supset v} \epsilon_v^i x_i \right)$, where $\epsilon_v^i = \pm 1$ depends on whether the edge i is ingoing or outgoing at v. A nice way to write down general expression for the gauge invariant states is the following. Consider the graph $\gamma' \geqslant \gamma$ obtained by (i) subdividing each edge $i \in \gamma$ into two parts i_s, i_t, where the sub-edge i_s meet the 'source' vertex s_i and i_t meets the 'target' vertex t_i of i, and (ii) by flipping the orientation of each i_t, so that the edges of the new graph γ' are all outgoing of the original

vertices of γ. This procedure defines a new element $u_{\gamma'} \in \mathcal{H}_{\gamma'}$ in the same equivalence class as u_γ, given by

$$u_{\gamma'}(x_{1_s}, x_{1_t}, \ldots, x_{n_s}, x_{n_t}) = \left(\prod_i \delta_{x_{i_s}} \star u_\gamma\right)(-x_{i_t}). \tag{22}$$

Then the projector onto gauge invariant states acts on $u_{\gamma'}$ by left $\star$-multiplication by the product of closure constraints $\widehat{C}_v = 8\pi \delta_0(\sum_{i_v \supset v} x_{i_v})$ at the vertices of γ:

$$\mathcal{P}_\gamma u = \left(\bigotimes_v \widehat{C}_v\right) \star u_{\gamma'}. \tag{23}$$

The action of the projectors $\mathcal{P}_\gamma$ is well defined on equivalence classes in $\cup_\gamma \mathcal{H}_\gamma$; hence, by construction, it is also well defined on the equivalence classes in $\cup_\gamma \mathcal{H}_{\star,\gamma}$. We may also check, directly from the definition (23), that the action of $\mathcal{P}_\gamma$ commutes with the action of **add**, **sub** and **inv**.

This only confirms the geometric interpretation of the Lie algebra variables x_i as fluxes associated with elementary surfaces dual to the edges of the graph γ, and closing around vertices of the same graph to form elementary 3-cells[9]. To be more precise, it is useful to think of reference frames associated with the vertices of the graph γ. For a given state, the group Fourier variables g_i associated with an oriented edge i should be thought of as the parallel transport between the frames of the 'source' and 'target' vertices s_i, t_i. The flux variable x_{i_s} (respectively x_{i_t}) in equation (22) is then naturally interpreted as the flux across an elementary surface intersecting the edge i at a single point, and then parallel-transported to the source vertex s_i (respectively to the target vertex t_i). These two flux variables, associated with the same edge, can then be identified with the relation $g_i x_{i_s} g_i^{-1}$. This relation is a consequence of the formula

$$(\delta_x \star_y e_g)(y) = e_g \star_x \delta_{gxg^{-1}}(y).$$

This geometrical interpretation is thus consistent with the action of plane waves and encoded into the star product.

4.3. Relation with spin network basis

It is interesting to investigate the relation between the Lie algebra variables x and the labels of the standard basis of states. Starting from the geometric interpretation of x as flux (or triad) variables, one would thus deduce from direct calculation the geometric interpretation of these labels. The relation with the usual spin network basis is made explicit using the Fourier transform of the Peter–Weyl theorem, see equation (16). This gives a basis for the dual states on a graph γ given by a product over the edges of dual Wigner functions:

$$\widehat{D}^{j_e}_{m_e n_e}(x) := \int \mathrm{d}g\, e_g(x) D^{j_e}_{m_e n_e}(g).$$

These functions, whose dependence upon the norm $r = |x|$ of x goes as $J_{d_j}(r)/r$, where J_{d_j} is the Bessel function of the first kind associated with the integer $d_j := 2j + 1$ (see for e.g [39]), are peaked on the value $r = d_j$, thus relating the spin j to the norm of the flux. The quantum labels corresponding to the direction variables of the fluxes may then be identified using Perelomov group coherent states $|j, \vec{n}\rangle = g_{\vec{n}}|j, j\rangle$, where $\vec{n} \in \mathcal{S}^2$ and $g_{\vec{n}}$ is an SU(2) element (say, the rotation with axis vector on the equator) mapping the north pole $(0, 0, 1)$ to

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[9] Note that the construction does not depend on the valence of the graph and thus does not need a simplicial setting for its geometric interpretation.

$\vec{n}$ by natural action on the 2-sphere $\mathcal{S}^2$. In such (overcomplete) coherent state basis, the dual Wigner functions

$$\widehat{D}_{\vec{n}\vec{n}'}^{j}(x) := \langle j, \vec{n}|\widehat{D}^{j}(x)|j, \vec{n}'\rangle$$

satisfy the property that

$$\widehat{D}_{\vec{n}\vec{n}'}^{j}(x) = \mathrm{e}_{g_{\vec{n}}g_{\vec{n}'}^{-1}} \star \widehat{D}_{\vec{n}'\vec{n}'}^{j} = \widehat{D}_{\vec{n}\vec{n}}^{j} \star \mathrm{e}_{g_{\vec{n}}^{-1}g_{\vec{n}'}}$$

where the diagonal matrix elements are given by

$$\widehat{D}_{\vec{n}\vec{n}}^{j}(x) = \int \mathrm{d}g\, \mathrm{e}_{g}\big(g_{\vec{n}}^{-1}xg_{\vec{n}}\big)D_{jj}^{j}(g). \tag{24}$$

Now, the dependence of these functions upon the directional part $\hat{x} = \vec{x}/|x|$ goes as $[\hat{x} \cdot \vec{n}]^{2j}$, and hence reaches its highest value for $\hat{x} = \pm\vec{n}$.

These considerations suggest the identification $\vec{x} = j\vec{n}$ of flux variables and the labels of the coherent states basis, which should hold true in a suitable semi-classical limit. This limit is the one where where both fluxes x and spins j are large while $|x|/j$ remains constant. Thus, for example, by rescaling $x \rightarrow x/\kappa$, $j \rightarrow j/\kappa$ in (24), one can recast the integrand on the right-hand side of (24) as an oscillatory phase, subject to saddle point analysis when $\kappa \rightarrow 0$; and it can be shown that the existence of a saddle point requires precisely that $\vec{x} = j\vec{n}$. The calculation is very similar to the one performed in section 3.2 of [14], so we do not repeat it here.

It worth noting that, interestingly, the regime of large fluxes can be reached using the *commutative* limit of the group Fourier transform. This commutative limit is obtained [32] by introducing a deformation parameter κ in the definition of the plane waves and the star product:

$$\mathrm{e}_{g}^{\kappa}(x) = \mathrm{e}^{\frac{i}{\kappa}\epsilon_{\theta}\sin\theta\,\vec{n}_{g}\cdot\vec{x}}, \qquad \mathrm{e}_{g_{1}}^{\kappa} \star_{\kappa} \mathrm{e}_{g_{2}}^{\kappa} = \mathrm{e}_{g_{1}g_{2}}^{\kappa},$$

where notations are the same as in (12), and by taking $\kappa \rightarrow 0$.

These results are thus consistent with the interpretation of the spin j as identifying eigenvalues of the (square of the) flux operators, thus of their norm. In four dimensions, this gives areas to the elementary surfaces with which the flux variables are associated. We also see that, in the semi-classical limit, the coherent state parameters $\vec{n}$ behave like the direction components of the flux variables $\vec{x}$, and thus admit the same interpretation as triad components[10].

In general, therefore, we can expect that any function of the quantum numbers $j, \vec{n}$ will acquire, in a semi-classical approximation, a functional dependence on them matching that of the function $u(x)$ on the non-commutative triad variables x, in the same approximation[11].

5. The U(1) case

Here we shortly want to explain the Group Fourier transform for U(1) and comment on the relation to the triad representation used in LQC (see e.g. [29, 30]). The U(1) case is in several

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[10] This gives further support to the recent constructions in the spin foam setting [14–16, 19] based on group coherent states and on their interpretation as metric variables; in particular, it suggests that imposing geometric restrictions on them in the definition of the dynamical amplitudes will ensure that such amplitudes will have nice geometric properties in a semi-classical regime, as confirmed by the asymptotic analysis of [20].

[11] The asymptotic analysis of the new spin foam amplitudes [20], showing how they take the form of a simplicial path integrals for gravity in the 'triad variables' $j\vec{n}$, can then be interpreted as suggesting the existence (possibly beyond the semi-classical regime) of a simplicial path integral expression for the same amplitudes in the non-commutative variables $\vec{x}$. This interpretation is of course strongly supported by the results of [23].

respects simpler than the SU(2) case but it can serve to understand the principle mechanisms. As for SU(2) we start by defining plane waves

$$e_\phi(x) = e^{-i\phi x} \tag{25}$$

where $x \in \mathbb{R}$. The Fourier transform $\mathcal{F}$ of a function $f(\phi)$ on U(1) (with the convention $-\pi < \phi \leqslant \pi$) is then defined as

$$\mathcal{F}(f)(x) = \int_{-\pi}^{\pi} d\phi f(\phi) e_\phi(x) = \int_{-\pi}^{\pi} d\phi f(\phi) e^{-i\phi x} \tag{26}$$

Note the similarity with the usual Fourier transform which is obtained by just restricting x from $\mathbb{R}$ to $\mathbb{Z}$. The image $\text{Im}\,\mathcal{F}$ is a certain set of continuous functions on $\mathbb{R}$, but certainly not all functions in $C(\mathbb{R})$ are hit by $\mathcal{F}$.

Im $\mathcal{F}$ can be equipped with a $\star$–product, which is dual to the convolution product on U(1). For plane waves, this product reads

$$(e_\phi \star e_{\phi'})(x) := e_{[\phi+\phi']}(x), \tag{27}$$

and extends to Im $\mathcal{F}$ by linearity. Here $[\phi + \phi']$ is the sum of the two angles modulus 2π such that $-\pi < [\phi + \phi'] \leqslant \pi$. In this way the star product is dual to group multiplication. Next, we define an inner product on Im $\mathcal{F}$ via

$$\langle u, v \rangle_\star := \int dx (\overline{u} \star v)(x) \quad \forall u, v \in \text{Im}\mathcal{F} \quad . \tag{28}$$

With this inner product one can check that $\mathcal{F}$ is a unitary transformation between $L^2(\text{U}(1))$ and Im $\mathcal{F}$.

The peculiar class of functions which build up Im $\mathcal{F}$ also leads to a different characterization of the $\star$–product: it turns out that $\langle u, v \rangle_\star$ is entirely fixed by a discrete set of values. This can be understood by comparing this Fourier transform with the usual one which is obtained from (26) by restricting x to be integer, $x \in \mathbb{Z}$. In this case the inverse transformation is given by

$$f(\phi) = \frac{1}{2\pi} \sum_{x \in \mathbb{Z}} \mathcal{F}(f)(x) e^{i\phi x}. \tag{29}$$

This formula indicates that for the function $u(x)$ in the image of $\mathcal{F}$, only the values $x \in \mathbb{Z}$ are relevant. Indeed, we will see below that the Lebesgue measure in x-space (together with the $\star$–product) reduces to a counting measure with support in $\mathbb{Z}$ (and the pointwise product) for functions $u \in \text{Im}\,\mathcal{F}$.

Using the formula for the inverse Fourier transform (29), the star product between the two functions $u_1 = \mathcal{F}(f_1)$ and $u_2 = \mathcal{F}(f_2)$ can be evaluated to

$$
\begin{aligned}
u_1 \star u_2(x) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\phi \, d\phi' e^{-i\phi' x} f_1(\phi) f_2(\phi' - \phi) \\
&= \sum_{x', x'' \in \mathbb{Z}} u_1(x') u_2(x'') \frac{\sin(\pi(x' - x''))}{\pi(x' - x'')} \frac{\sin(\pi(x'' - x))}{\pi(x'' - x)} \\
&= \sum_{x' \in \mathbb{Z}} u_1(x') u_2(x') \frac{\sin(\pi(x' - x))}{\pi(x' - x)},
\end{aligned}
\tag{30}
$$

where for the last line we used that

$$\frac{\sin(\pi(x' - x''))}{\pi(x' - x'')} = \delta_{x', x''} \tag{31}$$

for $x', x'' \in \mathbb{Z}$. The integral over x in $\frac{\sin(\pi(x'-x))}{\pi(x'-x)}$ results to 1 and therefore the inner product (28) is given by

$$\langle u, v \rangle_\star = \int \mathrm{d}x (\overline{u} \star v)(x) = \sum_{x \in \mathbb{Z}} \overline{u}(x)v(x). \tag{32}$$

This agrees with the inner product for the usual Fourier transform. As mentioned, the Lebesgue measure (to be understood together with the star multiplication) in the inner product (32) can be rewritten as a counting measure (together with point multiplication) for functions $u \in \mathrm{Im}\,\mathcal{F}$ which shows that we essentially have to deal with the Hilbert space of square summable sequences, that is $\hat{L}^2_\star(\mathbb{R}) \simeq \ell^2$. With this counting measure there is a large class of functions with zero norm inducing an equivalence relation between functions that differ only by terms of zero norm, that is functions that are vanishing on all $x \in \mathbb{Z}$. In every equivalence class one can define a standard representative by

$$u_s(x) = \sum_{x' \in \mathbb{Z}} u(x') \frac{\sin(\pi(x'-x))}{\pi(x'-x)} \ . \tag{33}$$

These standard representatives also span $\mathrm{Im}\,\mathcal{F}$; that is, the condition $u \in \mathrm{Im}\,\mathcal{F}$ picks a unique representative in the equivalence class. Furthermore, formula (33) defines the map that converts standard Fourier transformed functions to group Fourier transformed functions and is in precise analogy to the SU(2) case where we can use the 'dual' Peter–Weyl decomposition to show that functions in the image of $\mathcal{F}$ can be sampled by discrete values.

On $L^2(U(1))$ we have two elementary operators, the (left and right invariant) derivative $L = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\phi}$ and the holonomy operator $T_n := \mathrm{e}^{-\mathrm{i}\phi n}$, $n \in \mathbb{Z}$, that act as a multiplication operator. It is straightforward to check that these operators act dually as

$$\hat{L}\,u(x) = (x \star u)(x)$$
$$(\hat{T}_n\,u)(x) = u(x+n). \tag{34}$$

In the same way as for SU(2) one can construct Hilbert spaces over graphs and can also obtain cylindrical consistency of the group Fourier transform map.

In LQC [29, 30], a kind of mini-superspace reduction of LQG, one also uses a representation in which the (symmetry reduced) triad operator acts by multiplication and the holonomies act by translations. The spectrum of the multiplication operator is $\mathbb{R}$. Note that it is a discrete spectrum in the sense that the associated eigenfunctions have finite norm. This is possible as the Hilbert space used in LQC is non-separable. Note that the representation (34) used here is different. The action of $\hat{L}$ is via $\star$-multiplication and—as in $L^2(U(1))$—the spectrum is given by $\mathbb{Z}$.

The measure used in LQC can be defined through the inner product between two wavefunctions u and v in the following way. Such a wavefunction u can be understood as a map from a countable set $\{x_i\}_{i \in I_u} \subset \mathbb{R}$ for some index set I_u of countable cardinality to $\mathbb{C}$:

$$u : x_i \to u(x_i). \tag{35}$$

The union of two countable sets $\{x_i\}_{i \in I_u}$ and $\{x_i\}_{i \in I_v}$ defines another countable set which contains both previous sets. In this way, we obtain the structure of a partially ordered set similar to full LQG. Now one can extend each of the maps u, v to the union of the two sets by defining $u(x) := 0$ for all $x \notin \{x_i\}_{i \in I_u}$ and similarly for v. The inner product is given by

$$\langle u, v \rangle = \sum_{x \in \{x_i\}_{i \in I_u} \cup \{x_i\}_{i \in I_v}} \overline{u}(x)v(x). \tag{36}$$

Hence, the wavefunctions $u \in \mathrm{Im}\,\mathcal{F}$ based on one copy of $U(1)$ can be (isometrically) embedded into the LQC Hilbert space, but the latter space is obviously much bigger.

6. Outlook

In this paper, we have used tools from non-commutative geometry, more precisely the non-commutative group Fourier transform of [32, 33, 38], to define a new triad (flux) representation of LQG, which takes into account the fundamental non-commutativity of flux variables. We have shown first how this defines a unitary equivalent representation for states defined on given graphs (cylindrical functions), and then proven cylindrical consistency in this representation, thus defining the continuum limit and the full LQG Hilbert space. As one would expect, the new representation sees flux operators acting by $\star$-multiplication, while holonomies act as a (exponentiated) translation operator. We have then discussed further properties of the new representation, including the triad counterpart of gauge invariance, clarifying further its geometric meaning and the relation with the spin network basis (including the case in which group coherent states are used). Finally, we have discussed the analogous construction in the simpler case of U(1) emphasizing similarities and differences with the triad representation commonly used in loop quantum cosmology.

Let us conclude with a brief outlook on possible further developments. As we mentioned in the text, our construction has been limited, for simplicity, to the case of SO(3) states. The extension of the group Fourier transform to SU(2) has been considered in [33, 38] and we expect the generalization of our construction of a LQG triad representation to be straightforward, and probably most easily performed using the plane waves augmented by polarization vectors (identifying the hemisphere in SU(2) in which the plane wave $e_g(x)$ lives) defined in [38].

Perhaps more interesting is a fully covariant extension of the SU(2) structures we used to SO(4) or SL(2, $\mathbb{C}$) ones, depending on the spacetime signature. In fact, we can think of our non-commutative triad vectors as identifying the self-dual or the rotation sector of the SO(4) or SL(2, $\mathbb{C}$) algebra, and similarly for the group elements representing the conjugate connection. The SU(2) plane waves would then arise from SO(4) or SL(2, $\mathbb{C}$) plane waves after imposition of suitable constraints corresponding to the constraints that reduce the phase space of BF theory to that of gravity, in a Plebanski formulation of 4D gravity as a constrained BF theory. It is at this level that the role of the Immirzi parameter (absent in our construction) will be crucial. In identifying this covariant extension, one could take advantage of the detailed analysis of phase space variables and geometric constraints in [6], in the simplicial context, and of the work already done on simplicity constraints in the non-commutative metric representation of GFTs in [23]. This extension will most likely involve an embedding of the spatial SU(2) spin networks and cylindrical functions in spacetime obtained introducing unit vectors, interpreted as normals to the spatial hypersurface, located at the vertices of the graphs. The relevant structures would then be that of projected spin networks as studied in [40, 41] (see also [23]).

As we mentioned in the text, our construction has identified the Hilbert space of *continuum* loop quantum gravity in the new triad representation, by means of projective limits. It would be interesting, however, to obtain a better characterization of the resulting space in terms of some functional space of generalized flux fields, as we conjecture to be the case, in analogy to the usual construction of the L^2 space over generalized connections, endowed with the Ashtekar–Lewandowski measure. This will involve the definition of the relevant non-commutative C^*-algebra and the application of a generalization of the usual GNS construction (for some work in this direction, see [42]).

The new representation we have defined for LQG can be an important mathematical (and computational) tool for studying the semi-classical limit of the theory, using the expansion of the $\star$-product of functions in the Planck length (see [32]). In particular, this can be useful

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for a better understanding of quantum field theory for matter fields on a quantum spacetime, following [24], and more generally for the definition of matter coupling in LQG. This is indeed already facilitated by the very presence of explicit triad (metric) variables in the quantum states of the theory, which is true in the new representation.

Finally, the new triad representation brings the geometric meaning of the LQG states to the forefront, and suggests a different avenue for the construction of coherent states, on top of giving of course a new representation for the known ones. Both these two facts can be relevant for tackling the issue of defining the quantum dynamics of the theory in the canonical framework, for analyzing the relation to the one defined by the new spin foam models [16–19], and building up on the results of [23] in the group field theory setting.

Acknowledgments

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