Conformal extensions for stationary spacetimes

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Abstract

The construction of the cylinder at spatial infinity for stationary spacetimes is considered. Using a specific conformal gauge and frame, it is shown that the tensorial fields associated with the conformal Einstein field equations admit expansions in a neighbourhood of the cylinder at spatial infinity which are analytic with respect to some suitable time, radial and angular coordinates. It is then shown that the essentials of the construction are independent of the choice of conformal gauge. As a consequence, one finds that the construction of the cylinder at spatial infinity and the regular finite initial value problem for stationary initial data sets are, in a precise sense, as regular as they could be.

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1. Introduction

This paper discusses a certain conformal extension for vacuum stationary solutions for the Einstein field equations—the so-called *cylinder at spatial infinity*. This construction provides detailed information about the structure of this class of spacetimes in the region where spatial infinity touches null infinity. The analysis presented here is the key for the construction of asymptotically simple spacetimes from initial value problems on Cauchy hypersurfaces. In order to better understand the context of our analysis, we present a brief overview of the ideas and problems involved.

1.1. Asymptotically simple spacetimes

Penrose's notion of *asymptotic simplicity*—see e.g. [39, 40]—was introduced with the objective of providing a framework for the discussion of isolated systems in general relativity. The programme behind this idea is usually known as *Penrose's proposal*—see e.g. [29–31]. A vacuum spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ is said to be asymptotically simple if there exists a smooth,

oriented, time-oriented, causal spacetime $(\mathcal{M}, g_{\mu\nu})$ and a smooth function Θ (the *conformal factor*) on $\mathcal{M}$ such that

  (i) $\mathcal{M}$ is a manifold with boundary $\mathscr{I} \equiv \partial\mathcal{M}$;
 (ii) $\Theta > 0$ on $\mathcal{M}\backslash\mathscr{I}$, and $\Theta = 0$, $\mathrm{d}\Theta \neq 0$ on $\mathscr{I}$;
(iii) there exists an embedding $\Phi : \mathcal{M}\backslash\mathscr{I} \to \tilde{\mathcal{M}}$ such that $g_{\mu\nu} = \Theta^2(\Phi^{-1})^*\tilde{g}_{\mu\nu}$;
 (iv) each null geodesic of $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ acquires two distinct endpoints on $\mathscr{I}$.

In this definition, as well as in the rest of the paper the word *smooth* will be used as a synonym of C^∞. In order to simplify the notation, we will write $g_{\mu\nu} = \Theta^2\tilde{g}_{\mu\nu}$ instead of $g_{\mu\nu} = \Theta^2(\Phi^{-1})^*\tilde{g}_{\mu\nu}$. The point (iv) in the definition excludes black hole spacetimes—however, the discussion in this paper will be local to the conformal boundary $\mathscr{I}$, and hence (iv) will not be of relevance in our considerations.

The first natural examples of asymptotically simple spacetimes are solutions to the Einstein field equations which are *static* or, more generally, *stationary* near the conformal boundary. That this is the case is a consequence of the structural properties of the static and stationary field equations—see e.g. [22]. We will elaborate further on this in the following paragraphs. Now, in order to be of real physical value, the notion of asymptotic simplicity should also include spacetimes which are not static or stationary near the conformal boundary so as to allow for the discussion of gravitational radiation—the existence of this type of spacetimes has been shown in [18]. The examples constructed in [18] are somehow special, as they arise as the development of Cauchy initial data sets which are exactly Schwarzschildean in the asymptotic end. More generally, recent developments in the construction of solutions to the Einstein constraint equations by means of gluing methods—see e.g. [19–21]—allow us to construct asymptotically simple spacetimes from initial data sets which are exactly stationary in the asymptotic region.

Given this state of affairs, it is natural to ask whether there are more general types of asymptotically simple spacetimes than the ones described in the previous paragraph—see e.g. [31]. This question leads to the so-called *problem of spatial infinity*. If asymptotically simple spacetimes are to be constructed using some version of the Cauchy problem in general relativity, then one has to prescribe some initial data set $(\tilde{\mathcal{S}}, \tilde{h}_{ab}, \tilde{\chi}_{ab})$ on an *asymptotically Euclidean hypersurface*—for simplicity, here it will be assumed that $\tilde{\mathcal{S}}$ has the topology of $\mathbb{R}^3$. The question now is: how does one encode in $(\tilde{\mathcal{S}}, \tilde{h}_{ab}, \tilde{\chi}_{ab})$ that the development will be asymptotically simple? The examples of [18] suggest that this issue has to be related in some way to the behaviour of the initial data set in its asymptotic region.

1.2. Asymptotically Euclidean and regular initial data sets

As we are discussing properties of spacetimes by means of the conformally rescaled setting given by the notion of asymptotic simplicity, it is also natural to work with a conformally rescaled one-point compactification of the initial hypersurface $\tilde{\mathcal{S}}$. To this end, we recall the notion of asymptotically Euclidean and regular Riemannian manifolds. The pair $(\tilde{\mathcal{S}}, \tilde{h}_{ab})$ will be said to be *asymptotically Euclidean and regular* if there exists a three-dimensional, orientable, smooth compact manifold $(\mathcal{S}, h_{ab})$, a point $i \in \mathcal{S}$, a diffeomorphism $\phi : \mathcal{S}\backslash\{i\} \to \tilde{\mathcal{S}}$ and a function $\Omega \in C^2(\mathcal{S}) \cap C^\infty(\mathcal{S}\backslash\{i\})$ with the properties

$$\Omega(i) = 0, \quad D_a\Omega(i) = 0, \quad D_aD_b\Omega(i) = -2h_{ab}(i),$$
$$\Omega > 0 \quad \text{on } \mathcal{S}\backslash\{i\},$$
$$h_{ab} = \Omega^2\phi_*\tilde{h}_{ab}.$$

Again, in order to simplify the notation, the last condition will be written as $h_{ab} = \Omega^2\tilde{h}_{ab}$ so that $\mathcal{S}\backslash\{i\}$ and $\tilde{\mathcal{S}}$ are identified. The key property of asymptotically Euclidean and regular

manifolds is that suitable neighbourhoods of the point i (the *point at infinity*) are mapped into the asymptotic end of $\tilde{\mathcal{S}}$. Thus, one can use local differential geometry to discuss the asymptotic properties of the initial data set $(\tilde{\mathcal{S}}, \tilde{h}_{ab})$.

1.3. Static and stationary spacetimes

The notion of asymptotically Euclidean and regular manifolds has been crucial to understand the asymptotic properties of static and stationary spacetimes—see e.g. [13, 8, 26]—and to prove results concerning their multipolar structure—see e.g. [1, 5, 4, 6, 3, 14, 15, 32]. Stationary (and static) spacetimes are best discussed in terms of a *quotient manifold* $\tilde{\mathcal{X}}$ obtained by identifying points on $\tilde{\mathcal{M}}$ lying on the same orbit of the stationary (static) Killing vector ξ^μ. From this symmetry reduction procedure, one also obtains a metric $\tilde{\gamma}_{ab}$ for the quotient manifold $\tilde{\mathcal{X}}$. As the stationary spacetime arises as the development of an asymptotically Euclidean initial data set $(\tilde{\mathcal{S}}, \tilde{h}_{ab}, \tilde{\chi}_{ab})$, the pair $(\tilde{\mathcal{X}}, \tilde{\gamma}_{ab})$ will also be asymptotically Euclidean. Conversely, one can prescribe the leading asymptotic behaviour of the initial data set $(\tilde{\mathcal{S}}, \tilde{h}_{ab}, \tilde{\chi}_{ab})$ from assumptions on the asymptotic behaviour of $(\tilde{\mathcal{X}}, \tilde{\gamma}_{ab})$. For example, one can assume that $(\tilde{\mathcal{X}}, \tilde{\gamma}_{ab})$ is asymptotically Euclidean and regular and work on a point-compactified manifold $\mathcal{X}$ and a conformally rescaled quotient metric γ_{ab}—this is the assumption made, for example, in [13].

One of the key results of the theory of stationary spacetimes is that there exists coordinates in a suitably small neighbourhood of i for which γ_{ab}, Ω and the (rescaled) stationary potentials are analytic. This analyticity in a neighbourhood of the point at infinity of the compactified quotient manifold is the key to establish that static and stationary spacetimes are asymptotically simple. For static spacetimes, the analyticity on $\mathcal{X}$ is inherited by the point compactification $\mathcal{S}$ of time symmetric slices, and the conformal factor Ω of the conformally rescaled quotient metric γ_{ab} is used as the conformal factor for the whole spacetime, so that the spacetimes are asymptotically simple. The situation for stationary spacetimes is more delicate: in this case, the (analytic) conformal quotient metric γ_{ab} is no longer conformally related to the conformal metric h_{ab} of the t-constant slices. Furthermore, h_{ab} is no longer analytic, but only of class $C^{2,\alpha}$. Notwithstanding, it is still possible to construct a smooth conformal extension of the stationary spacetime. This result shows that although asymptotic simplicity is a property which can be naturally expected from stationary spacetimes, the fact that it holds is a consequence of the structural properties of the stationary equations at spatial infinity.

1.4. The cylinder at spatial infinity

In order to answer the question of whether there exist asymptotically simple spacetimes arising from initial data sets which are neither static nor stationary in a neighbourhood of infinity, one needs a framework that allows one to resolve and disentangle the delicate structure of spacetime in this region. Furthermore, as the strategy to construct asymptotically simple spacetimes is to make use of the Cauchy problem in general relativity, one would like to be able to formulate an initial value problem with data prescribed on the compact manifold $\mathcal{S}$ for various conformal fields which would directly render the conformally rescaled manifold $(\mathcal{M}, g_{\mu\nu})$. Appropriate tools for this construction are the *conformal Einstein field equations*—see e.g. [23–25]—and extensions thereof—see [27, 28, 30, 31]. However, the representation of spatial infinity as suggested by the point-compactification of the initial hypersurface $\tilde{\mathcal{S}}$ presents us with technical difficulties. The underlying reason is that for initial data sets with non-vanishing ADM mass, spatial infinity is a singular point of the conformal geometry—see e.g. [26]. At the level of the conformal field equations and the various fields they govern, this

singular behaviour of the conformal structure translates into the divergence of the so-called *rescaled Weyl tensor* at *i*.

In order to overcome the difficulties at spatial infinity that have been described in the previous paragraph, a new conformal representation of the region of spacetime near null and spatial infinity was introduced in [28]. This representation, based on general properties of conformal structures, together with the *extended conformal field equations* allows us to introduce a *regular finite initial value problem at spatial infinity* for which both the equations and their initial data are regular at the conformal boundary.

Whereas the standard (Penrose) compactification of spacetimes considers spatial infinity as a point, the approach used in [28, 31] represents spatial infinity as an extended set with the topology of $[-1, 1] \times \mathbb{S}^2$. This *cylinder at spatial infinity* is obtained as follows: one starts with the standard point-compactification $\mathcal{S}$ of an asymptotically Euclidean initial data set $\tilde{\mathcal{S}}$. As in previous paragraphs, $\mathcal{S}$ contains a special point *i* representing the infinity of $\tilde{\mathcal{S}}$. In a second stage, the point *i* is blown up to a 2-sphere. This blowing up is achieved by lifting a neighbourhood $\mathcal{B}$ of *i* to the bundle of orthonormal frames with group $O(3)$—or equivalently to the bundle of space-spinors with group $SU(2, \mathbb{C})$. In a final step, one uses a congruence of timelike *conformal geodesics* to obtain a conformal analogue of Gaussian coordinates in a spacetime neighbourhood of $\mathcal{B}$. Timelike conformal geodesics are conformal invariants which preserve their quality of being timelike under conformal transformations. Conformal Gaussian systems based on these curves provide a canonical conformal class of conformal factors for the development of the initial data. Remarkably, these conformal factors can be written entirely in terms of initial data quantities. Hence, the location of the conformal boundary is known *a priori*. The conformal boundary rendered by this class of canonical conformal factors contains a null infinity with the same structure as in the case of the standard Penrose compactification. Spatial infinity, however, extends in the time dimension—so that one can speak of the cylinder at spatial infinity proper. Of crucial relevance are the *critical sets* $\{\pm 1\} \times \mathbb{S}^2$—the collection of points where null and spatial infinity intersect. Null infinity and spatial infinity do not intersect tangentially at the critical points. As a consequence, some of the propagation equations implied by the conformal field equations degenerate at the critical points. The analysis in [28]—see also [45, 44, 46, 47]—has shown that, as a result, the solutions to the conformal field equations develop certain types of logarithmic singularities at the critical sets. These singularities form an intrinsic part of the conformal structure and cannot be regarded as a consequence of a bad gauge choice. The hyperbolic nature of the conformal propagation equations suggests that these singularities will propagate along null infinity, and thus, they will have an effect on the regularity of the conformal boundary.

The construction of the cylinder at spatial infinity bears some relation to other approaches in the analysis of the structure of spatial infinity. For example, the blow up of the point at infinity is closely related to Geroch's idea of directional dependent tensors—see [35, 36]. This idea was later retaken in the discussions given in [2, 42]. The cylinder at infinity is closely related to the *hyperboloid of spatial infinity* also introduced in [2], and later retaken by [7, 11] in a first attempt to combine geometric and partial differential equations points of view to study the structure of spatial infinity.

1.5. Static spacetimes and the cylinder at spatial infinity

In view that static and stationary spacetimes provide prime examples of asymptotically simple spacetimes, one also expects the associated construction of the cylinder at spatial infinity to be as smooth as it can be. This smoothness can be regarded as a consistency of the setting. If static or stationary spacetimes were to exhibit some type of non-smooth behaviour at the cylinder

at spatial infinity, these by necessity have to be associated with a bad gauge choice. In [31], a proof of the smoothness of the cylinder at spatial infinity for static spacetimes was given. Surprisingly, this proof is much more complicated than what one would expect given that firstly, the conformal fields are analytic in a neighbourhood of spatial infinity; and secondly, that the spacetimes are time independent. The difficulties in the analysis can be explained, in part, by the fact that the conformal geodesics used in the construction of the cylinder at spatial infinity are not aligned with the orbits of the timelike Killing vector. Nevertheless, the fact that one is considering time symmetric spacetimes simplifies the analysis as the quotient manifold can be identified with the slices of constant t so that the analyticity of the point i is inherited by spatial infinity i^0 and by all spacetime quantities.

It should be pointed out that the relevance of the analysis of static spacetimes carried out in [31] goes beyond its role as a consistency check of the framework. The analysis in [45, 46] suggests that static spacetimes have a special position among the class of spacetimes with a smooth compactification at spatial infinity. More precisely, it is conjectured that

Conjecture 1. *If an analytic time symmetric initial data set for the Einstein vacuum equation yields a development with a smooth null infinity, then the initial data set is exactly static in the neighbourhood of spatial infinity.*

The rigidity results of [49, 50] constitute further evidence in support of the conjecture.

1.6. Our main result

In this paper we extend the analysis of the cylinder at spatial infinity carried out in [31] to the case of stationary spacetimes. More precisely, we show that

Theorem. *Given an initial data set for the vacuum Einstein field equations which is stationary in a neighbourhood of infinity, the solution to the regular finite initial value problem for the conformal field equations at spatial infinity is smooth in a neighbourhood of the cylinder at spatial infinity, and in particular through the critical sets.*

The notion of initial data sets which are stationary in a neighbourhood of infinity will be clarified in the main text. Given one of these initial data sets, it follows that the domain of influence of the neighbourhood of infinity where the data is stationary is a stationary spacetime which, because of the asymptotic behaviour of the data, will be asymptotically flat in a suitable sense. In our discussion, we will adopt the point of view of the Cauchy problem in general relativity and regard the initial data sets as the primary concept. Thus, when we talk of a stationary spacetime, we think of it as the development of some stationary data. Furthermore, by 'stationary data' we understand data in a stationary spacetime. Stationary data are endowed with a 'transverse' solution to the Killing initial data equations (KID) which give rise to a stationary Killing vector in the development—see e.g. [9, 10]. From well-known theorems on the local existence of solutions to the Cauchy problem in general relativity, given a stationary initial data there is always, at least, a small slab of a corresponding stationary spacetime—see e.g. [16]. In fact, this development covers 'boost domains' reaching null infinity—see [17]. In what follows, we will consider a very specific type of stationary data. A particular advantage of the point of view of the Cauchy problem is that one has more understanding and control on the specification of boundary conditions on an initial data set than on a spacetime. In particular, one expects that an asymptotically Euclidean initial data set will give rise to some type of asymptotically flat spacetime.

The strategy to prove our main result is as follows. Starting with a generic asymptotically flat stationary spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$, one makes use of the analysis of [22] to construct a conformal completion, $(\breve{\Omega}, \breve{g}_{\mu\nu})$, of the stationary spacetime in a neighbourhood of spatial infinity. This completion is adapted to the stationarity of the spacetime and has a smooth null infinity. This representation is, however, not suitable for our purposes as spatial infinity is represented as a point. The conformal metric $\breve{g}_{\mu\nu}$ is given in terms of some asymptotically Cartesian coordinates. In a second stage, one performs a change of coordinates to a polar system in which asymptotic expansions can be analysed in a more convenient way. A subsequent change of the time coordinate and an associated conformal rescaling render a conformal representation, $(\bar{\Omega}, \bar{g}_{\mu\nu})$, in which spatial infinity already appears as an extended set with the topology of a cylinder. This representation is, in a strict sense, not a conformal completion as spatial infinity is not at a finite distance with respect to the conformal metric $\bar{g}_{\mu\nu}$—the metric becomes singular there. In order to deal with this behaviour, one introduces a suitable frame $v_{\boldsymbol{\alpha}}$. It is then shown that the components of the metric, $\bar{g}_{\boldsymbol{\alpha\beta}}$, with respect to this frame are regular at infinity. Furthermore, the components of key derived objects (the Schouten and Weyl tensors) are also shown to be regular. This is the most calculational involved part of our argument. Once the cylinder at spatial infinity has been obtained, one shows that the cylinder itself, and a neighbourhood of it can be ruled by means of a congruence of conformal geodesics. This cannot be shown explicitly, and thus, one has to resort to a perturbative argument. This construction leads to the *canonical* conformal factor Θ, related to $\bar{\Omega}$ through a further conformal factor Π. The congruence of conformal congruences also gives rise to a Weyl connection $\hat{\nabla}$. An abstract integration of the conformal evolution equations along the congruence of conformal geodesics shows that solutions to the conformal evolution equations extend smoothly through the cylinder at spatial infinity and also through a suitable neighbourhood of null infinity. In a final step, it is then shown that the construction is independent of the conformal gauge used to write the stationary initial data.

Contrasted with the result for static spacetimes given in [31], the main difficulties in proving our main result are as follows: the presence of a non-vanishing second fundamental form in the slices of constant t has as a consequence a Weyl tensor with a non-vanishing magnetic part; and crucially, the quotient manifold cannot be directly identified with the structures of the constant t slices. In particular, as already discussed, the analytic structure of the quotient manifold in a neighbourhood of infinity is not inherited by the slices of constant coordinate t. Instead, one obtains fields which are of the form $f + \rho g$ with f, g analytic and ρ a suitable radial coordinate—recall that the radial coordinate is not smooth in a neighbourhood of i with respect to Cartesian coordinates. It is of interest to note that our analysis requires the explicit computation up to quadrupolar order of the expansions of the relevant conformal fields.

Our argument assumes, *a priori*, the existence of the stationary spacetime, and makes statements about the smoothness of the spacetime in a certain gauge from the known smoothness in another gauge. A proof that makes use of only the conformal evolution and properties of stationary initial data sets would be much more complicated and would require a much deeper understanding of the properties of the conformal field equations and associated conformal structures than the one that is currently available. We expect our analysis to shed some light in this direction.

As in the case of static spacetimes, our main result, on the one hand, ensures that the construction of the cylinder at spatial infinity for spacetimes without time reflection symmetry does not have spurious gauge singularities, and on the other hand, it is expected to play a key role in the proof of a suitable generalization of conjecture 1.

Overview of the paper

In section 2, we present a concise discussion of the conformal field equations and conformal geodesics. The presentation in this section is aimed at providing a context for the analysis of the subsequent sections of this paper. Section 3 briefly summarizes the so-called regular initial value problem at spatial infinity. This discussion includes, in particular, the construction of the so-called cylinder at spatial infinity. Section 4 discusses results about stationary spacetimes which are relevant for our analysis. Particular attention is paid to their asymptotic expansions in both the quotient manifold and in a Cauchy slice. A 'standard' conformal completion of stationary spacetimes is discussed. Section 5 discusses an alternative conformal completion for stationary spacetimes. This particular completion ultimately leads to the cylinder at spatial infinity. A discussion of the asymptotic expansions for the relevant field quantities in this conformal completion are provided. In particular, it is shown that the components of the Schouten and Weyl tensors with respect to a particular frame are regular at infinity. Section 6 provides a discussion of the construction of conformal Gaussian systems in the neighbourhood of the cylinder at spatial infinity of stationary spacetimes. As a result of this analysis, it is shown that in a certain gauge the setting of the initial value problem at spatial infinity is as regular as it is to be expected. In a second stage, it is shown that the construction is independent of the particulars of the choice of a conformal gauge. This discussion completes the proof of our main result. Section 7 provides some concluding remarks concerning our analysis. Some lengthy expansions are presented separately from the main text in the appendix.

2. Conformal field equations and conformal geodesics

The *regular finite initial value problem at spatial infinity*, presented in section 3, was introduced by Friedrich in [28] and is based on a conformal representation of the Einstein field equations, known as the *extended conformal field equations*. In this section, we elaborate further on the ideas discussed in subsection 1.4 of the introduction, and we present a concise discussion of this system and of its associated structures. The presentation is geared towards the purpose of this paper.

2.1. Weyl connections

Let $\tilde{\mathcal{M}}$ denote a four-dimensional manifold endowed with a Lorentzian metric $\tilde{g}_{\mu\nu}$. A conformal rescaling of the metric is given by

$$\tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu},$$

where Θ is a positive function on $\tilde{\mathcal{M}}$. The conformal class $[\tilde{g}]$ is the collection of all metrics conformally related to $\tilde{g}_{\mu\nu}$:

$$[\tilde{g}] \equiv \{g_{\mu\nu} | g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}, \Theta > 0\}.$$

Let $\hat{\nabla}$ denote the covariant derivative operator of a torsion-free connection on $\tilde{\mathcal{M}}$. This connection is called a *conformal connection* or *Weyl connection* for $[\tilde{g}]$ if for $g_{\mu\nu} \in [\tilde{g}]$ one has that

$$\hat{\nabla}_\mu g_{\nu\lambda} = -2 f_\mu \, g_{\nu\lambda} \tag{1}$$

for some smooth 1-form f_μ. The connection $\hat{\nabla}$ preserves the conformal structure of $[\tilde{g}]$. Furthermore, it does not depend on the class representative. For example, if $\check{g}_{\mu\nu} = \check{\Theta}^2 g_{\mu\nu}$, then

$$\hat{\nabla}_\mu \check{g}_{\nu\lambda} = -2 \check{f}_\mu \check{g}_{\nu\lambda},$$

with

$$\check{f}_\mu = f_\mu - \check{\Theta}^{-1}\partial_\mu\check{\Theta}.$$

If ∇ is the Lévi–Cività connection of $g_{\mu\nu}$, then from equation (1) we have that the connections ∇ and $\hat{\nabla}$ define the difference tensor $\hat{\nabla} - \nabla = S(f)$, given by

$$S(f)_\mu{}^\rho{}_\nu = \delta_\mu{}^\rho f_\nu + \delta_\nu{}^\rho f_\mu - g_{\mu\nu}g^{\rho\lambda}f_\lambda, \tag{2}$$

where $\delta_\mu{}^\rho$ is the Kronecker delta. This shows that $\hat{\nabla}$ can be specified using ∇ and f_μ.

2.2. The extended conformal field equations

We now specialize to the case where $\tilde{g}_{\mu\nu}$ is a solution of Einstein vacuum field equations on $\tilde{\mathcal{M}}$, $\mathrm{Ric}[\tilde{g}_{\mu\nu}] = 0$. The Weyl connection $\hat{\nabla}$ and the Lévi–Cività connections ∇ and $\tilde{\nabla}$ are related by

$$\hat{\nabla} - \tilde{\nabla} = S(\tilde{f}), \qquad \nabla - \tilde{\nabla} = S(\Theta^{-1}\mathrm{d}\Theta), \qquad \hat{\nabla} - \nabla = S(f).$$

The Riemann tensor of the Weyl connection $\hat{\nabla}$ can be decomposed as

$$\hat{R}^\mu{}_{\nu\lambda\rho} = 2\big(g^\mu{}_{[\lambda}\hat{L}_{\rho]\nu} - g^\mu{}_\nu\hat{L}_{[\lambda\rho]} - g_{\nu[\lambda}\hat{L}_{\rho]}{}^\mu\big) + C^\mu{}_{\nu\lambda\rho},$$

where $\hat{L}_{\mu\nu}$ and $C^\mu{}_{\nu\lambda\rho}$ denote, respectively, the Schouten and Weyl tensors. The Schouten tensor is given by

$$\hat{L}_{\mu\nu} = \tfrac{1}{2}\hat{R}_{(\mu\nu)} - \tfrac{1}{4}\hat{R}_{[\mu\nu]} - \tfrac{1}{12}\hat{R}g_{\mu\nu}$$

where

$$\hat{R}_{\mu\nu} = \hat{R}^\rho{}_{\mu\rho\nu}, \qquad \hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}.$$

In what follows, it will be convenient to consider the 1-form

$$d_\mu \equiv \Theta\tilde{f}_\mu = \Theta f_\mu + \nabla_\mu\Theta,$$

and the rescaled conformal Weyl tensor

$$W^\mu{}_{\nu\lambda\rho} = \Theta^{-1}C^\mu{}_{\nu\lambda\rho}.$$

In order to deal with the possible direction dependence of the various fields near space-like infinity, it is convenient to use a frame formalism and a suitably chosen orthonormal frame field. For this, let us take a frame field $\{e_\alpha\}_{\alpha=0,1,2,3}$ satisfying

$$g_{\alpha\beta} \equiv g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1).$$

Let $\hat{\nabla}_\alpha$ and ∇_α denote, respectively, the covariant derivatives in the direction of e_α with respect to the connections $\hat{\nabla}$ and ∇. The respective connection coefficients are defined as $\hat{\nabla}_\alpha e_\beta = \hat{\Gamma}_\alpha{}^\gamma{}_\beta e_\gamma$ and $\nabla_\alpha e_\beta = \Gamma_\alpha{}^\gamma{}_\beta e_\gamma$. The frame coefficients with respect to an as yet unspecified coordinate system x^μ are given by $e^\mu{}_\alpha = \langle \mathrm{d}x^\mu, e_\alpha\rangle$. Using equation (2), we have

$$\hat{\Gamma}_\alpha{}^\gamma{}_\beta = \Gamma_\alpha{}^\gamma{}_\beta + \delta_\alpha{}^\gamma f_\beta + \delta_\beta{}^\gamma f_\alpha - g_{\alpha\beta}g^{\gamma\delta}f_\delta,$$

where $f_\alpha = e^\mu{}_\alpha f_\mu$. Then, if all tensor fields are expressed in terms of the frame field and the corresponding connection coefficients, the extended conformal field equations are equations for the unknowns

$$u \equiv (e^\mu{}_\alpha, \hat{\Gamma}_\alpha{}^\gamma{}_\beta, \hat{L}_{\alpha\beta}, W^\alpha{}_{\beta\gamma\delta}),$$

and are given by

$$[e_\alpha, e_\beta] = (\hat{\Gamma}_\alpha{}^\gamma{}_\beta - \hat{\Gamma}_\beta{}^\gamma{}_\alpha)e_\gamma, \tag{3a}$$

8

$$e_{\alpha}\big(\hat{\Gamma}_{\beta}{}^{\delta}{}_{\gamma}\big) - e_{\beta}\big(\hat{\Gamma}_{\alpha}{}^{\delta}{}_{\gamma}\big) - \big(\hat{\Gamma}_{\alpha}{}^{\epsilon}{}_{\beta} - \hat{\Gamma}_{\beta}{}^{\epsilon}{}_{\alpha}\big)\hat{\Gamma}_{\epsilon}{}^{\delta}{}_{\gamma} + \hat{\Gamma}_{\alpha}{}^{\delta}{}_{\epsilon}\hat{\Gamma}_{\beta}{}^{\epsilon}{}_{\gamma} - \hat{\Gamma}_{\beta}{}^{\delta}{}_{\epsilon}\hat{\Gamma}_{\alpha}{}^{\epsilon}{}_{\gamma}$$
$$= 2\{g^{\delta}{}_{[\alpha}\hat{L}_{\beta]\gamma} - g^{\delta}{}_{\gamma}\hat{L}_{[\alpha\beta]} - g_{\gamma[\alpha}\hat{L}_{\beta]}{}^{\delta}\} + \Theta W^{\delta}{}_{\gamma\alpha\beta}, \tag{3b}$$

$$\hat{\nabla}_{\alpha}\hat{L}_{\beta\gamma} - \hat{\nabla}_{\beta}\hat{L}_{\alpha\gamma} = d_{\delta}W^{\delta}{}_{\gamma\alpha\beta}, \tag{3c}$$

$$\nabla_{\delta}W^{\delta}{}_{\gamma\alpha\beta} = 0. \tag{3d}$$

In the last equation, called the *Bianchi equation*, one has to consider the relation between $\hat{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}$ and $\Gamma_{\alpha}{}^{\gamma}{}_{\beta}$, whence $f_{\alpha} = \frac{1}{4}\hat{\Gamma}_{\alpha}{}^{\beta}{}_{\beta}$. Note that no differential equations are given for the fields Θ and d_{α}. This is due to the conformal gauge freedom introduced into Einstein field equations by considering general Weyl connections and conformal metrics.

2.3. The conformal Gauss gauge

The fields Θ and d_{α} can be specified by means of a choice of gauge. To this end, we consider conformal geodesics. A conformal geodesic for $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ is a curve $x(\tau)$ in $\tilde{\mathcal{M}}$ and a 1-form $\tilde{f}(\tau)$ along the curve, which solve the following system of ordinary differential equations:

$$(\tilde{\nabla}_{\dot{x}}\dot{x})^{\mu} + S(\tilde{f})_{\lambda}{}^{\mu}{}_{\rho}\dot{x}^{\lambda}\dot{x}^{\rho} = 0, \tag{4a}$$

$$(\tilde{\nabla}_{\dot{x}}\tilde{f})_{\nu} - \tfrac{1}{2}\tilde{f}_{\mu}S(\tilde{f})_{\lambda}{}^{\mu}{}_{\nu}\dot{x}^{\lambda} = \tilde{L}_{\lambda\nu}\dot{x}^{\lambda}. \tag{4b}$$

Conformal geodesics are invariants of the conformal structure in the following sense: if $x(\tau)$, $\tilde{f}(\tau)$ solve equations (4a)–(4b) and b is a smooth 1-form field on $\tilde{\mathcal{M}}$, then the pair $x(\tau)$, $(\tilde{f} - b)|_{x(\tau)}$ solves the same equations with $\tilde{\nabla}$ replaced by $\hat{\nabla} = \tilde{\nabla} + S(\tilde{b})$ and $\tilde{L}$ by $\hat{L}$. This means that $x(\tau)$, and in particular the parameter τ, do not depend on the Weyl connection in the conformal class which is used to write the conformal geodesic equations.

Conformal geodesics, and in particular congruences of conformal geodesics, can be used to construct a special gauge for the conformal equations. For this, let $\tilde{\mathcal{S}}$ be a space-like hypersurface in the given vacuum solution $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$. We choose on $\tilde{\mathcal{S}}$ a positive 'conformal factor' Θ_*, a frame field $e_{\alpha*}$, and a 1-form $\tilde{f}_*$, such that $\tilde{g}(e_{\alpha*}, e_{\beta*}) = \Theta_*^{-2}\eta_{\alpha\beta}$ and e_{0*} is orthogonal to $\tilde{\mathcal{S}}$. Then, there exists through each point $x_* \in \tilde{\mathcal{S}}$ a unique conformal geodesic $x(\tau)$, $\tilde{f}(\tau)$ with $\tau = 0$ on $\tilde{\mathcal{S}}$ which satisfies there the initial conditions $\dot{x} = e_{0*}$, $\tilde{f} = \tilde{f}_*$. These curves define a smooth caustic free congruence in a neighbourhood $\mathcal{U}$ of $\tilde{\mathcal{S}}$ if all data are smooth. Furthermore, $\tilde{f}$ defines a smooth 1-form on $\mathcal{U}$ which supplies a Weyl connection $\hat{\nabla}$ on $\mathcal{U}$ given by $\hat{\nabla} = \tilde{\nabla} + S(\tilde{f})$. A smooth frame field e_{α} and a conformal factor Θ are then obtained on $\mathcal{U}$ by solving

$$\hat{\nabla}_{\dot{x}}e_{\alpha} = 0,$$
$$\hat{\nabla}_{\dot{x}}\Theta = \Theta\langle\tilde{f}, \dot{x}\rangle$$

for given initial conditions $e_{\alpha} = e_{\alpha*}$, $\Theta = \Theta_*$ on $\tilde{\mathcal{S}}$. The evolution equation for the conformal factor Θ is a consequence of the conformal geodesic equations (4a)–(4b) and the choice $\dot{x} = e_0$. The propagation equation for the frame is consistent with this choice and (4a). Furthermore, it follows that the frame field is orthonormal for the metric $g_{\mu\nu} = \Theta^2\tilde{g}_{\mu\nu}$. Dragging along the congruence the local coordinates x^a, $a = 1, 2, 3$ on $\tilde{\mathcal{S}}$, and setting $x^0 = \tau$ we obtain a coordinate system. In this gauge, one has in $\mathcal{U}$ that

$$\dot{x} = e_0 = \partial_{\tau}, \qquad \hat{\Gamma}_0{}^{\beta}{}_{\alpha} = 0, \qquad \hat{L}_{0\alpha} = 0. \tag{5}$$

This choice of coordinates, frame field and conformal gauge will be referred to as *the conformal Gauss system*. Remarkably, in this gauge, it is possible to obtain explicit expressions for Θ and d in terms of the initial data, given by

$$\Theta(\tau) = \Theta_*\left(1 + \langle \tilde{f}_*, e_{0*}\rangle\tau + \tfrac{1}{4}g^\sharp(\tilde{f}_*, \tilde{f}_*)\tau^2\right), \tag{6a}$$

$$d_0 = \dot{\Theta}, \qquad d_a = \Theta_*\langle \tilde{f}_*, e_{a*}\rangle, \qquad a = 1, 2, 3, \tag{6b}$$

where the quantities with the subscript '*' are constant along the conformal geodesics and $g^\sharp$ denotes the contravariant version of $g_{\mu\nu}$. These expressions, together with equations (3a)–(3d) provide a complete system of equations for u, called the *general conformal field equations*. Setting $\boldsymbol{\alpha} = 0$ in (3a)–(3d) and observing the gauge conditions (5), one obtains the following evolution equations:

$$\partial_\tau e^\mu{}_\alpha = -\hat{\Gamma}_\alpha{}^\beta{}_0 e^\mu{}_\beta, \tag{7a}$$

$$\partial_\tau \hat{\Gamma}_\alpha{}^\gamma{}_\beta = -\hat{\Gamma}_\delta{}^\gamma{}_\beta \hat{\Gamma}_\alpha{}^\delta{}_0 + g^\gamma{}_0\hat{L}_{\alpha\beta} + g^\gamma{}_\beta\hat{L}_{0\alpha} - g_{\beta 0}\hat{L}_\alpha{}^\gamma + \Theta W^\gamma{}_{\beta 0\alpha}, \tag{7b}$$

$$\partial_\tau \hat{L}_{\alpha\beta} = d_\gamma W^\gamma{}_{\alpha 0\beta}. \tag{7c}$$

If one extracts from the Bianchi equation (3d) a symmetric hyperbolic system, one gets symmetric hyperbolic reduced equations for those components of u which are not determined explicitly by the gauge conditions. The resulting system is called the *reduced conformal field equations*. It can be shown that for such a choice of reduced equations, any solution which satisfies (3a)–(3d) on a suitable spacelike hypersurface does indeed satisfy the complete set of field equations in the part of the domain of dependence of the initial data set where Θ is positive. These equations are equivalent to the Einstein field equations in the sense that if one has a solution of the conformal system, then one has a solution of Einstein field equations in the region where the conformal factor is positive and *vice versa*. The substantial advantage of using the conformal system is that one can deal with regions where the conformal factor vanish. Furthermore, in the Gauss gauge the location of this region can be prescribed *a priori*, giving us full control on the conformal boundary if the evolution extends far enough.

3. The regular finite initial value problem at space-like infinity

As mentioned in the introduction, the construction of the regular finite initial value problem at space-like infinity consists of two main steps. For this, one considers a hypersurface $\mathcal{S}$, which is a one-point conformal compactification of the Cauchy hypersurface $\tilde{\mathcal{S}}$ and therefore contains a geometrically distinguished point i—cf the discussion in subsection 1.2. In the first step, the point i is blown up into a sphere $\mathcal{I}^0$. In the second step, a congruence of conformal geodesics is used to describe the evolution of the conformal fields in a neighbourhood of $\mathcal{I}^0$. The construction is described in detail in [31], and it is implemented through the bundle of spin frames over $\mathcal{S}$ near i. In what follows, we present a summary of the aspects of a similar construction based on the bundle of orthonormal frames—see also [30].

3.1. The blow up of spatial infinity

Start by considering a space-like Cauchy hypersurface $\mathcal{S}$ with the intrinsic metric h_{ab} containing the distinguished point i. Next, choose a fixed oriented h-orthonormal frame e_a, $a = 1, 2, 3$, at i. Any other such frame at i is obtained by a rotation of e_a. That is, all other h-orthonormal frames at i are of the form $e_a(s) = s^b{}_a e_b$ with $s = \left(s^b{}_a\right) \in SO(3)$. In particular, $e_3(s)$ covers

all possible directions at i as one lets s exhaust $SO(3)$. For a given value of s, one distinguishes $e_{\mathbf{3}}(s)$ as the radial vector at i. Keeping s fixed, we construct the h-geodesic starting at i that has the tangent vector $e_{\mathbf{3}}(s)$ and denote by ρ the affine parameter on the geodesic that vanishes at i. The frame $e_{\boldsymbol{a}}(s)$ is then parallelly transported along the geodesic. For a particular value of the affine parameter ρ, the frame thus obtained will be denoted by $e_{\boldsymbol{a}}(\rho, s)$. We will consider only the region $|\rho| < a$, where a is chosen such that the metric ball $\mathcal{B}$ centred at i with radius a is a convex normal neighbourhood for the 3-metric h. The map from the set $(-a, a) \times SO(3)$ into the bundle $SO(\mathcal{S})$ of oriented orthonormal frames over $\mathcal{S}$, given by $(\rho, s) \to e_{\boldsymbol{a}}(\rho, s)$, defines a smooth embedding of a four-dimensional manifold into $SO(\mathcal{S})$.

In what follows, only non-negative values of ρ will be considered. Denote by $\hat{\mathcal{B}}$ the image of the set $[0, a) \times SO(3)$. The boundary of $\hat{\mathcal{B}}$ will be denoted by $\mathcal{I}^0$. One has that $\mathcal{I}^0 = \{\rho = 0\} \simeq SO(3)$. Finally, let π denote the restriction to $\hat{\mathcal{B}}$ of the projection of $SO(\mathcal{S})$ onto $\mathcal{S}$. In what follows it will be convenient to consider the subgroup $SO(2)$ of $SO(3)$, given by $SO(2) = \{s' \in SO(3) \, | \, s'^{\boldsymbol{b}}{}_{\mathbf{3}} e_{\boldsymbol{b}} = e_{\mathbf{3}}\}$—this is the subgroup of $SO(3)$ whose action leaves $e_{\mathbf{3}}$ invariant. Accordingly, if $s \in SO(3)$ and $s' \in SO(2)$ then $e_{\boldsymbol{a}}(s)$ and $e_{\boldsymbol{a}}(s\,s')$ are parallelly transported along the same geodesic. Hence, when we consider the projection π we have that $\pi(e_{\boldsymbol{a}}(\rho, s)) = \pi(e_{\boldsymbol{a}}(\rho, s\,s'))$, and therefore the map π has a factorization

$$\hat{\mathcal{B}} \xrightarrow{\ \pi'\ } \mathcal{B}' = \hat{\mathcal{B}}/SO(2) \xrightarrow{\ \pi''\ } \mathcal{B}.$$

On the one hand, the projection π'' maps the set $\pi'(\mathcal{I}^0) \simeq S^2$ onto i and on the other it implies a diffeomorphism of $\mathcal{B}' \backslash \pi'(\mathcal{I}^0)$ onto the punctured ball $\mathcal{B} \backslash \{i\}$. This diffeomorphism can be used to identify these sets. However, instead of this, it is convenient to pull back the initial data on $\mathcal{B}$ to $\hat{\mathcal{B}}$ via π. Then, $\hat{\mathcal{B}}$ becomes the initial manifold, ρ and s are used as coordinates on it, and its boundary $\mathcal{I}^0$ is a blow up of i. This manifold has an extra dimension when compared to the initial hypersurface $\mathcal{B}$. This extra dimension is given by the action of $SO(2)$. Since all fields have a well-defined transformation behaviour (spin weight) under such action, on the part of $\hat{\mathcal{B}}$ where $\rho > 0$ vector fields X, $c_{\boldsymbol{a}}(\rho, s)$, $\boldsymbol{a} = 1, 2, 3$, can be prescribed such that X is generated by the action of $SO(2)$ and the vector fields $c_{\boldsymbol{a}}(\rho, s)$ satisfy $T(\pi)c_{\boldsymbol{a}}(\rho, s) = e_{\boldsymbol{a}}(\rho, s)$—see [31]. These vector fields allow the introduction of a frame formalism.

3.2. Implementing the conformal Gauss gauge

In the second step of the construction, the development of the data is considered. For this, one uses a conformal Gauss system—see section 2. In what follows, it is assumed that the conformal compactification of the hypersurface $\tilde{\mathcal{S}}$ has been achieved by means of a conformal factor Ω. The initial data for the conformal factor for the Gauss system is set by requiring that

$$\Theta_* \equiv \kappa^{-1} \Omega, \tag{8}$$

where the function κ satisfies

$$\kappa = \rho \kappa', \qquad \kappa' \in C^\infty(\hat{\mathcal{B}}), \qquad \kappa' > 0, \qquad X\kappa' = 0, \qquad \kappa'|_{\mathcal{I}^0} = 1.$$

The change of the conformal factor induced by the function κ implies a map $\Xi : e_{\boldsymbol{a}} \to \kappa e_{\boldsymbol{a}}$ which maps the set $\hat{\mathcal{B}} \backslash \mathcal{I}^0$ bijectively onto a smooth submanifold $\mathcal{B}^*$ of the bundle of frame fields over $\mathcal{B}$. The diffeomorphism Ξ is used to carry the coordinates ρ, s and the vector fields X, $c_{\boldsymbol{a}}$ to $\mathcal{B}^*$. The projection of $\mathcal{B}^*$ onto $\mathcal{B}$ will be denoted again by π.

The construction of the conformal Gaussian system requires initial data for the 1-form f. For this, one takes

$$\langle f, \partial_\tau \rangle = 0, \qquad \pi^* f = \kappa^{-1} \, \mathrm{d}\kappa. \tag{9}$$

The reduced field equations ($7a$)–($7c$) and the symmetric hyperbolic system implied by ($3d$) can be interpreted as equations in the development of $\mathcal{B}^*$. This development will be denoted by $\tilde{\mathcal{N}}$, and is a five-dimensional manifold smoothly embedded in the bundle of frame fields over $\tilde{\mathcal{M}}$. The manifold $\tilde{\mathcal{N}}$ is a $SO(2)$ bundle over the spacetime. Its projection sending $\tilde{\mathcal{N}}$ onto $\tilde{\mathcal{M}}$ will again be denoted by π. The coordinates ρ, s and the vector fields X, c_a are pushed forward with the flow of the conformal geodesics ruling $\tilde{\mathcal{N}}$ in such a way that X generates the kernel of π. The parameter $x^0 \equiv \tau$ defines a further independent coordinate with $x^0 = \tau = 0$ on $\mathcal{B}^*$. The tangent vector field of this congruence is denoted by ∂_τ. To interpret the reduced field equations as equations on $\tilde{\mathcal{N}}$, it is assumed that frame fields e_α are vector fields on $\tilde{\mathcal{N}}$ which are defined at a frame field (∂_τ, c_a) by the requirements that (i) they project onto the frame field defined by (∂_τ, c_a) on $\tilde{\mathcal{M}}$; (ii) they do not pick up components in the X direction. The unknowns in the reduced field equations are then interpreted as vector-valued functions on $\tilde{\mathcal{N}}$.

3.3. The conformal boundary

From equations ($6a$) and (9) together with the gauge choice $\langle \tilde{f}_*, e_{0*} \rangle = 0$, it follows that

$$\Theta = \Theta_* \left(1 - \tau^2 \frac{\kappa_*^2}{\omega_*^2} \right) \qquad \text{on} \quad \tilde{\mathcal{N}}, \tag{10}$$

where the function ω is given by

$$\omega = \frac{2\Omega}{\sqrt{|D_a \Omega D^a \Omega|}} \qquad \text{on} \quad \mathcal{B}^*.$$

As before, subscripts '$*$' imply that the relevant functions are constant along the conformal geodesics. Explicit expressions for d_α can be obtained from ($6b$) using that f and $\tilde{f}$ are related by $f = \tilde{f} - \Theta^{-1} d\Theta$.

An important property of the construction described in the previous lines is that if the initial data set for the reduced conformal field equations has a smooth limit as $\rho \to 0$, then it can be smoothly extended into the coordinate range $\rho \leqslant 0$. Similarly, Θ and d_α take smooth limits as $\rho \to 0$ and can then be extended smoothly into a range where $\rho \leqslant 0$. It follows that the initial value problem for the reduced field equations can be extended smoothly into a range where $\rho \leqslant 0$ in such a way that the reduced equations are still a symmetric hyperbolic system. If the development of the initial value problem just formulated extends far enough, then the following regions of the development can be distinguished:

$$\tilde{\mathcal{N}} \equiv \left\{ |\tau| < \frac{\omega}{\kappa}, 0 < \rho < a, s \in SO(3) \right\},$$
$$\bar{\mathcal{N}} \equiv \left\{ |\tau| \leqslant \frac{\omega}{\kappa}, 0 \leqslant \rho < a, s \in SO(3) \right\},$$

where ω/κ is a function of ρ and s. One also has the four-dimensional submanifolds

$$\mathscr{I}^+ \equiv \left\{ \tau = +\frac{\omega}{\kappa}, 0 < \rho < a, s \in SO(3) \right\},$$
$$\mathscr{I}^- \equiv \left\{ \tau = -\frac{\omega}{\kappa}, 0 < \rho < a, s \in SO(3) \right\},$$
$$\mathcal{I} \equiv \left\{ |\tau| < 1, \rho = 0, s \in SO(3) \right\},$$

and the three-dimensional submanifolds

$$\mathcal{I}^+ \equiv \left\{ \tau = +1, \rho = 0, s \in SO(3) \right\},$$
$$\mathcal{I}^- \equiv \left\{ \tau = -1, \rho = 0, s \in SO(3) \right\},$$
$$\mathcal{I}^0 \equiv \left\{ \tau = 0, \rho = 0, s \in SO(3) \right\},$$

where it has been observed that $\omega/\kappa \to 1$ as $\rho \to 0$. It can be verified that

$$
\begin{aligned}
&\Theta > 0 \quad \text{on} \quad \tilde{\mathcal{N}}, \\
&\Theta = 0, \quad \mathrm{d}\Theta \neq 0 \quad \text{on} \quad \mathscr{I}^- \cup \mathscr{I}^+ \cup \mathcal{I}, \\
&\Theta = 0, \quad \mathrm{d}\Theta = 0 \quad \text{on} \quad \mathcal{I}^- \cup \mathcal{I}^+.
\end{aligned}
$$

The initial hypersurface $\mathcal{B}^*$ is given by

$$
\mathcal{B}^* = \{\tau = 0, 0 < \rho < a, s \in SO(3)\}.
$$

Its closure in $\bar{\mathcal{N}}$ is given by

$$
\bar{\mathcal{B}} \equiv \{\tau = 0, 0 \leqslant \rho < a, s \in SO(3)\} = \mathcal{B}^* \cup \mathcal{I}^0.
$$

Factoring out the group $SO(2)$ projects $\tilde{\mathcal{N}}$ onto the set $\tilde{\mathcal{M}}$, representing the 'physical spacetime'.

Observation. The data on $\mathcal{B}^*$ have a unique smooth extension to $\bar{\mathcal{B}}$. Furthermore, the solution on $\bar{\mathcal{N}}$ depends only on this data as the set $\mathcal{I}$ is a total characteristic of the system of equations, and therefore the solution there depends only on the data on $\mathcal{I}^0$. The set $\mathcal{I}$, referred to as the *cylinder at spacelike infinity*, represents a boundary of the spacetime $\tilde{\mathcal{N}}$, and may be understood as a blow up of spacelike infinity i^0. Of particular importance are the sets $\mathcal{I}^+$ and $\mathcal{I}^-$, *the critical sets*, as the system of evolution equations degenerates there. This degeneracy makes it very difficult to make any statement about the smoothness of the solution to the initial value problem stated above, and it has even been shown that for general initial data these sets are not regular—see e.g. [28, 31].

In view of the discussion in the previous paragraphs, the objective of this paper is as follows: to show that for stationary asymptotically flat initial data the solutions to the regular finite initial value problem are smooth in a neighbourhood of $\mathcal{I}$, and in particular are smooth through $\mathcal{I}^+$ and $\mathcal{I}^-$.

4. Stationary asymptotically flat spacetimes

In what follows, let $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu}, \xi^\mu)$ denote a stationary spacetime. That is, $\tilde{\mathcal{M}}$ is a four-dimensional manifold, $\tilde{g}_{\mu\nu}$ is a Lorentzian metric on $\tilde{\mathcal{M}}$ with signature $(+, -, -, -)$ satisfying the Einstein vacuum field equations, and ξ^μ is a time-like Killing vector field with complete orbits. As discussed in section 1.3, instead of working with the four-dimensional manifold $\tilde{\mathcal{M}}$, it is more convenient to consider the quotient manifold, $\tilde{\mathcal{X}}$, of $\tilde{\mathcal{M}}$ with respect to the trajectories of the Killing vector ξ^μ—see [12, 33, 34].

4.1. The stationary metric in terms of quantities on the quotient manifold

Locally, the metric $\tilde{g}_{\mu\nu}$ can be written in terms of quantities defined on the quotient manifold $\tilde{\mathcal{X}}$: a scalar V, a 1-form $\tilde{\beta}_a$ and a Riemannian metric $\tilde{\gamma}_{ab}$. More precisely, one has that

$$
\tilde{g}_{\mu\nu}\, \mathrm{d}\tilde{x}^\mu\, \mathrm{d}\tilde{x}^\nu = V(\mathrm{d}t + \tilde{\beta}_a\, \mathrm{d}\tilde{x}^a)(\mathrm{d}t + \tilde{\beta}_b\, \mathrm{d}\tilde{x}^b) - V^{-1}\tilde{\gamma}_{ab}\, \mathrm{d}\tilde{x}^a\, \mathrm{d}\tilde{x}^b, \tag{11}
$$

where V, $\tilde{\beta}_a$ and $\tilde{\gamma}_{ab}$ depend only on the spatial coordinates $\tilde{x}^a$.

In order to obtain the field equations implied on $\tilde{\mathcal{X}}$ by Einstein vacuum field equations on $\tilde{\mathcal{M}}$, it is convenient to consider the quantity $\tilde{\omega}_a$, defined on $\tilde{\mathcal{X}}$ by

$$
\tilde{\omega}_a = -V^2 \tilde{\epsilon}_{abc} \tilde{D}^b \tilde{\beta}^c, \tag{12}
$$

where $\tilde{D}$ is the covariant derivative associated with $\tilde{\gamma}_{ab}$. Then, the Einstein vacuum field equations on $\tilde{\mathcal{M}}$ imply

$$
\tilde{D}_{[a}\tilde{\omega}_{b]} = 0.
$$

If one further assumes $\tilde{\mathcal{X}}$ to be simply connected (our analysis will concentrate in a neighbourhood of infinity), then there exists a scalar field ω such that

$$\tilde{D}_a\omega = \tilde{\omega}_a.$$

In what follows, we will consider 'gauge' transformations of the form

$$\tilde{\beta}_a \to \tilde{\beta}_a + \partial_a f,$$

where f is a scalar field on $\tilde{\mathcal{X}}$. Clearly, $\tilde{\omega}_a$ does not change under these transformations—cf equation (12). Moreover, the metric remains unchanged if one sets $t \to t - f$.

4.2. The Hansen potentials

In order to write down the stationary field equations, it is convenient to introduce the so-called *Hansen potentials*:

$$\tilde{\phi}_M = \frac{V^2 + \omega^2 - 1}{4V}, \qquad \tilde{\phi}_S = \frac{\omega}{2V}, \qquad \tilde{\phi}_K = \frac{V^2 + \omega^2 + 1}{4V}.$$

They are not independent as

$$\tilde{\phi}_M^2 + \tilde{\phi}_S^2 - \tilde{\phi}_K^2 = -\tfrac{1}{4}.$$

The vacuum field equations on $\tilde{\mathcal{M}}$ then imply on $\tilde{\mathcal{X}}$:

$$\tilde{\Delta}\tilde{\phi}_\alpha = 2\tilde{R}[\tilde{\gamma}]\tilde{\phi}_\alpha, \qquad \alpha = M, S, K, \tag{13a}$$

$$\tilde{R}_{ab}[\tilde{\gamma}] = 2(\tilde{D}_a\tilde{\phi}_M\tilde{D}_b\tilde{\phi}_M + \tilde{D}_a\tilde{\phi}_S\tilde{D}_b\tilde{\phi}_S - \tilde{D}_a\tilde{\phi}_K\tilde{D}_b\tilde{\phi}_K). \tag{13b}$$

The latter will be regarded as field equations for $\tilde{\gamma}_{ab}$, $\tilde{\phi}_M$ and $\tilde{\phi}_S$ on $\tilde{\mathcal{X}}$. They are equivalent to Einstein vacuum field equations on $\tilde{\mathcal{M}}$ in the sense that $\tilde{\mathcal{M}}$ can be reconstructed as a stationary spacetime if $\tilde{\gamma}_{ab}$, $\tilde{\phi}_M$ and $\tilde{\phi}_S$ are given.

4.3. The 3+1 form of the stationary metric

As mentioned in section 1.3 of the introduction, although the field equations take the simple form (13a)–(13b) in $\tilde{\mathcal{X}}$, our main interest is to consider the Cauchy problem with data arising from stationary spacetimes. For this, one needs to consider the $3 + 1$ decomposition of the spacetime metric $\tilde{g}_{\mu\nu}$ with respect to a particular spacelike hypersurface of $\tilde{\mathcal{M}}$. If we choose $\tilde{\mathcal{S}}$ as defined by $t =$ constant, then $\tilde{g}_{\mu\nu}$ has a $3 + 1$ decomposition with respect to $\tilde{\mathcal{S}}$ given by

$$\tilde{g}_{\mu\nu}\,\mathrm{d}\tilde{x}^\mu\,\mathrm{d}\tilde{x}^\nu = \tilde{N}^2\,\mathrm{d}t^2 - \tilde{h}_{ab}(\tilde{N}^a\,\mathrm{d}t + \mathrm{d}\tilde{x}^a)(\tilde{N}^b\,\mathrm{d}t + \mathrm{d}\tilde{x}^b), \tag{14}$$

where $\tilde{N}$, $\tilde{N}^a$, $\tilde{h}_{ab}$ denote, respectively, the lapse function, the shift vector and the intrinsic metric of the hypersurface $\tilde{\mathcal{S}}$, respectively. Comparing the line elements (11) and (14), one finds that

$$\tilde{N}^2 = \frac{V}{1 - V^2\tilde{\beta}_a\tilde{\beta}^a}, \qquad \tilde{N}^a = -\frac{V^2\tilde{\beta}^a}{1 - V^2\tilde{\beta}_b\tilde{\beta}^b}, \qquad \tilde{h}_{ab} = V^{-1}\tilde{\gamma}_{ab} - V\tilde{\beta}_a\tilde{\beta}_b. \tag{15}$$

We will adopt the convention of moving indices of objects in $\tilde{\mathcal{X}}$ with the quotient metric $\tilde{\gamma}_{ab}$. The relations (15) allow us to go back and forth between quantities defined on $\tilde{\mathcal{X}}$ and quantities defined on $\tilde{\mathcal{S}}$.

14

4.4. Asymptotic flatness and its consequences

Following the usual assumptions about asymptotic flatness for stationary spacetimes, it will be assumed that $(\tilde{\mathcal{X}}, \tilde{\gamma}_{ab}, \tilde{\phi}_M, \tilde{\phi}_S)$ is asymptotically Euclidean and regular in the sense discussed in section 1.2 of the introduction—see e.g. [14, 15]. More precisely, it is assumed that there exists a manifold $\mathcal{X}$, such that $\mathcal{X} = \tilde{\mathcal{X}} \cup \{i\}$, where i is a point. As our analysis is local to infinity, we only need that the asymptotic end can be compactified. Thus, $\mathcal{X}$ could be, on the whole, noncompact. Furthermore, it is assumed that for some real constant $B^2 > 0$, the conformal factor

$$\Omega = \tfrac{1}{2}B^{-2}\big[\big(1 + 4\tilde{\phi}_M^2 + 4\tilde{\phi}_S^2\big)^{\frac{1}{2}} - 1\big] \tag{16}$$

is $C^{2,\alpha}$ on $\mathcal{X}$ and satisfies

$$\Omega(i) = 0, \qquad D_a\Omega(i) = 0.$$

In addition it will be assumed that

$$\gamma_{ab} = \Omega^2 \tilde{\gamma}_{ab} \tag{17}$$

extends to a $C^{4,\alpha}$ metric on $\mathcal{X}$ and satisfies

$$D_a D_b \Omega(i) = 2\gamma_{ab}(i),$$

where D is the Lévi–Cività covariant derivative of the 3-metric γ_{ab}.

The conformal rescaling of the metric given by (17) suggests the following definition of rescaled potentials:

$$\phi_\alpha = \Omega^{-\frac{1}{2}}\tilde{\phi}_\alpha, \qquad\qquad \alpha = M, S, K.$$

The motivation behind the introduction of conformally rescaled fields is the following theorem by Beig and Simon [14]—see also [38].

Theorem 1 (Theorem 1 of [14]). *For any asymptotically flat solution $(\tilde{\gamma}_{ab}, \tilde{\phi}_M, \tilde{\phi}_S)$ of the stationary equations (13a)–(13b), there exists a chart defined in some neighbourhood of i in $\mathcal{X}$ such that $(\gamma_{ab}, \phi_M, \phi_S, \Omega)$ are analytic.*

Remark. Given the chart indicated by the previous theorem, one can make a coordinate transformation to γ-normal coordinates x^a centred at i. The fields $(\gamma_{ab}, \phi_M, \phi_S, \Omega)$ are also analytic with respect to the normal coordinates x^a—this follows from the Cauchy–Kovalewska theorem applied to the equations of the radial geodesics written in the analytic coordinates given by theorem 1. It is important to note that theorem 1 does not make any assertion about the smoothness of other quantities on $\mathcal{X}$, like V, ϕ_K, β_a or quantities defined on a hypersurface of the spacetime. An analysis of the regularity of these and other related quantities has been carried out in [22].

In what follows, we will require several results from [22]. These will be presented here for completeness and quick reference. In the following, let the radial coordinate ρ be defined by

$$\rho \equiv \left(\sum_{i=1}^{3}(x^a)^2\right)^{1/2}. \tag{18}$$

In [22] it was found that the non-analyticity of the relevant functions is of a very special type and it depends only on the coordinate ρ. Accordingly, one defines the following function space.

Definition (Definition 2.2 of [22]). *We define the space E^ω as the set*

$$E^\omega = \{f = f_1 + \rho f_2 : f_1, f_2 \in C^\omega\},$$

where C^ω denotes the set of analytic functions in a neighbourhood of i.

Associated with the latter definition, one has the following.

Lemma 1 (Lemma 2.3 of [22]). *Let $f, g \in E^\omega$; then*

(i) $f + g \in E^\omega$,
(ii) $fg \in E^\omega$,
(iii) *if* $f \neq 0$, *then* $1/f \in E^\omega$.

Obviously, if $f \in C^\omega$, then $f \in E^\omega$. The main result of the analysis in [22] is that most of the relevant quantities belong to E^ω. In particular, one has that

Lemma 2 (Lemmas 2.4 and 2.5 of [22]). *In the normal coordinates implied by theorem 1, one has that*

$$V \in E^\omega.$$

Furthermore, there exists a choice of gauge for which the 1-form β_a has the following form:

$$\beta_a = \beta_a^1 + \frac{\beta_a^2}{\rho}, \tag{19}$$

where β_a^1, β_a^2 are analytic functions of x^a given by

$$\beta_a^1 = e_{abc} f_1^b x^c, \qquad \beta_a^2 = e_{abc} f_2^b x^c, \tag{20}$$

where f_1^a, f_2^a are analytic and e_{abc} is the flat volume element. In particular, it follows that $\beta_a x^a = 0$.

4.5. A first conformal compactification of stationary spacetimes

As discussed in [22], the fields Ω and V can be used to construct a first conformal compactification of the *physical manifolds* $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{S}}$. For this one introduces a conformal factor $\check{\Omega}$ via

$$\check{\Omega} \equiv V^{1/2}\Omega.$$

Note that, as defined, $\check{\Omega}$ is not analytic as V is not analytic. The associated rescaled metrics are then given by

$$\check{g}_{\mu\nu} = \check{\Omega}^2 \tilde{g}_{\mu\nu}, \qquad \check{h}_{ab} = \check{\Omega}^2 \tilde{h}_{ab}.$$

Hence, one has that

$$\check{g}_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = V^2\Omega^2(\mathrm{d}t + \beta_a\, \mathrm{d}x^a)(\mathrm{d}t + \beta_b\, \mathrm{d}x^b) - \gamma_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b. \tag{21}$$

Moreover, one also has a $3 + 1$ decomposition with respect to the hypersurfaces $\tilde{\mathcal{S}} = \{t = \text{constant}\}$:

$$\check{g}_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = N^2\, \mathrm{d}t^2 - \check{h}_{ab}(N^a\, \mathrm{d}t + \mathrm{d}x^a)(N^b\, \mathrm{d}t + \mathrm{d}x^b). \tag{22}$$

By comparison with equation (14), one gets that

$$N = \check{\Omega}\tilde{N}, \qquad N^a = \tilde{N}^a, \tag{23a}$$

$$\check{h}_{ab} = \gamma_{ab} - V^2\Omega^2\beta_a\beta_b, \tag{23b}$$

where $\check{h}_{ab}$ is the intrinsic metric on $\tilde{\mathcal{S}}$. The following result is fundamental for our subsequent analysis.

16

Theorem 2 (Theorem 2.6 of [22])**.** *Assume β_a is given by lemma 2. Then, in some neighbourhood of i, the metric $\check{h}_{ab}$ has the form*

$$\check{h}_{ab} = \check{h}^1_{ab} + \rho^3 \check{h}^2_{ab}, \tag{24}$$

where $\check{h}^1_{ab}$, $\check{h}^1_{ab} \in C^\omega$.

Remark. The later result implies that the conformal 3-metric $\check{h}_{ab}$ is in $C^{2,\alpha}$. Therefore, the conformal factor $\check{\Omega}$ can be used to define a conformal compactification $\mathcal{S}$ of the Cauchy slice $\tilde{\mathcal{S}}$ plus the point at infinity i, in the same way as made for $\tilde{\mathcal{X}}$. This implies that the pair $(\tilde{\mathcal{S}}, \check{h}_{ab})$ admits a $C^{2,\alpha}$ compactification—that is, the pair is asymptotically Euclidean and regular in the sense discussed in section 1.2 of the introduction. The decomposition of β_a given by equation (19) in lemma 2, and thus, the decomposition of $\check{h}$ given by equation (24) is preserved under the transformation

$$\beta_a \to \beta_a + \partial_a f, \qquad f \in E^\omega.$$

If one imposes the condition $x^a \beta_a = 0$, then one fixes $\partial_a f$. Accordingly, β_a, as given by equation (19), is the unique possible choice of the 1-form β_a that satisfies this condition.

Let $\tilde{\chi}_{ab}$, $\check{\chi}_{ab}$ denote, respectively, the extrinsic curvatures of $\tilde{\mathcal{S}}$ with respect to the metrics $\tilde{g}_{\mu\nu}$ and $\check{g}_{\mu\nu}$. One has that $\tilde{\chi}_{ab} = \check{\Omega}^{-1}\check{\chi}_{ab}$. Furthermore, we define

$$\check{\psi}_{ab} \equiv \check{\Omega}^{-1}\tilde{\chi}_{ab} = \check{\Omega}^{-2}\check{\chi}_{ab}.$$

The behaviour of $\check{\psi}_{ab}$ near i is given by the following result.

Theorem 3 (Theorem 2.7 of [22])**.** *Assume β_a as given by lemma 2. Then, in some neighbourhood of i, the tensor $\check{\psi}_{ab}$ has the form*

$$\check{\psi}_{ab} = \rho^{-5} f x_{(a} \beta^2_{b)} + \rho^{-3} \check{\psi}^1_{ab},$$

where $\check{\psi}^1_{ab} \in E^\omega$, $f \in E^\omega$ and β^2_a is given by (20). Furthermore, $\rho^8 \check{\psi}_{ab} \check{\psi}^{ab} \in E^\omega$.

4.6. Detailed expansions at infinity

The key for our present analysis is that if a quantity belongs to the space E^ω, then although it is not analytic, it nevertheless has in a neighbourhood of infinity an analytic expansion in terms of the radial coordinate ρ and the angular coordinates. In what follows, it will be necessary not only to know that relevant fields belong to E^ω, but also to know the first orders of the expansion in a neighbourhood of i. In [41], the first orders of the asymptotic expansions of the unrescaled fields $\tilde{\phi}_M$, $\tilde{\phi}_S$, $\tilde{\phi}_K$ and $\tilde{\gamma}_{ab}$ have been explicitly given in terms of constant tensors $M, S, M_a, S_a, M_{ab}, S_{ab}$, etc. These expansions read

$$\tilde{\phi}_M = \frac{M}{\tilde{r}} + \frac{M_a \tilde{x}^a}{\tilde{r}^3} + \frac{M_{ab}\tilde{x}^a \tilde{x}^b}{2\tilde{r}^5} + \frac{M(M^2+S^2)}{\tilde{r}^3} + O(\tilde{r}^{-4}),$$

$$\tilde{\phi}_S = \frac{S}{\tilde{r}} + \frac{S_a \tilde{x}^a}{\tilde{r}^3} + \frac{S_{ab}\tilde{x}^a \tilde{x}^b}{2\tilde{r}^5} + \frac{S(M^2+S^2)}{\tilde{r}^3} + O(\tilde{r}^{-4}),$$

$$\tilde{\phi}_K = \frac{1}{2} + \frac{M^2+S^2}{\tilde{r}^2} + \frac{2MM_a x^a}{\tilde{r}^4} + \frac{2SS_a x^a}{\tilde{r}^4} + O(\tilde{r}^{-4}),$$

$$\tilde{\gamma}_{ab} = \delta_{ab} - \frac{M^2}{\tilde{r}^4}(\delta_{ab}\tilde{r}^2 - \tilde{x}_a \tilde{x}_b) - \frac{2MM_{(a}\tilde{x}_{b)}}{\tilde{r}^4} - \frac{2MM_c \tilde{x}^c \delta_{ab}}{\tilde{r}^4} + \frac{4MM_c \tilde{x}^c \tilde{x}_a \tilde{x}_b}{\tilde{r}^6}$$
$$+ \frac{S^2}{\tilde{r}^4}(\delta_{ab}\tilde{r}^2 + \tilde{x}_a \tilde{x}_b) + \frac{2SS_{(a}\tilde{x}_{b)}}{\tilde{r}^4} + \frac{2SS_c \tilde{x}^c \delta_{ab}}{\tilde{r}^4} - \frac{4SS_c \tilde{x}^c \tilde{x}_a \tilde{x}_b}{\tilde{r}^6} + O(\tilde{r}^{-4}),$$

where indices in the coordinates and constant tensors are moved with the flat metric δ_{ab}.

Remark. Asymptotic flatness implies that the angular momentum monopole S has to vanish. Furthermore, by a suitable choice of the origin for the coordinates $\tilde{x}^a$, one can make $M_a = 0$. In order to simplify our computations, we assume this choice of coordinates.

Following the discussion in the previous subsections, we want to consider the fields in a neighbourhood of i written in terms of normal coordinates centred at i. This is done in several steps. First, one changes coordinates from $\tilde{x}^a$ to y^a, via

$$\tilde{x}^a = -\frac{y^a}{r^2}, \qquad r^2 = \delta_{ab} y^a y^b = \frac{1}{\tilde{r}^2}.$$

Then, using expression (16) for the conformal factor with $B = M$, one finds that

$$\Omega = r^2 + M^2 r^4 + \frac{1}{M^2}(S_a y^a)^2 r^2 + \frac{1}{M} M_{ab} y^a y^b r^2 + O(r^5).$$

Furthermore, from the definition of the conformal potentials it follows that

$$\phi_M = M + \frac{1}{2}M^3 r^2 - \frac{1}{2M}(S_a y^a)^2 + O(r^3),$$

$$\phi_S = -S_a y^a + \frac{1}{2}S_{ab} y^a y^b + O(r^3),$$

$$\phi_K = \frac{1}{2r} + \frac{3}{4}M^2 r - \frac{1}{4M^2 r}(S_a y^a)^2 - \frac{1}{4Mr}M_{ab} y^a y^b + O(r^2).$$

Next, we give the expansion for the conformally rescaled metric γ_{ab}. However, instead of giving these expansions with respect to the coordinates y^a, we use normal coordinates, x^a, centred at i—as it was done in the remark after theorem 1. For this one requires that after the coordinate transformation $y^a \to x^a$, the metric satisfies

$$\gamma_{ab} x^b = \delta_{ab} x^b.$$

A lengthy computation renders

$$\gamma_{ab} = \delta_{ab} + \frac{\rho^2}{3M^2}(M^4(\delta_{ab} - e_a e_b) + 2M(M_{ab} + \delta_{ab} M_{cd} e^c e^d - 2e_{(a} M_{b)c} e^c)$$
$$+ 2(S_a S_b + \delta_{ab}(S_c e^c)^2 - 2e_{(a} S_{b)} S_c e^c)) + O(\rho^3),$$

where ρ is defined by (18) and $e^a \equiv x^a/\rho$. The leading terms of the expansion of the inverse metric are given by

$$\gamma^{ab} = \delta^{ab} - \frac{\rho^2}{3M^2}(M^4(\delta^{ab} - e^a e^b) + 2M(M^{ab} + \delta^{ab} M_{cd} e^c e^d - 2e^{(a} M^{b)c} e_c)$$
$$+ 2(S^a S^b + \delta^{ab}(S_c e^c)^2 - 2e^{(a} S^{b)} S_c e^c)) + O(\rho^3).$$

The transformation between the coordinates y^a and x^a is given by

$$y^a = x^a + \frac{1}{3M^2}\rho^3(-M^4 e^a + M(M^{ab} e_b - 2e^a M_{bc} e^b e^c) + S^a S_b e^b - 2e^a (S_b e^b)^2) + O(\rho^4), \tag{25a}$$

$$r = \rho - \frac{1}{3M^2}\rho^3(M^4 + M M_{ab} e^a e^b + (S_a e^a)^2) + O(\rho^4). \tag{25b}$$

Using (25a)–(25b), one can express the rescaled potentials in normal coordinates. One finds that

$$\phi_M = M + \frac{1}{2M}\rho^2(M^4 - (S_a e^a)^2) + O(\rho^3),$$

$$\phi_S = -S_a e^a \rho + \frac{1}{2}\rho^2 S_{ab} e^a e^b + O(\rho^3),$$

$$\phi_K = \frac{1}{2\rho} + \frac{1}{12M^2}\rho(11M^4 - M M_{ab} e^a e^b - (S_a e^a)^2) + O(\rho^2).$$

18

For later use, it is also convenient to calculate the expansion of other quantities. Namely,

$$V = 1 + 2M\rho + 2M^2\rho^2 + \frac{1}{3M}\rho^3(4M^4 + MM_{ab}e^a e^b - 2(S_a e^a)^2) + O(\rho^4),$$

$$\beta_a = e_{abc}\, e^c(2S^b + O(\rho)).$$

Using formula (21), one finds that the four-dimensional spacetime metric and its inverse are given by

$$(\check{g}_{\mu\nu}) = \begin{pmatrix} V^2\Omega^2 & V^2\Omega^2\beta_a \\ V^2\Omega^2\beta_b & -\gamma_{ab} + V^2\Omega^2\beta_a\beta_b \end{pmatrix},$$

$$(\check{g}^{\mu\nu}) = \begin{pmatrix} \dfrac{1 - V^2\Omega^2\beta_c\beta^c}{V^2\Omega^2} & \beta^a \\ \beta^b & -\gamma^{ab} \end{pmatrix},$$

where $\beta^a \equiv \gamma^{ab}\beta_b$. Moreover, one has the following expansions for the components of the metric and its inverse:

$$\check{g}_{tt} = \rho^4 + 4M\rho^5 + \frac{2}{3M^2}\rho^6(13M^4 + MM_{ab}e^a e^b + (S_a e^a)^2) + O(\rho^7),$$

$$\check{g}_{ta} = \rho^4 e_{abc}\, e^c(2S^b + O(\rho)),$$

$$\check{g}_{ab} = -\delta_{ab} - \frac{\rho^2}{3M^2}(M^4(\delta_{ab} - e_a e_b) + 2M(M_{ab} + \delta_{ab}M_{cd}e^c e^d - 2e_{(a}M_{b)c}e^c)$$
$$+ 2(S_a S_b + \delta_{ab}(S_c e^c)^2 - 2e_{(a}S_{b)}S_c e^c)) + O(\rho^3),$$

$$\check{g}^{tt} = \frac{1}{\rho^4} - \frac{4M}{\rho^3} + \frac{2}{3M^2\rho^2}(11M^4 - MM_{ab}e^a e^b - (S_a e^a)^2) + O(\rho^{-1}),$$

$$\check{g}^{ta} = e^{abc}e_c(2S_b + O(\rho)),$$

$$\check{g}^{ab} = -\delta^{ab} + \frac{\rho^2}{3M^2}(M^4(\delta^{ab} - e^a e^b) + 2M(M^{ab} + \delta^{ab}M_{cd}e^c e^d - 2e^{(a}M^{b)c}e_c)$$
$$+ 2(S^a S^b + \delta^{ab}(S_c e^c)^2 - 2e^{(a}S^{b)}S_c e^c)) + O(\rho^3).$$

One also finds that

$$\check{\Omega} = V^{1/2}\Omega = \rho^2 + M\rho^3 + \frac{1}{3M^2}\rho^4\left(\frac{5}{2}M^4 + MM_{ab}e^a e^b + (S_a e^a)^2\right) + O(\rho^5).$$

5. Conformal extension of stationary vacuum spacetimes

In this section, we introduce a conformal extension of vacuum stationary spacetimes which is well adapted for the analysis of the structure of spatial infinity. To this end, we start from the conformal extension of the stationary metric given by the line element (21). In order to analyse the regularity of the relevant fields, we will make use of the first order expansions in a neighbourhood of i, which we collect as we consider the corresponding quantities. The present analysis is based on a similar analysis for static spacetimes given in [31].

Recall that x^a, $a = 1, 2, 3$, are normal coordinates of the quotient metric γ_{ab}. We have defined

$$\rho \equiv \left(\sum_{a=1}^3 (x^a)^2\right)^{1/2}, \qquad e^a \equiv \frac{x^a}{\rho} \quad \text{for } \rho > 0.$$

For constant t, the surfaces of constant ρ are diffeomorphic to a two-dimensional sphere. Accordingly, arbitrary coordinates ψ^A, $A = 2, 3$, on the 2-sphere $\mathbb{S}^2 = \{|x| = 1\}$ can be used to parametrize e^a. We then write

$$e^a = e^a(\psi^A), \qquad \mathrm{d}e^a = e^a{}_{,\psi^A}\mathrm{d}\psi^A.$$

The coordinates ψ^A can be chosen such that e^a depends analytically on them. Consistent with the previous definitions, one has that $x^a = \rho\, e^a(\psi^A)$. Therefore, the conformal 3-metric $\check{h}_{ab}$ given by equation (23b) takes the form

$$\check{h} = \mathrm{d}\rho^2 + \rho^2 k,$$

where k denotes the two-dimensional metric on the surfaces of constant ρ given by

$$k = k_{AB}\,\mathrm{d}\psi^A\,\mathrm{d}\psi^B \equiv \check{h}_{ab}(\rho)\,\mathrm{d}e^a\,\mathrm{d}e^b.$$

Remark. Note that as $\rho \to 0$, the metric $\check{h}$ approaches the standard Euclidean metric in normal coordinates and the metric k approaches the standard line element $\mathrm{d}\sigma^2 = k(0, \psi^A)$ on the two-dimensional unit sphere in the coordinates ψ^A.

5.1. Coordinates for the analysis of the cylinder at spatial infinity

The coordinates (t, ρ, ψ^A) are well adapted to the description of spatial infinity as a point. In order to resolve the structure of the cylinder at spatial infinity, it is convenient to introduce new coordinates $(\bar{\tau}, \bar{\rho}, \psi^A)$. The coordinate change is inspired by an analysis of the Minkowski spacetime—see e.g. [28, 43].

　　We define $x^0 = t$, $x^{0'} = \bar{\tau}$, $x^{1'} = \bar{\rho}$, $x^{A'} = \psi^A$ and consider the map $\Phi : x^{\mu'} \to x^\mu(x^{\mu'})$ given by

$$t(x^{\mu'}) = x^0(x^{\mu'}) = \int_{(1-\bar{\tau})\bar{\rho}}^{\bar{\rho}} \frac{\mathrm{d}s}{(V\Omega)[se^a(\psi^A)]},$$

$$x^a(x^{\mu'}) = (1 - \bar{\tau})\bar{\rho}e^a[\psi^A],$$

where the squared brackets indicate the arguments of a given function. One explicitly finds that

$$\mathrm{d}t = \frac{\bar{\rho}}{(V\Omega)[(1-\bar{\tau})\bar{\rho}e^a]}\,\mathrm{d}\bar{\tau} + \left(\frac{1}{(V\Omega)[\bar{\rho}e^a]} - \frac{1-\bar{\tau}}{(V\Omega)[(1-\bar{\tau})\bar{\rho}e^a]}\right)\mathrm{d}\bar{\rho} + l, \qquad (26a)$$

$$\mathrm{d}x^a = -\bar{\rho}e^a\,\mathrm{d}\bar{\tau} + (1 - \bar{\tau})e^a\,\mathrm{d}\bar{\rho} + (1 - \bar{\tau})\bar{\rho}\,\mathrm{d}e^a, \qquad (26b)$$

where

$$l = l_A\,\mathrm{d}\psi^A, \qquad l_A = \int_{(1-\bar{\tau})\bar{\rho}}^{\bar{\rho}} \left(\frac{1}{(V\Omega)[se^a]}\right)_{,\psi^A}\mathrm{d}s, \qquad \mathrm{d}e^a = (e^a)_{,\psi^A}\,\mathrm{d}\psi^A.$$

The differentials $\mathrm{d}t$, $\mathrm{d}x^a$ are independent for $0 \leqslant \bar{\tau} < 1$ and $\bar{\rho}$ between zero and a small enough number. Therefore, one can consider the $x^{\mu'}$ as smooth coordinates on an open neighbourhood of space-like infinity in $\{t \geqslant 0\}$. For later use, we notice the relation

$$\rho = (1 - \bar{\tau})\bar{\rho},$$

which will be used to simplify the notation.

　　The main purpose of the coordinate transformation is to remove the coefficient $V^2\Omega^2$ from the time–time component of the metric $\check{g}_{\mu\nu}$ and to introduce a convenient parametrization set at which $\rho = 0$, so that it seems to have an extension in the time direction.

5.2. A frame adapted to spatial infinity

We define a set of frame fields $v_{\boldsymbol{\alpha}}$ and their associated coframe fields $\alpha^{\boldsymbol{\alpha}}$ by

$$v_{\mathbf{0}} = \partial_{\bar{\tau}}, \qquad v_{\mathbf{1}} = \bar{\rho}\partial_{\bar{\rho}}, \qquad v_{\mathbf{A}} = \partial_{\psi^A},$$

$$\alpha^{\mathbf{0}} = \mathrm{d}\bar{\tau}, \qquad \alpha^{\mathbf{1}} = \frac{1}{\bar{\rho}}\,\mathrm{d}\bar{\rho}, \qquad \alpha^{\mathbf{A}} = \mathrm{d}\psi^A.$$

To change from the frame field associated with our original coordinates to the frame $v_{\boldsymbol{\alpha}}$ one needs the inner products $v^{\mu}{}_{\boldsymbol{\alpha}} = \langle \mathrm{d}x^{\mu}, v_{\boldsymbol{\alpha}} \rangle$. From expressions (26a) and (26b), one sees that

$$v^t{}_{\mathbf{0}} = \frac{\bar{\rho}}{(V\Omega)[\rho e^a]}$$
$$= \frac{\bar{\rho}}{\rho^2} - 2M\frac{\bar{\rho}}{\rho} + \frac{1}{3M^2}\bar{\rho}(5M^4 - MM_{ab}e^a e^b - (S_a e^a)^2) + O(\bar{\rho}^2),$$

$$v^t{}_{\mathbf{1}} = \frac{\bar{\rho}}{(V\Omega)[\bar{\rho}e^a]} - \frac{\rho}{(V\Omega)[\rho e^a]}$$
$$= -\frac{\bar{\tau}}{\rho} + \frac{1}{3M^2}\bar{\tau}\bar{\rho}(5M^4 - MM_{ab}e^a e^b - (S_a e^a)^2) + O(\bar{\rho}^2),$$

$$v^t{}_{\mathbf{A}} = l_A,$$

$$v^a{}_{\mathbf{0}} = -\bar{\rho}e^a,$$

$$v^a{}_{\mathbf{1}} = \rho e^a,$$

$$v^a{}_{\mathbf{A}} = \rho e^a{}_{,\psi^A}.$$

The frame and coframe fields distort the length of the radial component of the tensorial fields they are contracted with. This distortion will be of importance in what follows when discussing objects that are singular at spatial infinity.

5.3. A conformal metric containing the cylinder at spatial infinity

Let $\bar{g}_{\mu\nu}$ denote a metric conformal to $\breve{g}_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ defined by

$$\bar{g}_{\mu\nu} = \frac{1}{\rho^2}\breve{g}_{\mu\nu} = \bar{\Omega}^2\tilde{g}_{\mu\nu}.$$

It follows that

$$(\bar{g}_{\mu\nu}) = \begin{pmatrix} V^2\Omega^2/\rho^2 & V^2\Omega^2\beta_a/\rho^2 \\ V^2\Omega^2\beta_b/\rho^2 & (-\gamma_{ab} + V^2\Omega^2\beta_a\beta_b)/\rho^2 \end{pmatrix}$$

and

$$\bar{g}_{tt} = \rho^2 + 4M\rho^3 + \frac{2}{3M^2}\rho^4(13M^4 + MM_{ab}e^a e^b + (S_a e^a)^2) + O(\rho^5),$$

$$\bar{g}_{ta} = \rho^2 e_{abc}(2S^b + O(\rho))e^c,$$

$$\bar{g}_{ab} = -\frac{1}{\rho^2}\delta_{ab} - \frac{1}{3M^2}[M^4(\delta_{ab} - e_a e_b) + 2M(M_{ab} + \delta_{ab}M_{cd}e^c e^d - 2e_{(a}M_{b)c}e^c)$$
$$+ 2(S_a S_b + \delta_{ab}(S_c e^c)^2 - 2e_{(a}S_{b)}S_c e^c)] + O(\rho).$$

The inverse metric $\bar{g}^{\mu\nu}$ is given by

$$(\bar{g}^{\mu\nu}) = \begin{pmatrix} \rho^2(1 - V^2\Omega^2\beta_c\beta^c)/V^2\Omega^2 & \rho^2\beta^a \\ \rho^2\beta^b & -\rho^2\gamma^{ab} \end{pmatrix}$$

and

$$\bar{g}^{tt} = \frac{1}{\rho^2} - \frac{4M}{\rho} + \frac{2}{3M^2}(11M^4 - MM_{ab}e^a e^b - (S_a e^a)^2) + O(\rho^0),$$

$$\bar{g}^{ta} = \rho^2 e^{abc}(2S_b + O(\rho))e_c,$$

$$\bar{g}^{ab} = -\rho^2 \delta^{ab} + \frac{\rho^4}{3M^2}[M^4(\delta^{ab} - e^a e^b) + 2M(M^{ab} + \delta^{ab}M_{cd}e^c e^d - 2e^{(a}M^{b)c}e_c)$$
$$+ 2(S^a S^b + \delta^{ab}(S_c e^c)^2 - 2e^{(a}S^{b)}S_c e^c)] + O(\rho^5).$$

Note that the metric $\bar{g}_{\mu\nu}$ is singular at $\rho = 0$. Thus, the points for which $\rho = 0$ are at an infinite distance with respect to this metric—hence, one does not obtain a finite representation of spatial infinity. However, this singular behaviour is counteracted by the use of components with respect to the frame and coframe basis introduced in the previous subsection.

It is also noticed that the conformal factor $\bar{\Omega}$ has the following expansion:

$$\bar{\Omega} = \frac{1}{\rho}V^{1/2}\Omega = \rho + M\rho^2 + \frac{1}{6M^2}\rho^3(5M^4 + 2MM_{ab}e^a e^b + 2(S_a e^a)^2) + O(\rho^4).$$

In what follows, we will also require the components of the metric $\bar{g}$ with respect to the frame v_α:

$$\bar{g}_{\alpha\beta} = \langle \Phi^*(\bar{g}); v_\alpha, v_\beta \rangle$$
$$= \langle (\bar{g}_{\mu\nu} \circ \Phi)dx^\mu dx^\nu; v_\alpha, v_\beta \rangle$$
$$= (\bar{g}_{tt} \circ \Phi)\langle dt, v_\alpha \rangle\langle dt, v_\beta \rangle + 2(\bar{g}_{ta} \circ \Phi)\langle dt, v_{(\alpha} \rangle\langle dx^a, v_{\beta)} \rangle + (\bar{g}_{ab} \circ \Phi)\langle dx^a, v_\alpha \rangle\langle dx^b, v_\beta \rangle$$
$$= (\bar{g}_{tt} \circ \Phi)v^t{}_\alpha v^t{}_\beta + 2(\bar{g}_{ta} \circ \Phi)v^t{}_{(\alpha}v^a{}_{\beta)} + (\bar{g}_{ab} \circ \Phi)v^a{}_\alpha v^b{}_\beta.$$

These components are explicitly given by

$$\bar{g}_{00} = 0,$$

$$\bar{g}_{01} = \frac{(V\Omega)[\rho e^a]}{(1 - \bar{\tau})^2(V\Omega)[\bar{\rho}e^a]},$$

$$\bar{g}_{0A} = \frac{(V\Omega)[\rho e^a]}{(1 - \bar{\tau})^2 \bar{\rho}}(l_A + \rho\beta_A),$$

$$\bar{g}_{11} = \frac{(V\Omega)[\rho e^a]}{(1 - \bar{\tau})^2(V\Omega)[\bar{\rho}e^a]}\left(\frac{(V\Omega)[\rho e^a]}{(V\Omega)[\bar{\rho}e^a]} - 2(1 - \bar{\tau})\right),$$

$$\bar{g}_{1A} = \frac{(V\Omega)[\rho e^a]}{(1 - \bar{\tau})^2 \bar{\rho}}\left(\frac{(V\Omega)[\rho e^a]}{(V\Omega)[\bar{\rho}e^a]} - (1 - \bar{\tau})\right)(l_A + \rho\beta_A),$$

$$\bar{g}_{AB} = \frac{(V\Omega)^2[\rho e^a]}{(1 - \bar{\tau})^2 \bar{\rho}^2}(l_A l_B + 2\rho\beta_{(A}l_{B)}) - k_{AB},$$

where $\beta_A = \beta_a e^a{}_{,\psi^A}$.

The expansions discussed in the previous paragraphs imply that

$$\bar{g}_{\alpha\beta} = g^*_{\alpha\beta} + O(\bar{\rho}^2),$$

with

$$(g^*_{\alpha\beta}) = \begin{pmatrix} 0 & 1 - 2M\bar{\rho}\bar{\tau} & 0 \\ 1 - 2M\bar{\rho}\bar{\tau} & -(1 - \bar{\tau})(1 + \bar{\tau} - 4M\bar{\rho}\bar{\tau}^2) & 0 \\ 0 & 0 & -k_{AB}(0) \end{pmatrix}.$$

In order to obtain these expressions, we have used that $k_{AB}(0) = \delta_{ab}e^a{}_{,\psi^A}e^b{}_{,\psi^B}$, $\delta_{ab}e^a e^b = 1$ and $\delta_{ab}e^a e^b{}_{,\psi^A} = 0$. The inverse metric is given by

$$\bar{g}^{\alpha\beta} = g^{*\alpha\beta} + O(\bar{\rho}^2), \tag{27}$$

with

$$
(g^{*\alpha\beta}) = \begin{pmatrix}
\dfrac{(1-\bar{\tau})(1+\bar{\tau}-4M\bar{\rho}\bar{\tau}^2)}{(1-2M\bar{\rho}\bar{\tau})^2} & \dfrac{1}{1-2M\bar{\rho}\bar{\tau}} & 0 \\[2mm]
\dfrac{1}{1-2M\bar{\rho}\bar{\tau}} & 0 & 0 \\[2mm]
0 & 0 & -k^{AB}(0)
\end{pmatrix}.
\tag{28}
$$

Remark 1. One sees that although the conformal metric $\bar{g}_{\mu\nu}$ is singular at $\rho = 0$, its components measured with respect to the frame of subsection 5.2 are regular and indicate the existence of an extended set with the topology of a cylinder at spatial infinity—its sections of constant $\bar{\tau}$ correspond to 2-spheres.

Remark 2. The stationary metric $\tilde{g}$ possesses the Killing vector $\xi = \partial_t$. As the conformal factor Ω and ρ do not depend on t, ξ is also a Killing vector of $\bar{g}$. In the new coordinates, it takes the form

$$
\begin{aligned}
\xi &= \frac{(V\Omega)[\bar{\rho}e^a]}{\bar{\rho}}\big((1-\bar{\tau})\partial_{\bar{\tau}} + \bar{\rho}\partial_{\bar{\rho}}\big) \\
&= \frac{(V\Omega)[\bar{\rho}e^a]}{\bar{\rho}}\big((1-\bar{\tau})v_{\mathbf{0}} + v_{\mathbf{1}}\big).
\end{aligned}
$$

Remark 3. The metric $\bar{g}$ is a conformal representation of the metric $\check{g}$ which allows us to construct an extension of $\mathcal{M}$ in a neighbourhood of spatial infinity. For this, we replace the hypersurface $\mathcal{S}$ by the manifold with boundary $\bar{\mathcal{S}}$, by adding to $\tilde{\mathcal{S}}$ the two-dimensional surface $\partial\bar{\mathcal{S}}$. The points of this two-dimensional surface are thought of as ideal end points of radial curves on $\tilde{\mathcal{S}}$ as $\bar{\rho} \to 0$. The coordinates $\bar{\rho}, \psi^A$ extend by definition to smooth coordinates on $\bar{\mathcal{S}}$ and on $\partial\bar{\mathcal{S}}$ we have $\bar{\rho} = 0$. The coordinates ψ^A, although not specified, are supposed to cover the $\mathbb{S}^2$. The construction described in this paragraph provides an alternative implementation of the blow up of spatial infinity discussed in subsection 1.4 of the introduction.

Working by analogy with the discussion of section 3, one defines the following regions in terms of the range of the coordinates $\bar{\tau}, \bar{\rho}$ and ψ^A:

$$
\begin{aligned}
\tilde{\mathcal{M}}' &\equiv \{0 \leqslant \bar{\tau} < 1, 0 < \bar{\rho}\}, \\
\bar{\mathcal{M}}' &\equiv \{0 \leqslant \bar{\tau} \leqslant 1, 0 \leqslant \bar{\rho}\}, \\
\mathscr{I}^{+\prime} &\equiv \{\bar{\tau} = 1, \bar{\rho} > 0\}, \\
\mathcal{I}' &\equiv \{0 \leqslant \bar{\tau} < 1, \bar{\rho} = 0\}, \\
\mathcal{I}^{+\prime} &\equiv \{\bar{\tau} = 1, \bar{\rho} = 0\}, \\
\mathcal{I}^{0\prime} &\equiv \partial\bar{\mathcal{S}} = \{\bar{\tau} = 0, \bar{\rho} = 0\}, \\
\bar{\mathcal{I}}' &\equiv \mathcal{I}' \cup \mathcal{I}^{+\prime}.
\end{aligned}
$$

The names for these sets have been chosen in accordance with the related sets defined in section 3, although they differ in some aspects. One can readily verify that in terms of the new coordinates, frame and coframe fields, the metric $\bar{g}$ extends smoothly through the sets $\mathscr{I}^{+\prime}$ and $\bar{\mathcal{I}}'$. Accordingly, $\bar{\mathcal{M}}'$ provides a suitable extension of $\mathcal{M}$. In order to use this extension to show that the construction of the cylinder at infinity for stationary spacetimes is as smooth as expected, we also need information regarding the Schouten tensor and the conformal Weyl tensor, which we derive in the following subsections. Finally, note that in terms of the coordinates $(\bar{\tau}, \bar{\rho}, \psi^A)$, null infinity appears to be parallel to the surfaces of constant $\bar{\tau}$, and in particular the initial hypersurface $\bar{\mathcal{S}}$.

5.4. Expansions of the Schouten tensor

In this section, we discuss expansions of the Schouten tensor of the metric $\bar{g}$. This is related to the Schouten tensor of the metric $\tilde{g}$ by

$$\bar{L}_{\mu\nu} = \tilde{L}_{\mu\nu} - \frac{1}{\bar{\Omega}}\bar{\nabla}_\mu\bar{\nabla}_\nu\bar{\Omega} + \frac{1}{2\bar{\Omega}^2}\bar{\nabla}^\lambda\bar{\Omega}\bar{\nabla}_\lambda\bar{\Omega}\bar{g}_{\mu\nu}.$$

In what follows, we will consider the components of $\bar{L}$ in the frame $v_{\boldsymbol{\alpha}}$:

$$\begin{aligned}
\bar{L}_{\boldsymbol{\alpha\beta}} &= \langle \Phi^*(\bar{L}); v_{\boldsymbol{\alpha}}, v_{\boldsymbol{\beta}}\rangle \\
&= (\bar{L}_{tt}\circ\Phi)v^t{}_{\boldsymbol{\alpha}}v^t{}_{\boldsymbol{\beta}} + 2(\bar{L}_{ta}\circ\Phi)v^t{}_{\boldsymbol{\alpha}}v^a{}_{\boldsymbol{\beta}} + (\bar{L}_{ab}\circ\Phi)v^a{}_{\boldsymbol{\alpha}}v^b{}_{\boldsymbol{\beta}}.
\end{aligned}$$

As $\tilde{g}$ is a solution to the vacuum Einstein field equation, it follows that $\tilde{L}_{\mu\nu} = 0$. Furthermore, in view that $\bar{\Omega}$ does not depend on t, one obtains

$$\bar{L}_{\mu\nu} = -\frac{1}{\bar{\Omega}}(\partial_\mu\partial_\nu\bar{\Omega} - \bar{\Gamma}_\mu{}^a{}_\nu\partial_a\bar{\Omega}) + \frac{1}{2\bar{\Omega}^2}\bar{g}^{ab}\partial_a\bar{\Omega}\partial_b\bar{\Omega}\bar{g}_{\mu\nu}.$$

Alternatively, one can write

$$\begin{aligned}
\bar{L}_{tt} &= \frac{1}{\bar{\Omega}}\bar{\Gamma}_t{}^a{}_t\partial_a\bar{\Omega} + \frac{1}{2\bar{\Omega}^2}\bar{g}^{ab}\partial_a\bar{\Omega}\partial_b\bar{\Omega}\bar{g}_{tt}, \\
\bar{L}_{ta} &= \frac{1}{\bar{\Omega}}\bar{\Gamma}_t{}^b{}_a\partial_b\bar{\Omega} + \frac{1}{2\bar{\Omega}^2}\bar{g}^{bc}\partial_b\bar{\Omega}\partial_c\bar{\Omega}\bar{g}_{ta}, \\
\bar{L}_{ab} &= -\frac{1}{\bar{\Omega}}\left(\partial_a\partial_b\bar{\Omega} - \bar{\Gamma}_a{}^c{}_b\,\partial_c\bar{\Omega}\right) + \frac{1}{2\bar{\Omega}^2}\bar{g}^{cd}\partial_c\bar{\Omega}\partial_d\bar{\Omega}\bar{g}_{ab}.
\end{aligned}$$

Using the expansions for the conformal factor, the first and second derivatives of the conformal factor and the Christoffel symbols of the conformal metric given in the appendix, one obtains

$$\begin{aligned}
\bar{L}_{tt} &= \frac{1}{2}\rho^2 + 4M\rho^3 + O(\rho^4), \\
\bar{L}_{ta} &= -2\rho^2 e_{abc}e^b S^c + O(\rho^3), \\
\bar{L}_{ab} &= \frac{1}{2\rho^2}(\delta_{ab} - 2e_a e_b) + \frac{M}{\rho}(\delta_{ab} - 3e_a e_b) + O(\rho^0).
\end{aligned}$$

One then has all the ingredients to calculate the components $\bar{L}_{\boldsymbol{\alpha\beta}}$. One obtains

$$\begin{aligned}
\bar{L}_{\boldsymbol{00}} &= O(\bar{\rho}^2), \\
\bar{L}_{\boldsymbol{01}} &= \frac{1}{2} + M(2 - 3\bar{\tau})\bar{\rho} + O(\bar{\rho}^2), \\
\bar{L}_{\boldsymbol{0A}} &= O(\bar{\rho}^2), \\
\bar{L}_{\boldsymbol{11}} &= -\frac{1}{2}(1 - \bar{\tau}^2) - 2M(1 + 2\bar{\tau}^2)(1 - \bar{\tau})\bar{\rho} + O(\bar{\rho}^2), \\
\bar{L}_{\boldsymbol{1A}} &= O(\bar{\rho}^2), \\
\bar{L}_{\boldsymbol{AB}} &= \frac{1}{2}k_{\boldsymbol{AB}} + M(1 - \bar{\tau})\bar{\rho}k_{\boldsymbol{AB}} + O(\bar{\rho}^2),
\end{aligned}$$

where it has been used that

$$\delta_{ab}e^a{}_{,\psi^A}e^b{}_{,\psi^B} = k_{AB}(0) + O(\bar{\rho}^2), \quad \delta_{ab}e^a e^b{}_{,\psi_A} = 0.$$

Summarizing, one has proven the following result:

Lemma 3. *The components, $\bar{L}_{\boldsymbol{\alpha\beta}}$, of the Schouten tensor $\bar{L}_{\mu\nu}$ in the frame $v_{\boldsymbol{\alpha}}$ are regular (i.e. non-divergent) at $\mathcal{I}'$.*

5.5. The Weyl tensor

In this section, we verify that the conformal Weyl tensor has a smooth limit at $\bar{\rho} \to 0$. For this, we start by recalling the decomposition of the Weyl tensor in terms of its electric and magnetic parts with respect to the normal to a hypersurface.

5.5.1. The electric–magnetic decomposition.
Let $\tilde{\mathcal{M}}$ be the spacetime with metric $\tilde{g}_{\mu\nu}$ and $\tilde{\mathcal{S}}$ a space-like hypersurface with unit normal $\tilde{n}^{\mu}$. The induced metric on $\tilde{\mathcal{S}}$ by $\tilde{g}_{\mu\nu}$ is given by

$$\tilde{h}_{\mu\nu} = \tilde{g}_{\mu\nu} - \tilde{n}_{\mu}\tilde{n}_{\nu}.$$

For convenience, one defines

$$\tilde{p}_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - \tilde{n}_{\mu}\tilde{n}_{\nu}, \qquad \tilde{\epsilon}_{\mu\nu\lambda} \equiv \tilde{n}^{\rho}\tilde{\epsilon}_{\rho\mu\nu\lambda}. \tag{29}$$

The $\tilde{n}$-electric and $\tilde{n}$-magnetic parts of the conformal Weyl tensor are given, respectively, by

$$\tilde{c}_{\nu\rho} \equiv \tilde{C}_{\mu\nu\lambda\rho}\tilde{n}^{\mu}\tilde{n}^{\lambda}, \qquad \tilde{c}^{*}_{\nu\rho} \equiv \tilde{C}^{*}_{\mu\nu\lambda\rho}\tilde{n}^{\mu}\tilde{n}^{\lambda}, \tag{30}$$

where $\tilde{C}^{*}_{\mu\nu\lambda\rho}$ denotes the dual of the conformal Weyl tensor: $\tilde{C}^{*}_{\mu\nu\lambda\rho} = \frac{1}{2}\tilde{C}_{\mu\nu\sigma\chi}\tilde{\epsilon}^{\sigma\chi}{}_{\lambda\rho}$. The $\tilde{n}$-electric and $\tilde{n}$-magnetic parts of the Weyl tensor are symmetric, trace-free and spatial:

$$\tilde{n}^{\mu}\tilde{c}_{\mu\nu} = 0, \qquad \tilde{n}^{\mu}\tilde{c}^{*}_{\mu\nu} = 0.$$

Given two tensors of rank 2, $f_{\mu\nu}$ and $k_{\mu\nu}$, their *Kulkarni–Nomizu* product is defined as

$$(f \oslash k)_{\mu\nu\lambda\rho} = 2(f_{\mu[\lambda}k_{\rho]\nu} - f_{\nu[\lambda}k_{\rho]\mu}).$$

In terms of the Kulkarni–Nomizu product, the conformal Weyl tensor is given by

$$\begin{aligned}\tilde{C}_{\mu\nu\lambda\rho} &= 2\big(\tilde{p}_{\nu[\lambda}\tilde{c}_{\rho]\mu} - \tilde{p}_{\mu[\lambda}\tilde{c}_{\rho]\nu} - \tilde{n}_{[\lambda}\tilde{c}^{*}_{\rho]\delta}\tilde{\epsilon}^{\delta}{}_{\mu\nu} - \tilde{n}_{[\mu}\tilde{c}^{*}_{\nu]\delta}\tilde{\epsilon}^{\delta}{}_{\lambda\rho}\big) \\ &= (\tilde{p} \oslash \tilde{c}^{*})^{*}_{\mu\nu\lambda\rho} - (\tilde{p} \oslash \tilde{c})_{\mu\nu\lambda\rho}.\end{aligned}$$

5.5.2. Conformal rescalings.
If $\tilde{g}_{\mu\nu}$ is a solution of the Einstein vacuum field equations, then the first and second fundamental forms of $\tilde{\mathcal{S}}$ induced by $\tilde{g}_{\mu\nu}$ satisfy the Gauss and the Codazzi equations. These equations allow us to write the pull-back of $\tilde{c}_{\mu\nu}$ and $\tilde{c}^{*}_{\mu\nu}$ to $\tilde{\mathcal{S}}$ in terms of the initial data quantities. More precisely,

$$\tilde{c}_{ab} = -r_{ab}[\tilde{h}] + \chi_{c}{}^{c}\tilde{\chi}_{ab} - \tilde{\chi}_{ca}\tilde{\chi}_{b}{}^{c}, \qquad \tilde{c}^{*}_{ab} = -\tilde{D}_{c}\tilde{\chi}_{d(a}\tilde{\epsilon}_{b)}{}^{cd}. \tag{31}$$

If $\bar{g}_{\mu\nu} = \bar{\Omega}^{2}\tilde{g}_{\mu\nu}$, then it is well known that $\bar{C}^{\mu}{}_{\nu\lambda\rho} = \tilde{C}^{\mu}{}_{\nu\lambda\rho}$. The rescaled conformal Weyl tensor is given by $\bar{W}^{\mu}{}_{\nu\lambda\rho} = \bar{\Omega}^{-1}\bar{C}^{\mu}{}_{\nu\lambda\rho}$, and therefore

$$\bar{W}_{\mu\nu\lambda\rho} = \bar{\Omega}\tilde{C}_{\mu\nu\lambda\rho}.$$

The $\bar{n}$-electric and $\bar{n}$-magnetic parts of $\bar{W}$ are defined in accordance with (30) as

$$\bar{w}_{\nu\rho} = \bar{W}_{\mu\nu\lambda\rho}\bar{n}^{\mu}\bar{n}^{\lambda}, \qquad \bar{w}^{*}_{\nu\rho} = \bar{W}^{*}_{\mu\nu\lambda\rho}\bar{n}^{\mu}\bar{n}^{\lambda}.$$

Now, recalling that the $\bar{g}$-unit normal to $\tilde{\mathcal{S}}$ is given by $\bar{n}^{\mu} = \bar{\Omega}^{-1}\tilde{n}^{\mu}$, one obtains

$$\bar{w}_{\mu\nu} = \bar{\Omega}^{-1}\tilde{c}_{\mu\nu}, \qquad\qquad \bar{w}^{*}_{\mu\nu} = \bar{\Omega}^{-1}\tilde{c}^{*}_{\mu\nu}. \tag{32}$$

From definition (29) one readily obtains

$$\begin{aligned}\bar{p}_{\mu\nu} &= \bar{\Omega}^{2}\tilde{p}_{\mu\nu}, \\ \bar{W}_{\mu\nu\lambda\rho} &= (\bar{p} \oslash \bar{w}^{*})^{*}_{\mu\nu\lambda\rho} - (\bar{p} \oslash \bar{w})_{\mu\nu\lambda\rho} \\ &= 2\big(\bar{p}_{\nu[\lambda}\bar{w}_{\rho]\mu} - \bar{p}_{\mu[\lambda}\bar{w}_{\rho]\nu} - \bar{n}_{[\lambda}\bar{w}^{*}_{\rho]\delta}\bar{\epsilon}^{\delta}{}_{\mu\nu} - \bar{n}_{[\mu}\bar{w}^{*}_{\nu]\delta}\bar{\epsilon}^{\delta}{}_{\lambda\rho}\big).\end{aligned}$$

5.5.3. Regularity at $\mathcal{I}'$.　In what follows, it will be shown that the components of the Weyl tensor $\bar{W}_{\mu\nu\lambda\rho}$ with respect to the frame v_{α} given by

$$\bar{W}_{\alpha\beta\gamma\delta} = \langle \Phi^*(\bar{W}); v_{\alpha}, v_{\beta}, v_{\gamma}, v_{\delta} \rangle$$
$$= (\bar{W}_{\mu\nu\lambda\rho} \circ \Phi) v^{\mu}{}_{\alpha} v^{\nu}{}_{\beta} v^{\lambda}{}_{\gamma} v^{\rho}{}_{\delta}$$

do not diverge as $\bar{\rho} \to 0$. In order to do this, one needs to discuss the expansions of $\bar{n}_{\mu}$, $\bar{\epsilon}^{\mu}{}_{\nu\lambda}$, $\bar{p}_{\mu\nu}$, $\bar{w}_{\mu\nu}$ and $\bar{w}^*_{\mu\nu}$.

We note that the hypersurface to be considered is given in our coordinates by

$$\tilde{\mathcal{S}} = \{\phi(x^{\mu}) = t - constant = 0\}.$$

It follows that the normal vector and covector are given by

$$\bar{n}_t = \frac{V\Omega}{\rho(1 - V^2\Omega^2\beta_c\beta^c)^{1/2}}, \qquad \bar{n}_a = 0,$$

$$\bar{n}^t = \frac{\rho(1 - V^2\Omega^2\beta_c\beta^c)^{1/2}}{V\Omega}, \qquad \bar{n}^a = \frac{\rho V\Omega\beta^a}{(1 - V^2\Omega^2\beta_c\beta^c)^{1/2}}, \tag{33}$$

whence

$$\bar{n}_t = \rho + 2M\rho^2 + \frac{1}{3M^2}\rho^3(7M^4 + MM_{ab}e^ae^b + (S_ae^a)^2) + O(\rho^4),$$

$$\bar{n}_a = 0,$$

$$\bar{n}^t = \rho^{-1} - 2M + \frac{1}{3M^2}\rho(5M^4 - MM_{ab}e^ae^b - (S_ae^a)^2) + O(\rho^4),$$

$$\bar{n}^a = -2\rho^3 e^{abc}e_bS_c + O(\rho^4).$$

To compute $\bar{\epsilon}^{\mu}{}_{\nu\lambda}$, one notes the relations $\bar{\epsilon}^{\mu}{}_{\nu\lambda} = \bar{g}^{\mu\rho}\bar{\epsilon}_{\rho\nu\lambda}$ and that $\bar{\epsilon}_{\mu\nu\lambda} = \bar{n}^{\sigma}\bar{\epsilon}_{\sigma\mu\nu\lambda}$, where

$$\bar{\epsilon}_{\sigma\mu\nu\lambda} = |\det(\bar{g}_{\mu\nu})|^{\frac{1}{2}}\eta_{\sigma\mu\nu\lambda} = \frac{1}{\rho^2}(1 + 2M\rho + O(\rho^2))\eta_{\sigma\mu\nu\lambda},$$

with $\eta_{\sigma\mu\mu\lambda}$ the totally antisymmetric tensor of rank 4. Next one evaluates $\bar{p}_{\mu\nu} = \bar{g}_{\mu\nu} - 2\bar{n}_{\mu}\bar{n}_{\nu}$ to obtain

$$(\bar{p}_{\mu\nu}) = \begin{pmatrix} -\dfrac{V^2\Omega^2}{\rho^2}\dfrac{(1 + V^2\Omega^2\beta_c\beta^c)}{(1 - V^2\Omega^2\beta_c\beta^c)} & \dfrac{1}{\rho^2}V^2\Omega^2\beta_a \\ \dfrac{1}{\rho^2}V^2\Omega^2\beta_b & \dfrac{1}{\rho^2}(-\gamma_{ab} + V^2\Omega^2\beta_a\beta_b) \end{pmatrix},$$

from which

$$\bar{p}_{tt} = -\rho^2 - 4M\rho^3 - \frac{2}{3M^2}\rho^4(13M^4 + MM_{ab}e^ae^b + (S_ae^a)^2) + O(\rho^5),$$

$$\bar{p}_{ta} = -2\rho^2 e_{abc}e^bS^c + O(\rho^3),$$

$$\bar{p}_{ab} = -\rho^{-2}\delta_{ab} - \frac{1}{3M^2}(M^4(\delta_{ab} - e_ae_b) + 2M(M_{ab} + \delta_{ab}M_{cd}e^ce^d - 2e_{(a}M_{b)c}e^c)$$
$$+ 2(S_aS_b + \delta_{ab}(S_ce^c)^2 - 2e_{(a}S_{b)}S_ce^c)) + O(\rho).$$

In order to calculate expansions for the electric and magnetic parts $\bar{w}_{\mu\nu}$ and $\bar{w}^*_{\mu\nu}$, one makes use of expressions (32) and (31). From the formulae for conformal transformations, one has

$$\tilde{r}_{ab} = \bar{r}_{ab} + \bar{\Omega}^{-1}\bar{D}_a\bar{D}_b\bar{\Omega} + \bar{\Omega}^{-1}\bar{D}^c\bar{D}_c\bar{\Omega}\bar{h}_{ab} - 2\bar{\Omega}^{-2}\bar{D}^c\bar{\Omega}\bar{D}_c\bar{\Omega}\bar{h}_{ab}$$

and

$$\bar{\chi}_{ab} = \bar{\Omega}(\tilde{\chi}_{ab} + \bar{\Sigma}\tilde{h}_{ab}), \tag{34}$$

where $\bar{\Sigma} = \bar{n}^\mu \, \partial_\mu \bar{\Omega}$. Hence, one obtains

$$\bar{w}_{ab} = -\bar{\Omega}^{-1}\big(\bar{r}_{ab} + \bar{\Omega}^{-1}\bar{D}_a\bar{D}_b\bar{\Omega} + \bar{\Omega}^{-1}\bar{D}^c\bar{D}_c\bar{\Omega}\bar{h}_{ab} - \bar{\chi}_c{}^c\bar{\chi}_{ab} + \bar{\chi}_{ca}\bar{\chi}_b{}^c$$
$$+ \bar{\Omega}^{-1}\bar{\Sigma}\,\bar{\chi}_{ab} + \bar{\Omega}^{-1}\bar{\Sigma}\,\bar{\chi}_c{}^c\bar{h}_{ab} - 2\bar{\Omega}^{-2}\bar{D}^c\bar{\Omega}\bar{D}_c\bar{\Omega}\bar{h}_{ab} - 2\bar{\Omega}^{-2}\bar{\Sigma}^2\bar{h}_{ab}\big). \tag{35}$$

The seemingly more singular terms in the last expression can be eliminated using the Hamiltonian constraint

$$0 = \tilde{r} - \big(\tilde{\chi}_c^c\big)^2 + \tilde{\chi}_{cd}\tilde{\chi}^{cd}$$
$$= \bar{\Omega}^2\bar{r} + 4\bar{\Omega}\bar{D}^c\bar{D}_c\bar{\Omega} - 6\bar{D}^c\bar{\Omega}\bar{D}_c\bar{\Omega} - \bar{\Omega}^2\big(\bar{\chi}_c^c\big)^2 + \bar{\Omega}^2\bar{\chi}_{cd}\bar{\chi}^{cd} + 4\bar{\Omega}\bar{\Sigma}\,\bar{\chi}_c^c - 6\bar{\Sigma}^2,$$

so that equation (35) transforms into

$$\bar{w}_{ab} = -\bar{\Omega}^{-1}\big(\bar{r}_{ab} - \tfrac{1}{3}\bar{r}\bar{h}_{ab} + \bar{\Omega}^{-1}\big(\bar{D}_a\bar{D}_b\bar{\Omega} - \tfrac{1}{3}\bar{D}^c\bar{D}_c\bar{\Omega}\bar{h}_{ab}\big)$$
$$- (\bar{\chi}_c{}^c - \bar{\Omega}^{-1}\bar{\Sigma})\big(\bar{\chi}_{ab} - \tfrac{1}{3}\bar{\chi}_c{}^c\bar{h}_{ab}\big) + \bar{\chi}_{ca}\bar{\chi}_b{}^c - \tfrac{1}{3}\bar{\chi}_{cd}\bar{\chi}^{cd}\bar{h}_{ab}\big). \tag{36}$$

To calculate $\bar{w}_{ab}^*$, one needs to use the conformal transformation for the derivative of a $(0, 2)$-tensor:

$$\tilde{D}_a\tilde{\chi}_{bc} = \bar{D}_a\tilde{\chi}_{bc} + \bar{\Omega}^{-1}(2\partial_a\bar{\Omega}\tilde{\chi}_{bc} + \partial_b\bar{\Omega}\tilde{\chi}_{ac} + \partial_c\bar{\Omega}\tilde{\chi}_{ba} - \bar{h}_{ab}\bar{h}^{de}\partial_e\bar{\Omega}\tilde{\chi}_{dc} - \bar{h}_{ac}\bar{h}^{de}\partial_d\bar{\Omega}\tilde{\chi}_{db}).$$

The latter, together with $\bar{\epsilon}_{abc} = \bar{\Omega}^3\tilde{\epsilon}_{abc}$, finally yield

$$\bar{w}_{ab}^* = -\bar{\Omega}^{-1}\bar{D}_c\bar{\chi}_{d(a}\bar{\epsilon}_{b)}{}^{cd}. \tag{37}$$

In order to compute $\bar{w}_{\mu\nu}$ and $\bar{w}_{\mu\nu}^*$ from expressions (36) and (37), one uses that $\tilde{n}^\mu\tilde{c}_{\mu\nu} = 0$ and $\tilde{n}^\mu\tilde{c}_{\mu\nu}^* = 0$ so that $\bar{n}^\mu\bar{w}_{\mu\nu} = 0$ and $\bar{n}^\mu\bar{w}_{\mu\nu}^* = 0$. With this and the symmetry of the tensors, one obtains

$$\bar{w}_{tt} = \frac{\bar{n}^a\bar{n}^b\bar{w}_{ab}}{(\bar{n}^t)^2}, \qquad \bar{w}_{ta} = -\frac{\bar{n}^b\bar{w}_{ba}}{\bar{n}^t},$$
$$\bar{w}_{tt}^* = \frac{\bar{n}^a\bar{n}^b\bar{w}_{ab}^*}{(\bar{n}^t)^2}, \qquad \bar{w}_{ta}^* = -\frac{\bar{n}^b\bar{w}_{ba}^*}{\bar{n}^t}.$$

In order to evaluate formula (36) one makes use of

$$\bar{r}_{ab} = \partial_c\bar{\Gamma}_a{}^c{}_b - \partial_a\bar{\Gamma}_c{}^c{}_b + \bar{\Gamma}_a{}^c{}_b\bar{\Gamma}_c{}^d{}_d - \bar{\Gamma}_c{}^d{}_b\bar{\Gamma}_d{}^c{}_a$$
$$= \frac{1}{\rho^2}(\delta_{ab} - e_ae_b) - \frac{2}{3M^2}(M^4(\delta_{ab} + e_ae_b) + M(M_{ab} + 2\delta_{ab}M_c^c - 3\delta_{ab}M_{cd}e^ce^d$$
$$+ 4e_{(a}M_{b)c}e^c + S_aS_b + 2\delta_{ab}S_cS^c - 3\delta_{ab}(S_ce^c)^2 + 4e_{(a}S_{b)}S_ce^c) + O(\rho)$$

and

$$\bar{D}_a\bar{D}_b\bar{\Omega} = \partial_a\partial_b\bar{\Omega} - \bar{\Gamma}_a{}^c{}_b\partial_c\bar{\Omega}$$
$$= \frac{1}{\rho}e_ae_b + 4Me_ae_b + \frac{1}{3M^2}\rho\bigg(4e_{(a}S_{b)}S_ce^c + e_ae_b(S_ce^c)^2 + M^4\bigg(\delta_{ab} + \frac{43}{2}e_ae_b\bigg)$$
$$+ M(4M_{ab} + e_ae_bM_{cd}e^ce^d + 4e_{(a}M_{b)c}e^c) + 4S_aS_b\bigg) + O(\rho^2).$$

To complete the analysis, one also needs expansions for $\bar{\chi}_{ab}$. To this end, we make use of the tensor $\check{\chi}_{ab}$ and the results in [22]. More precisely,

$$\check{\chi}_{ab} = \check{\Omega}^2\check{\psi}_{ab}, \qquad \check{\psi}_{ab} = \frac{3}{\rho^3}e_{(a}e_{b)cd}S^ce^d + O(\rho^{-2}),$$

so that

$$\check{\chi}_{ab} = 3\rho e_{(a}e_{b)cd}S^ce^d + O(\rho^2).$$

27

To get from $\check{\chi}_{ab}$ to $\bar{\chi}_{ab}$, one uses the corresponding conformal transformation formulae to find that

$$\bar{\chi}_{ab} = 3e_{(a}e_{b)cd}S^c e^d + O(\rho).$$

Finally, it is noticed that

$$\bar{\Sigma} = \bar{n}^\mu \partial_\mu \bar{\Omega} = O(\rho^4), \qquad \bar{\epsilon}_a{}^{bc} = \rho e_a{}^{bc} + O(\rho^2).$$

It follows then from formula (36) that

$$\bar{w}_{ab} = \rho^{-2}M(\delta_{ab} - 3e_a e_b) + O(\rho^0),$$
$$\bar{w}_{ta} = 2M\rho^2 e_{abc}e^b S^c + O(\rho^3) = -M\rho^2 \beta_a + O(\rho^3),$$
$$\bar{w}_{tt} = 4M\rho^6(S_a S^a - (S_a e^a)^2) + O(\rho^7).$$

Similarly, one finds from (37) that

$$\bar{w}^*_{ab} = \frac{3}{2}\rho^{-1}((\delta_{ab} + e_a e_b)S_c e^c - 4e_{(a}S_{b)}) + O(\rho^0),$$
$$\bar{w}^*_{ta} = 3\rho^3 e_{abc}e^b S^c S_d e^d + O(\rho^4),$$
$$\bar{w}^*_{tt} = -6\rho^7(S_a S^a - (S_a e^a)^2)S_b e^b + O(\rho^8).$$

In view of the previous discussion, one has all the ingredients to compute the leading terms of the expansions of the components $\bar{W}_{\alpha\beta\gamma\delta}$. Due to the length of the calculation this has been done in a tensor manipulation program. One obtains the following.

Lemma 4. *The components, $\bar{W}_{\alpha\beta\gamma\delta}$, of the Weyl tensor $\bar{W}_{\mu\nu\lambda\rho}$ in the frame v_α are regular (i.e. non-divergent) at $\mathcal{I}'$.*

6. Stationary vacuum solutions near the cylinder at space-like infinity

Once the regularity of the various conformal fields at $\mathcal{I}'$ has been shown, the last step in our analysis is very similar to the discussion in [31]. The proof consists of several parts: first, one starts by giving explicitly a solution to the conformal geodesic equations on $\bar{\mathcal{I}}'$; in a second step, a stability argument is used to show that the construction of the cylinder at spacelike infinity is regular in a neighbourhood of $\bar{\mathcal{I}}$; finally, one needs to show that the whole construction does not depend on the choice of conformal factor on the initial hypersurface.

Remark. It is important to stress the differences in the regularity of static and stationary fields at spatial infinity. In the static case, all relevant fields are analytic. By contrast, as shown in [22], in the strictly stationary case, the relevant fields are never analytic as functions of asymptotically Cartesian coordinates. However, as already seen, the stationary fields have an analytic expansion in terms of radial and angular coordinates. This is the type of coordinates used in both the general construction of the cylinder at spacelike infinity and the extension discussed in section 5. As a consequence, all the relevant fields are analytic in these coordinates.

In what follows, the word analyticity will be used to describe analytic behaviour with respect to the radial and angular coordinates.

6.1. Setting the conformal Gauss system

We consider now the regular finite initial value problem at spacelike infinity for stationary data. For this, we make use of the initial hypersurface

$$\tilde{\mathcal{S}} = \{t = 0\},$$

and set the initial conditions on $\tilde{\mathcal{S}}$ that generate the desired conformal Gauss gauge system.

6.1.1. Initial data for the canonical conformal factor. The initial data for the conformal factor, Θ_*, is prescribed—cf equation (8)—by means of the function

$$\kappa \equiv 2\breve{\Omega}|\breve{D}_a\breve{\Omega}\breve{D}^a\breve{\Omega}|^{-1/2} \tag{38}$$

so that

$$\Theta_* = \kappa^{-1}\breve{\Omega}.$$

It follows that

$$\kappa = \rho + O(\rho^2),$$
$$\Theta_* = \tfrac{1}{2}|\breve{D}_a\breve{\Omega}\breve{D}^a\breve{\Omega}|^{1/2} = \rho + O(\rho^2).$$

Hence, the conformal metric evaluated on $\tilde{\mathcal{S}}$ and the induced metric are given, respectively, by

$$g_{\mu\nu} = \Theta_*^2\tilde{g}_{\mu\nu}, \qquad h_{ab} = \Theta_*^2\tilde{h}_{ab} = \kappa^{-2}\breve{h}_{ab}.$$

6.1.2. Initial data for the tangent vector to the congruence of conformal geodesics. Initial conditions for the tangent vector $\dot{x} = \mathrm{d}x/\mathrm{d}\tau$, where τ is the parameter of the conformal geodesic, are set to be

$$\dot{x} \perp \tilde{\mathcal{S}}, \qquad g(\dot{x}, \dot{x}) = 1, \quad \text{on } \tilde{\mathcal{S}}.$$

The normal covector to the hypersurface $\tilde{\mathcal{S}}$, normalized with respect to $\bar{g}_{\mu\nu}$, has been found previously in (33). The relation between $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ is given by $g_{\mu\nu} = \kappa^{-2}\rho^2\bar{g}_{\mu\nu}$, which implies $g^{\mu\nu} = \kappa^2\rho^{-2}\bar{g}^{\mu\nu}$, and therefore

$$n_\mu = \frac{\rho}{\kappa}\bar{n}_\mu.$$

As $n_\alpha = n_\mu v^\mu{}_\alpha$, we have on $\tilde{\mathcal{S}}$

$$n_0 = 1 + O(\bar{\rho}), \qquad n_1 = 0, \qquad n_A = 0.$$

The covector n^α is readily found by using that $n^\alpha = g^{\alpha\beta}n_\beta$ and relations (27) and (28), which give

$$n^0 = 1 + O(\bar{\rho}), \qquad n^1 = 1 + O(\bar{\rho}), \qquad n^A = 0.$$

We finally have

$$\dot{x} = n^\alpha v_\alpha \equiv X^\alpha v_\alpha, \tag{39}$$

where

$$X^\alpha = \delta_0{}^\alpha + \delta_1^\alpha + O(\bar{\rho}). \tag{40}$$

6.1.3. Initial data for the 1-form f. The initial data for the 1-form f is chosen in agreement with condition (9) so that

$$\langle f, \partial_\tau \rangle = 0, \quad \text{pull back of } f \text{ to } \tilde{\mathcal{S}} = \kappa^{-1}d\kappa.$$

It then follows that $\langle f, \dot{x} \rangle = 0$. This property, together with the choice (38) for the function κ—see equation (10)—gives

$$\Theta = \Theta_*(1 - \tau^2). \tag{41}$$

29

6.2. Relating the various conformal gauges

The analysis performed in section 5 was carried out in terms of the metric $\bar{g}_{\mu\nu}$. Now, we proceed to discuss how this metric and its Lévi–Cività connection, $\bar{\nabla}$, are related to the metric $g_{\mu\nu}$ and its Lévi–Cività connection, ∇. From the relation $\bar{g}_{\mu\nu} = \bar{\Omega}^2 \tilde{g}_{\mu\nu}$, it follows that

$$g_{\mu\nu} = \Pi^2 \bar{g}_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}, \qquad \Pi \equiv \bar{\Omega}^{-1}\Theta.$$

The relations between the physical Lévi–Cività connection $\tilde{\nabla}$, the unphysical Lévi–Cività connections $\bar{\nabla}$, ∇, and the Weyl connection $\hat{\nabla}$ are given by

$$\hat{\nabla} = \tilde{\nabla} + S(\tilde{f}),$$
$$\hat{\nabla} = \nabla + S(f),$$
$$\hat{\nabla} = \bar{\nabla} + S(\bar{f}),$$
$$\nabla = \tilde{\nabla} + S(\Theta^{-1}\, \mathrm{d}\Theta),$$
$$\bar{\nabla} = \tilde{\nabla} + S(\bar{\Omega}^{-1}\, \mathrm{d}\bar{\Omega}),$$

where

$$\bar{f} = f + \Pi^{-1}\, \mathrm{d}\Pi = f + \Theta^{-1}\, \mathrm{d}\Theta - \bar{\Omega}^{-1}\, \mathrm{d}\bar{\Omega}.$$

From the particular form of the conformal factor Θ given by equation (41), one has that $\langle \mathrm{d}\Theta, \dot{x} \rangle = 0$ on $\tilde{\mathcal{S}}$. As $\bar{\Omega}$ is independent of t and ∂_t is orthogonal $\tilde{\mathcal{S}}$, it follows that $\langle \mathrm{d}\bar{\Omega}, \dot{x} \rangle = 0$. Thus, $\langle \bar{f}, \dot{x} \rangle = 0$ on $\tilde{\mathcal{S}}$. Finally, from

$$\Pi = \frac{\bar{\rho}}{\kappa} \quad \text{on } \tilde{\mathcal{S}}, \tag{42}$$

it follows that on $\tilde{\mathcal{S}}$ (where $t = 0$ so that $\tau = 0$ and $\rho = \bar{\rho}$) one has

$$\begin{aligned}
\bar{f}_{\boldsymbol{\alpha}} &= \langle \Phi^*(\bar{f}), v_{\boldsymbol{\alpha}} \rangle \\
&= (\bar{f}_\mu \circ \Phi) v^\mu{}_{\boldsymbol{\alpha}} \\
&= \bar{\rho}^{-1} \partial_\mu \bar{\rho}\, v^\mu{}_{\boldsymbol{\alpha}} \\
&= \bar{\rho}^{-1} \partial_a \bar{\rho}\, v^a{}_{\boldsymbol{\alpha}},
\end{aligned}$$

where in the last line it was used that $\partial_t \bar{\rho}|_{t=0} = 0$. As

$$\partial_a \bar{\rho} = \delta_{ab} e^b, \qquad \delta_{ab} e^a e^b = 1, \qquad \delta_{ab} e^a e^b_{\psi^A} = 0,$$

we have

$$\bar{f}_{\mathbf{0}} = -1, \qquad \bar{f}_{\mathbf{1}} = 1, \qquad \bar{f}_A = 0.$$

In other words, one obtains

$$\bar{f} = \bar{f}_{\boldsymbol{\alpha}} \alpha^{\boldsymbol{\alpha}}, \qquad \text{with} \quad \bar{f}_{\boldsymbol{\alpha}} = -\delta_{\boldsymbol{\alpha}}{}^0 + \delta_{\boldsymbol{\alpha}}{}^1 \quad \text{on } \tilde{\mathcal{S}}. \tag{43}$$

A property of conformal Gaussian systems is that $\langle f, \dot{x} \rangle = 0$ in the whole of the spacetime. Hence, one has that

$$\dot{\Pi} = \Pi \langle \bar{f}, \dot{x} \rangle \tag{44}$$

along the conformal geodesics. This last equation, together with equation (42), allows us to determine Π if the contraction $\langle \bar{f}, \dot{x} \rangle$ is known.

6.3. Solving the conformal geodesic equations

In this section, we provide a discussion of the conformal geodesic equations with respect to the metric $\bar{g}$ and of its solutions. A solution to these equations is given by a spacetime curve $x(\tau) = (\bar{\tau}(\tau), \bar{\rho}(\tau), \psi^A(\tau))$ and a 1-form $\bar{f}(\tau)$ along the curve. If expressed in terms of the frame fields, v_{α}, and coframe fields, α^{α}, the functions involved in the conformal geodesic equations are the components $\bar{g}_{\alpha\beta}$, $\bar{g}^{\alpha\beta}$, $\bar{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}$ and $\bar{L}_{\alpha\beta}$. These functions extend by analyticity through $\bar{\mathcal{I}}'$ into a domain where $\bar{\rho} < 0$. If one assumes such an extension, one obtains the so-called *extended conformal geodesic equations*. It is important to point out that the initial data on $\bar{\mathcal{S}}$ are analytic. Therefore, one can consider these equations in a neighbourhood of $\bar{\mathcal{I}}'$. Moreover, it turns out that the restriction of the equations to $\mathcal{I}^{0'}$ can be solved explicitly. The solution one obtains is universal in the sense that it is the same for all stationary solutions with non-vanishing mass. More precisely, one has the following lemma, whose proof is the same as that of lemma 7.2 in [31].

Lemma 5. *The solution to the restriction of the extended conformal geodesic equations to $\bar{\mathcal{I}}'$ with the components $\bar{L}_{\alpha\beta}$ as given in subsection 5.4 and initial data $x = (0, 0, \psi^{A'})$, $X^{\alpha} = \delta_0{}^{\alpha} + \delta_1{}^{\alpha}$ and $\bar{f}_{\alpha} = -\delta_{\alpha}{}^0 + \delta_{\alpha}{}^1$ is given by*

$$x(\tau) = (\bar{\tau}(\tau), 0, \psi^A(\tau)) = (\tau, 0, \psi^{A'}),$$

$$\bar{f}_{\mathbf{0}} = -\frac{1}{1+\tau}, \qquad \bar{f}_{\mathbf{1}} = 1, \qquad \bar{f}_A = 0.$$

This solution extends by analyticity to a domain $0 \leqslant \tau \leqslant 1 + 2\epsilon$ for some $\epsilon > 0$. The extension to $\bar{\mathcal{I}}'$ of the conformal factor Π determined by equations (44) and (42) takes on $\bar{\mathcal{I}}'$ the value $\Pi = 1$.

The previous lemma not only gives precise information about the conformal geodesics ruling $\bar{\mathcal{I}}'$, but it also shows that these geodesics extend analytically beyond $\tau = 1$. These facts are, in turn, used to show that there exists a solution of the conformal geodesic equations near $\bar{\mathcal{I}}'$, and that this solution extends for sufficiently large values of the parameter τ.

Consider a smooth extension of $\tilde{\mathcal{S}}$ into a range where $\bar{\rho} < 0$ such that $(\bar{\rho}, \psi^A)$ extend as smooth coordinates. We denote this extension by $\bar{\mathcal{S}}_{\text{ext}}$. Now, if the extension $\bar{\mathcal{S}}_{\text{ext}} \setminus \tilde{\mathcal{S}}$ is small enough, then the initial conditions for the conformal geodesic equations (39), (40) and (43) extend analytically to $\bar{\mathcal{S}}_{\text{ext}}$—the precise range of $\bar{\rho}$ is not required as long as it is small enough. Therefore, the conformal geodesic equations determine near $\bar{\mathcal{S}}_{\text{ext}}$ an analytic congruence of solutions to the extended conformal geodesic equations. From lemma 5, and making use of the well-known results of the theory of ordinary differential equations—see e.g. [37]—it follows that, using the same ϵ as in the lemma, there exists $\rho_{\#} > 0$ such that for the initial data

$$\bar{\tau} = 0, \qquad \bar{\rho} = \rho', \qquad \psi^A(0) = \psi^{A'}, \qquad \text{with} \quad |\rho'| < \rho_{\#},$$

and what is implied at these points by equations (39), (40) and (43), the solution of the extended conformal geodesic equations

$$\bar{\tau} = \bar{\tau}(\tau, \rho', \psi^{A'}), \quad \bar{\rho} = \bar{\rho}(\tau, \rho', \psi^{A'}), \quad \psi^A = \psi^A(\tau, \rho', \psi^{A'}), \quad \bar{f}_{\alpha} = \bar{f}_{\alpha}(\tau, \rho', \psi^{A'}) \tag{45}$$

exists for the values $0 \leqslant \tau \leqslant 1 + \epsilon$ of their natural parameter and the function Π is positive in the given range of ρ' and τ. Moreover, the Jacobian of the map $(\tau, \rho', \psi^{A'}) \rightarrow x^{\mu}(\tau, \rho', \psi^{A'})$ takes the value $1 + \bar{\tau}$ on $\bar{\mathcal{I}}'$ and for sufficiently small $\rho_{\#} > 0$ it does not vanish in the range $0 \leqslant \tau \leqslant 1 + \epsilon$, $|\bar{\rho}| \leqslant \rho_{\#}$. Therefore, the functions (τ, ρ', ψ^A) define a smooth coordinate system in a neighbourhood $\mathcal{O}'$ of $\bar{\mathcal{I}}'$ in $\bar{\mathcal{M}}'$. In particular, relation (41) implies that the curves

with $\rho' > 0$ cross $\mathscr{I}^{+'}$ for $\tau = 1$. The set $\mathcal{O}'$ contains the following special regions:

$$\mathcal{O}' \cap \mathscr{I}^{+'} = \{\tau = 1, \bar{\rho}' > 0\},$$
$$\mathcal{I}' = \{0 \leqslant \tau < 1, \bar{\rho}' = 0\},$$
$$\mathcal{I}^{+'} = \{\tau = 1, \rho' = 0\},$$

and it is ruled by conformal geodesics. As a consequence of this discussion, in terms of the frame fields $v_{\boldsymbol{\alpha}}$ and the coframe fields $\alpha^{\boldsymbol{\alpha}}$, the metric g, the connection coefficients $\hat{\Gamma}_{\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}}{}_{\boldsymbol{\gamma}}$ of the Weyl connection $\hat{\nabla}$ and the components of the tensor fields $\hat{L}_{\boldsymbol{\alpha\beta}}$, $f_{\boldsymbol{\alpha}}$, $W_{\boldsymbol{\alpha\beta\gamma\delta}}$ extend in the new coordinates as analytic fields to $\mathcal{O}'$.

Finally, the conformal geodesics on $\mathcal{O}'$ and the fields discussed in the previous paragraph can be used to implement the construction of the manifold $\bar{\mathcal{N}}$ as described in section 3. This is done by solving linear ordinary differential equations along the conformal geodesics, with the given analytical initial data on $\bar{\mathcal{S}}$. One obtains the following lemma:

Lemma 6. *For stationary asymptotically flat initial data (as described in section 4) the construction of section 3 leads to a conformal representation of the stationary vacuum spacetime which, in a neighbourhood $\mathcal{O} \subset \bar{\mathcal{N}}$ of the set $\bar{\mathcal{I}}$, is real analytic in the radial and angular coordinates.*

6.4. The conformal gauge for the initial data

The conformal representation discussed in the previous lemma has made use of the 3-metric $\check{h}$ and the conformal factor $\check{\Omega}$ on $\tilde{\mathcal{S}}$. It remains to be verified that the whole construction is robust with respect to rescalings on the conformal initial data of the form

$$\check{h} \to \check{h} = \vartheta^2 \check{h}, \qquad \check{\Omega} \to \check{\Omega} = \vartheta \check{\Omega}, \tag{46}$$

where ϑ is an analytic, positive conformal factor. These rescalings correspond to a change in the conformal gauge, and imply a harmless change of the normal coordinates $x^a \to x^{a'}$ with $x^{a'}(0) = 0$ and an associated change $e_a \to e'_a$ of the frame vector fields tangent to $\tilde{\mathcal{S}}$, which will be propagated along the new conformal geodesics. It is necessary to understand how the congruence of conformal geodesics corresponding to $\check{\Omega}$ relates to the congruence corresponding to $\check{\Omega}$.

The conformal rescaling given by (46) also implies the transitions

$$\kappa \to \check{\kappa} = \frac{\vartheta \kappa}{\varsigma}, \qquad \Theta_* \to \check{\Theta}_* = \varsigma \Theta_*,$$

where

$$\varsigma \equiv \left| 1 - 3\vartheta^{-1} \frac{\check{D}_a \check{\Omega} \check{D}^a \vartheta}{\Delta_{\check{h}} \check{\Omega}} - \frac{3}{2} \vartheta^{-2} \check{\Omega} \frac{\check{D}_a \vartheta \check{D}^a \vartheta}{\Delta_{\check{h}} \check{\Omega}} \right|^{1/2}.$$

The function ς extends to $\bar{\mathcal{S}}$ as an analytic function of $(\bar{\rho}, \psi^A)$. From the initial conditions for the $\check{\Omega}$-congruence of conformal geodesics one gets that

$$\dot{\check{x}} = \varsigma^{-1} \dot{x}, \qquad \check{f}_{\tilde{\mathcal{S}}} = f_{\tilde{\mathcal{S}}} + \vartheta^{-1} \mathrm{d}\vartheta - \varsigma^{-1} \, \mathrm{d}\varsigma,$$

where the subscripts indicate the pull-back to $\tilde{\mathcal{S}}$. It can be verified that

$$\dot{\check{x}} \perp \tilde{\mathcal{S}}, \qquad \check{\Theta}^2 \tilde{g}(\dot{x}, \dot{x}) = 1.$$

In what follows, we consider the equations for the $\check{\Omega}$-congruence in terms of g and its Lévi–Città connection ∇. As a result of the conformal invariance of conformal geodesics, it follows

that the spacetime curves do not change (as set points) writing the equations in this form. Furthermore, their parameter $\check{\tau}$ remains unchanged. The 1-form is transformed according to

$$\check{f} \to f^* = \check{f} - (\Theta\check{\Theta})^{-1}d(\Theta\check{\Theta}),$$

and therefore

$$\langle f^*, \dot{x} \rangle = 0, \qquad f^*_{\tilde{S}} = f_{\tilde{S}} + \vartheta^{-1}d\vartheta, \quad \text{on } \tilde{S}.$$

If one expresses the 1-form f^* in terms of the g-orthonormal frame e_{α} satisfying $e_0 \perp \tilde{S}$, one obtains

$$f^*_0 = 0, \qquad f^*_a = f_a + \vartheta^{-1}\langle d\vartheta, e_a \rangle, \qquad a = 1, 2, 3.$$

The fields $\dot{x}$ and f^*_{α} are the initial data for the $\check{\Omega}$-congruence written in terms of g, e_{α} and ∇. As $\varsigma \to 1$ and $\langle d\vartheta, e_a \rangle = O(\bar{\rho})$ as $\bar{\rho} \to 0$, then

$$\check{\Theta} \to \Theta, \qquad \dot{x} \to \dot{x}, \qquad f^*_{\alpha} \to f_{\alpha} \qquad \text{as} \quad \bar{\rho} \to 0. \tag{47}$$

This means that the limits of the initial data for both congruences coincide on $\mathcal{I}^{0'}$, and therefore the corresponding curves are identical on $\bar{\mathcal{I}}'$.

Now, one can go back to the arguments used to show the smoothness of the construction in terms of the $\check{\Omega}$-congruence and apply them to the $\check{\Omega}$-congruence. One concludes that in a certain neighbourhood $\mathcal{O}' \subset \bar{\mathcal{M}}'$ of $\bar{\mathcal{I}}'$ the gauge and construction of section 3 for the $\check{\Omega}$-congruence is as smooth and regular as the one based on the $\check{\Omega}$-congruence. This result is summarized in the following lemma:

Lemma 7. *For stationary asymptotically flat spacetimes, the construction of the set $\bar{\mathcal{I}}'$ is independent of the choice of the conformal factor Ω. The set $\mathcal{I}'$ coincides with the projection $\pi'(\mathcal{I})$ of the cylinder at space-like infinity as defined in section 3.*

This concludes the proof of our main theorem—cf subsection 1.6.

7. Conclusions

The discussion in the previous section has shown that for initial data sets which are stationary in the asymptotic region, the construction of the cylinder at spatial infinity is as regular as one would expect it to be. As a consequence, the solutions to the associated regular initial value problem at spatial infinity are regular at the critical sets $\mathcal{I}^{\pm}$ notwithstanding the degeneracy of a subset of the evolution equations at these sets. As the length of our analysis shows, this is by no means an obvious result, and it makes evident the delicate interplay between geometry and properties of differential equations that the conformal framework allows to resolve. Moreover, it brings to the forefront the special role played by stationary solutions in the class of solutions to the Einstein field equations admitting a smooth compactification at null infinity.

It is worth pointing out that the analysis carried out in this paper is essentially a spacetime one. A proof of our main theorem that relies only on properties of stationary data and the conformal evolution would be, by necessity, much more complicated and would require an understanding of the structure of the conformal field equations that is not yet available.

It is expected that our analysis will play an essential role in the construction of suitable non-time symmetric generalizations of the rigidity results for asymptotically simple spacetimes in [49, 50]. In this respect, it will also be of interest to obtain a parametrization of a large class of initial data sets for which it is easy to recognize when the data is, in fact, stationary. This type of characterizations may well require the consideration of other properties of stationary solutions at spatial infinity which have not been touched upon here—most notably, whether stationary data sets satisfy some generalization of the regularity conditions of [28, 48].

Appendix. Various expansions

In this appendix, we collect the expansions of some auxiliary quantities used in this paper.

Recalling that $\bar{\Omega} = \rho^{-1} V^{\frac{1}{2}} \Omega$, one obtains

$$\bar{\Omega} = \rho + M\rho^2 + \frac{1}{6M^2}\rho^3(5M^4 + 2MM_{ab}e^a e^b + 2(S_a e^a)^2) + O(\rho^4),$$

$$\partial_a \bar{\Omega} = e_a + 2Me_a\rho + \frac{1}{6M^2}\rho^2(15M^4 e_a + 2Me_a M_{bc}e^b e^c$$
$$+ 4MM_{ab}e^b + 2e_a(S_b e^b)^2 + 4S_a(S_b e^b)) + O(\rho^3),$$

$$\partial_a \partial_b \bar{\Omega} = \frac{1}{\rho}(\delta_{ab} - e^a e^b) + 2M\delta_{ab} + \frac{1}{3M^2}\rho\left(\frac{15}{2}M^4(\delta_{ab} + e_a e_b)\right.$$
$$+ M(2M_{ab} + 4e_{(a}M_{b)c}e^c + (\delta_{ab} - e_a e_b)M_{cd}e^c e^d)$$
$$\left. + 2S_a S_b + 4e_{(a}S_{b)}S_c e^c + (\delta_{ab} - e_a e_b)(S_c e^c)^2\right) + O(\rho^2).$$

Also

$$\bar{g}^{ab}\partial_a\bar{\Omega}\partial_b\bar{\Omega} = -\rho^2 - 4M\rho^3 - \frac{1}{M^2}\rho^4(9M^4 + 2MM_{ab}e^a e^b + 2(S_a e^a)^2) + O(\rho^5).$$

The Christoffel symbols of $\bar{g}$ are given by

$$\bar{\Gamma}_\mu{}^\rho{}_\nu = \frac{1}{2}\bar{g}^{\rho\lambda}(\partial_\mu\bar{g}_{\lambda\nu} + \partial_\nu\bar{g}_{\mu\lambda} - \partial_\lambda\bar{g}_{\mu\nu}).$$

It follows that

$$\bar{\Gamma}_t{}^a{}_t = -\frac{1}{2}\bar{g}^{ab}\partial_b\bar{g}_{tt}$$
$$= \rho^3 e^a + 6M\rho^4 e^a + \frac{2}{3M^2}\rho^5\left(26M^4 e^a + e^a MM_{bc}e^b e^c\right.$$
$$\left. + MM^a{}_b e^b + e^a(S_b e^b)^2 + S^a S_b e^b\right) + O(\rho^6),$$

$$\bar{\Gamma}_t{}^b{}_a = \frac{1}{2}\left(\bar{g}^{bt}\partial_a\bar{g}_{tt} + \bar{g}^{bc}\partial_a\bar{g}_{tc} - \bar{g}^{bc}\partial_c\bar{g}_{ta}\right)$$
$$= -\rho^3(2S^c e_{ca}{}^b + e^c S^d e_{cda}e^b + e_c S_d e^{cdb}e_a) + O(\rho^4),$$

$$\bar{\Gamma}_a{}^c{}_b = \bar{g}^{ct}\partial_{(a}\bar{g}_{b)t} + \frac{1}{2}\bar{g}^{cd}(\partial_a\bar{g}_{bd} + \partial_b\bar{g}_{ad} - \partial_d\bar{g}_{ab})$$
$$= \rho^{-1}(\delta_{ab}e^c - 2\delta_{(a}^c e_{b)}) + \frac{1}{3M^2}\rho\left(-M^4(\delta_{ab}e^c + e_a e_b e^c - 2\delta_{(a}^c e_{b)})\right.$$
$$+ 2M(2e_{(a}M_{b)}{}^c - e^c M_{ab} + 2\delta_{(a}^c M_{b)d}e^d - 2\delta_{ab}M_d^c e^d$$
$$+ \delta_{ab}e^c M_{de}e^d e^e - 2e_{(a}e^c M_{b)d}e^d) + 2(2e_{(a}S_{b)}S^c - e^c S_a S_b$$
$$\left. + 2\delta_{(a}^c S_{b)}S_d e^d - 2\delta_{ab}S^c S_d e^d + \delta_{ab}e^c(S_d e^d)^2 - 2e_{(a}e^c S_{b)}S_d e^d)\right) + O(\rho^2).$$

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