Direct Sampling of Negative Quasiprobabilities of a Squeezed State

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Although squeezed states are nonclassical states, so far, their nonclassicality could not be demonstrated by negative quasiprobabilities. In this work we derive pattern functions for the direct experimental determination of so-called nonclassicality quasiprobabilities. The negativities of these quantities turn out to be necessary and sufficient for the nonclassicality of an arbitrary quantum state and are therefore suitable for a direct and general test of nonclassicality. We apply the method to a squeezed vacuum state of light that was generated by parametric down-conversion in a second-order nonlinear crystal.

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Introduction.—In quantum optics and quantum information science, the notion of nonclassicality describes the distinguished difference between classical and quantum physics. Here, a quantum state is referred to as nonclassical if one is not able to model the outcomes of experimentally measured optical field correlation functions by classical electrodynamics. Considering solely pure states, the famous coherent states $|\alpha\rangle$ are the only classical states according to this notation, which makes them the closest analogue to the classical oscillator [1]. Sudarshan [2] and Glauber [3] showed that the density operator of an arbitrary quantum state can formally be written as a statistical mixture of coherent states,

$$\hat{\rho} = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|. \tag{1}$$

If $P(\alpha)$ resembles the properties of a classical probability function, the state is simply a classical mixture of the (classical) coherent states, e.g., a thermal state. In general, the P function may attain negative values—often in connection with a strongly singular behavior. In such cases the corresponding quantum state is referred to as a nonclassical one [4].

The main problem of this definition of nonclassicality lies in the singularities of the P function, which definitely prevent the experimental reconstruction of $P(\alpha)$. Only for special quantum states one may approximately obtain this quasiprobability [5]. Therefore, different criteria for the detection of nonclassicality have been developed. Some of them are simple, such as squeezing [6], classical limits on probabilities [7], or negativities in the Wigner function [8], but they are only sufficient for nonclassicality. Others are necessary and sufficient, but they consist of an infinite hierarchy of inequalities [9,10]. Recently, nonclassicality quasiprobabilities have been introduced, which provide a complete and simple method for the verification of nonclassicality [11]: For any nonclassical state, there exists a regular nonclassicality quasiprobability whose negativities unambiguously reflect its nonclassicality. In [12], the experimental applicability, as a matter of principle, was demonstrated on a nonclassical but less problematic state, which also had a negative Wigner function.

In this Letter, we prove the nonclassicality of a Gaussian squeezed state by reconstructing negative quasiprobabilities from data taken by a balanced homodyne detector. Squeezed states are of special interest in this context, since their commonly used quasiprobabilities, such as the Wigner function, are non-negative and still satisfy the properties of classical probability densities. We avoid any Fourier transformation of the data, which was used in [12], and present a method of direct sampling of nonclassicality quasiprobabilities from the measured data. For this purpose, we use the concept of pattern functions [13], which provide an estimate of the quasiprobability together with its variance. This method applies to the experimental characterization of nonclassicality of any quantum states, the only limitation being statistical uncertainties.

Quasiprobabilities of squeezed states.—Squeezed states are prominent examples of nonclassical states, which can be easily generated in the laboratory. Although nonclassicality is defined by negativities of the P function, its general verification by negativities of any commonly used quasiprobability distribution is impossible. For instance, the Wigner function of a squeezed state with quadrature variances V_x and V_p reads as

$$W_{\rm sv}(x,p)=\frac{1}{2\pi\sqrt{V_xV_p}}\exp\left\{-\frac{x^2}{2V_x}-\frac{p^2}{2V_p}\right\},\tag{2}$$

clearly being a Gaussian. In contrast, the P function may formally be written as

$$P_{\rm sv}(\alpha)=e^{-[(V_x-V_p)/8]\{\partial^2/\partial\alpha^2+\partial^2/\partial\alpha^{*2}-2[(V_x+V_p-2)/(V_x-V_p)](\partial/\partial\alpha)(\partial/\partial\alpha^*)\}}\delta(\alpha),\tag{3}$$

representing one of the most singular representations of a quantum state, with infinitely high orders of derivatives of the δ distribution. Moreover, the s-parametrized quasiprobabilities [14] of a squeezed state are either Gaussian (and non-negative) or strongly singular. Based on a simple condition for the characteristic function of the P function [15], the nonclassicality can be easily verified [16]. However, this condition is only sufficient and the question remains of whether there exists any well-behaved quasiprobability, allowing a complete characterization of the nonclassicality of squeezed states by the failure of being interpreted as a classical probability.

Nonclassicality quasiprobabilities.—The starting point of our discussion is the characteristic function of the P function,

$$\Phi(\beta)=\langle:e^{\beta\hat a^\dagger-\beta^*\hat a}:\rangle=\langle e^{i|\beta|\hat x[\arg(\beta)-\pi/2]}\rangle e^{|\beta|^2/2},\tag{4}$$

with $\hat x(\varphi)$ the quadrature operator of the optical field at phase φ. In order to obtain a regular phase-space distribution, we filter $\Phi(\beta)$ in the form

$$\Phi_\Omega(\beta)=\Phi(\beta)\Omega_w(\beta).\tag{5}$$

The filter $\Omega_w(\beta)$ has to satisfy the following conditions to be useful for nonclassicality detection [11].

(1) The filtered characteristic function $\Phi_\Omega(\beta)$ should be integrable for an arbitrary quantum state $\Phi(\beta)$, such that its Fourier transform—the nonclassicality quasiprobability—exists as a regular function.

(2) Negativities in the Fourier transform shall unambiguously be due to the nonclassicality of the state. Conversely, the nonclassicality quasiprobability shall be non-negative for any classical state. This requires that the filter $\Omega_w(\beta)$ has a non-negative Fourier transform.

(3) If the width parameter approaches infinity, the filtered characteristic function $\Phi_\Omega(\beta)$ should converge to the characteristic function of the P function, $\Phi(\beta)$. This can be realized by the conditions $\Omega_w(\beta)=\Omega_1(\beta/w)$ and $\Omega_1(0)=1$.

(4) The filter should be nonzero everywhere, $\Omega_w(\beta)\neq 0$, such that no information about the quantum state is lost due to the filtering in Eq. (5).

Under these conditions, the nonclassicality quasiprobability is defined as the Fourier transform of the filtered characteristic function,

$$P_\Omega(\alpha)=\frac{1}{\pi^2}\int d^2\beta e^{\alpha\beta^*-\alpha^*\beta}\Phi(\beta)\Omega_w(\beta).\tag{6}$$

Negativities of the quasiprobability are signatures of nonclassicality of the state and the negativity of its P function, since the filter is constructed in such a way that it does not introduce negativities by itself. In the present work, we construct a filter from the autocorrelation of the function $\omega(\beta)=\exp(-|\beta|^4)$,

$$\Omega_1(\beta)=\frac{1}{\mathcal{N}}\int d^2\beta'\omega(\beta')\omega(\beta'+\beta),\tag{7}$$

where the normalization constant $\mathcal{N}$ is chosen to obey $\Omega_1(0)=1$. The width is introduced via $\Omega_w(\beta)=\Omega_1(\beta/w)$. This filter satisfies all criteria mentioned above; for the proof see [11].

Derivation of a pattern function.—Pattern functions provide an efficient technique to directly estimate a physical quantity together with its uncertainty. From balanced homodyne detection, we obtain quadrature values $x_j(\varphi)$ measured for certain phases φ. They obey the quadrature distributions $p(x;\varphi)$, which satisfy $\int p(x;\varphi)dx=1$. The quadratures are normalized such that the vacuum quadratures have a variance $V_{\rm vac}=1$. Now the nonclassicality quasiprobability $P_\Omega(\alpha)$ with a certain width parameter w is written as the statistical mean of the pattern function $f_\Omega(x,\varphi;\alpha,w)$, averaged over the quadrature distributions:

$$P_\Omega(\alpha)=\int_{-\infty}^{\infty}dx\int_0^\pi d\varphi\frac{p(x;\varphi)}{\pi}f_\Omega(x,\varphi;\alpha,w).\tag{8}$$

For this purpose, we note that due to Eq. (4), the characteristic function of the P function of the state can be calculated from the quadrature distribution via

$$\Phi(\beta)=\int_{-\infty}^{\infty}dxp\left(x;\arg(\beta)-\frac{\pi}{2}\right)e^{i|\beta|x}e^{|\beta|^2/2}.\tag{9}$$

It is convenient to rewrite the integral in Eq. (6) in polar coordinates $\beta=be^{i\varphi}$. Here, we restrict φ to $[0,\pi)$ and extend b to $(-\infty,\infty)$. Then we obtain

$$P_\Omega(\alpha)=\frac{1}{\pi^2}\int_{-\infty}^{\infty}db\int_0^\pi d\varphi|b|e^{2i|\alpha|b\sin(\arg(\alpha)-\varphi)}\Phi(be^{i\varphi})\times\Omega_w(be^{i\varphi}).\tag{10}$$

The filter is chosen to be independent of the phase, i.e. $\Omega_w(be^{i\varphi})\equiv\Omega_w(b)$. Now we insert Eq. (9) and obtain

$$P_\Omega(\alpha)=\int_{-\infty}^{\infty}dx\int_0^\pi d\varphi\frac{p(x;\varphi)}{\pi}\int_{-\infty}^{\infty}db\frac{|b|}{\pi}e^{ibx}e^{b^2/2}\times e^{2i|\alpha|b\sin(\arg(\alpha)-\varphi-(\pi/2))}\Omega_w(b).\tag{11}$$

This equation defines the pattern function

$$f_\Omega(x, \varphi; \alpha, w) = \int_{-\infty}^{\infty} db \frac{|b|}{\pi} e^{ibx} e^{2i|\alpha|b \sin(\arg(\alpha)-\varphi-(\pi/2))} \times e^{b^2/2}\Omega_w(b), \quad (12)$$

which has to be used in Eq. (8).

Equation (8) gives rise to the following interpretation: Suppose we have measured N quadrature-phase pairs (x_i, φ_i), whose joint probability distribution is $\frac{1}{\pi} p(x; \varphi)$. Here the phases φ are assumed to be uniformly distributed in $[0, \pi)$, while the quadratures obey the quadrature distribution $p(x; \varphi)$, conditioned on the value of the phase φ. Then the quasiprobability $P_\Omega(\alpha)$ can be calculated as the expectation value of the pattern function $f_\Omega(x, \varphi; \alpha, w)$. For experimental data, we replace the expectation value by its empirical estimate,

$$P_\Omega(\alpha) \approx \frac{1}{N} \sum_{i=1}^{N} f_\Omega(x_i, \varphi_i; \alpha, w). \quad (13)$$

Its variance can be obtained naturally as the mean square deviation of the numbers $f_\Omega(x_i, \varphi_i; \alpha, w)$.

If the phases, at which quadratures are measured, are scanned in $[0, \pi]$ or drawn randomly from a uniform distribution, one can calculate the nonclassicality quasiprobability directly as the empirical mean of the pattern function. This mean is taken over all pairs (x_i, φ_i) of quadrature and phase. In our experiment, we only obtained quadratures at 21 fixed phase angles. In this case, one may not simply replace the integral over the phase φ by a sum: This leads to systematic deviations, since the pattern function is varying rapidly with respect to the phase, in particular, if $|\alpha|$ becomes large. For a detailed discussion and solution of this problem, we refer to the supplemental material [17].

Experimental setup.—The squeezed vacuum states of light were generated by type-I degenerate parametric down-conversion [optical parametric amplification (OPA)] inside an optical resonator. The latter was a standing wave resonator with a linewidth of 25 MHz containing a noncritically phase matched second-order nonlinear crystal (7% Mg:$LiNbO_3$). The OPA process was continuously pumped by 50 mW of second harmonic light yielding a classical power amplification factor of 6. Both the length (resonance frequency) of the resonator as well as the orientation of the squeezing ellipse were stably controlled by electronic servo loops. With this setup we directly measured a squeezed variance of -4.5 dB and an antisqueezed variance of $+7.2$ dB with respect to the unity vacuum variance. From these measurements we inferred an overall efficiency of 75% and an initial squeezing factor of -8.2 dB.

The quadrature amplitudes of the squeezed state were measured by balanced homodyne detection (BHD). The visibility of the squeezed field and the spatially filtered [mode cleaner (MC), Fig. 1] local oscillator was 98.9% and

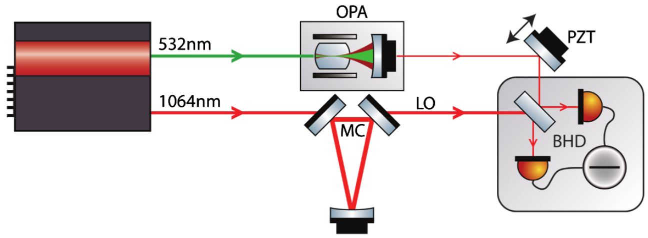

FIG. 1 (color online). Simplified sketch of the experimental setup. A spatially filtered continuous-wave field at 1064 nm served as a local oscillator (LO) for balanced homodyne detection (BHD) and a phase-locked second harmonic field at 532 nm as the pump for the parametric squeezed light source (OPA). MC, spatial mode cleaner; PZT, piezoelectrically actuated mirror for adjusting the quadrature amplitude phase of the BHD.

was limited by OPA crystal inhomogeneities. The quadrature phase of the BHD was adjusted by servo loop controlled micropositioning of steering mirrors in one of the optical input paths. The photoelectric signals of the two individual BHD photodiodes were electronically mixed down at 7 MHz and low pass filtered with a bandwidth of 400 kHz to address a mode showing good squeezing and a high detector dark noise clearance of the order of 20 dB. The resulting signals were fed into a PC-based data acquisition system and sampled with 1×10^6 samples per second and 14 bit resolution and finally subtracted yielding the quadrature amplitude data. For a more detailed description of the main parts of the setup, we refer to [18,19].

Experimental results.—The examined squeezed vacuum state is characterized by the variances $V_x = 0.36$ and $V_p = 5.28$. We acquired 10^5 quadrature values for each of the 21 quadrature phases, the latter being equally spaced in $[0, \pi]$. The values of the quasiprobability $P_\Omega(\alpha)$ as well as their standard deviation $\sigma(P_\Omega(\alpha))$ are estimated from the pattern function as given in Eq. (12). The filter width is chosen such that the significance of the negativity is optimized. Our figure of merit is defined as

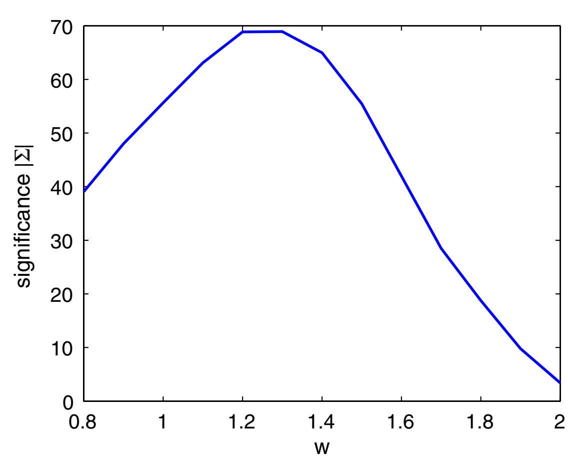

FIG. 2 (color online). Absolute value of the significance $\Sigma(w)$ of the negativity of the quasiprobability versus the filter width w.

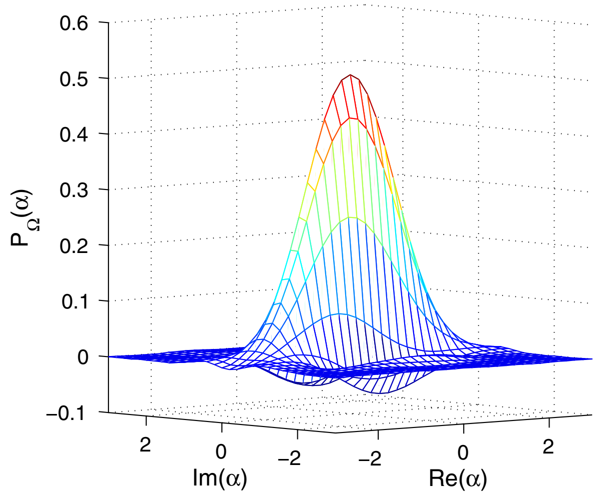

FIG. 3 (color online). Nonclassicality quasiprobability for a squeezed vacuum state. The data are directly sampled from our balanced homodyne data and clearly show negative values.

$$\Sigma(w) = \min_{\alpha}\left[\frac{P_\Omega(\alpha)}{\sigma(P_\Omega(\alpha))}\right], \tag{14}$$

with $\Sigma(w)$ being negative if $P_\Omega(\alpha)$ is negative for some α. By construction of the quasiprobability, a significant negativity clearly indicates nonclassicality of the state. The larger the absolute value of $\Sigma(w)$, the larger the significance of the negativity. This quantity can be optimized with respect to w. In Fig. 2, we show the dependence of the significance on the filter width w. The larger the filter width, the larger the nonclassical effects of the state manifested in negativities, but also the larger the statistical uncertainty grows. Therefore, an optimum width exists, which is achieved here for $w = 1.3$.

Figure 3 shows the nonclassicality quasiprobability obtained from the experimental data. We find that along the axis of $\mathrm{Im}(\alpha)$ the quasiprobability oscillates and becomes

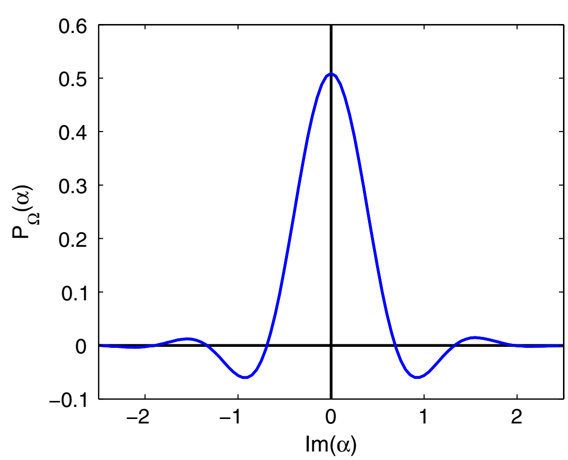

FIG. 4 (color online). Cross section of the nonclassicality quasiprobability for our squeezed vacuum state. Note that the uncertainty in the data is less than the linewidth chosen here.

clearly negative. This uncovers the nonclassicality of the squeezed state in a general sense, beyond the phase-sensitive reduction of the quadrature variance. It also includes the information on other kinds of effects, such as higher-order squeezing of the types considered in [10,20], and others; see [16]. In Fig. 4, we show a cross section of Fig. 3 along this axis. We clearly observe distinct negativities. The standard deviation is less than 1.1×10^{-3} for all points and therefore covered by the width of the line. We also calculated a systematic error due to the finite set of examined phases, which is less than 3.6×10^{-4} for all points along this axis [17]. $P_\Omega(\alpha)$ attains the minimum at $\alpha = 0.9i$ with $P_\Omega(\alpha_{\mathrm{min}}) = -0.059\,89$ and $\sigma(P_\Omega(\alpha_{\mathrm{min}})) = 0.9 \times 10^{-3}$, therefore leading to a significance of $|\Sigma| = 69$ standard deviations. Hence, this is a very clear demonstration of the nonclassicality of the examined state by means of negativities of a nonclassicality quasiprobability, which is not possible for commonly used quasiprobabilities such as the Wigner function.

Conclusions.—In our work, we introduced a method for the direct sampling of nonclassicality quasiprobabilities of arbitrary quantum states from measured quadrature amplitudes. By applying our method to a squeezed state, whose P function belongs to the most singular ones, we experimentally verified nonclassicality in its general sense, i.e., through negative quasiprobabilities. Our method is not only capable of estimating the negativity of the quasiprobability, but also its statistical uncertainty, in a surprisingly simple manner.

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[1] E. Schrödinger, Naturwissenschaften **14**, 664 (1926).
[2] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[3] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[4] U. M. Titulaer and R. J. Glauber, Phys. Rev. **140**, B676 (1965).
[5] T. Kiesel, W. Vogel, V. Parigi, A. Zavatta, and M. Bellini, Phys. Rev. A **78**, 021804(R) (2008).
[6] D. F. Walls, Nature (London) **306**, 141 (1983).
[7] A. Rivas and A. Luis, Phys. Rev. A **79**, 042105 (2009).
[8] A. Kenfack and K. Zyczkowski, J. Opt. B **6**, 396 (2004).
[9] T. Richter and W. Vogel, Phys. Rev. Lett. **89**, 283601 (2002).
[10] E. V. Shchukin and W. Vogel, Phys. Rev. A **72**, 043808 (2005).
[11] T. Kiesel and W. Vogel, Phys. Rev. A **82**, 032107 (2010).
[12] T. Kiesel, W. Vogel, M. Bellini, and A. Zavatta, Phys. Rev. A **83**, 032116 (2011).
[13] G. M. D'Ariano, C. Macchiavello, and M. G. A. Paris, Phys. Rev. A **50**, 4298 (1994).
[14] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).
[15] W. Vogel, Phys. Rev. Lett. **84**, 1849 (2000).

[16] T. Kiesel, W. Vogel, B. Hage, J. DiGuglielmo, A. Samblowski, and R. Schnabel, Phys. Rev. A **79**, 022122 (2009).

[17] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.107.113604 for detailed information on the numerical evaluation of the pattern function and the sampling procedure for a discrete set of phases.

[18] A. Franzen, B. Hage, J. DiGuglielmo, J. Fiurášek, and R. Schnabel, Phys. Rev. Lett. **97**, 150505 (2006).

[19] B. Hage, A. Franzen, J. DiGuglielmo, P. Marek, J. Fiurášek, and R. Schnabel, New J. Phys. **9**, 227 (2007).

[20] C. K. Hong and L. Mandel, Phys. Rev. Lett. **54**, 323 (1985).